Noncommutative Algebra

Lecture Notes

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Preface

These notes grew out of a graduate course in noncommutative algebra that I taught at Iowa State University in the Fall semester of 2020. I wrote the lectures by hand, and the handwritten notes for all twenty-seven lectures were made available to the students on the course web page. This text is a typeset and revised version of those notes.

I have tried to keep the character of lecture notes. Each chapter corresponds to one lecture (the two lectures on the Diamond Lemma have been merged into a single chapter), the tone is informal, and the emphasis is on examples and on a handful of ideas that recur throughout: presentations by generators and relations, skew polynomial rings and generalized Weyl algebras, gradings and filtrations, the Diamond Lemma, tensor products and Morita theory, Clifford theory, growth and Gelfand–Kirillov dimension, and quantum algebras and deformations. Arguments that were only sketched in class have been completed where they are short; longer ones have been turned into exercises or replaced by precise references.

I would like to thank the students who took part in the course for their questions and comments, in particular Dylan Fillmore and Kirin Martin.

How this text was prepared

The handwritten notes were transcribed into LaTeX and revised with the assistance of Claude, an AI model developed by Anthropic. Claude typeset the text and diagrams, and it went through the notes lecture by lecture, listing errors, gaps, and inconsistencies and proposing one or more ways to resolve each of them: for instance by completing a proof, correcting or sharpening a statement, adding a reference, or turning an argument into an exercise. I reviewed these proposals and decided, for each one, which to adopt, and a record of those decisions was kept. Responsibility for the content of this text, including any remaining errors, is mine.

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Notation

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$\Bbbk$	the base field	2
$\mathbb{C}G$	group algebra of a group G	3
A^{op}	opposite algebra of A	3
$U(\mathfrak{g})$	universal enveloping algebra of a Lie algebra $\mathfrak{g}$	5
$\mathfrak{gl}_n$	general linear Lie algebra of $n \times n$ matrices	5
$\mathfrak{sl}_n$	special linear Lie algebra of traceless $n \times n$ matrices	6
$T(V)$	tensor algebra of a vector space V	6
$\langle R \rangle_{\mathrm{ideal}}$	ideal generated by a subset R of an algebra	6
$\langle X \rangle_{\mathrm{mon}}$	monoid generated by X (free monoid if X is an abstract set)	6
$\langle X \rangle_{\mathrm{grp}}$	group generated by X	6
$\langle X \rangle_{\mathrm{alg}}$	algebra generated by X (free algebra if X is an abstract set)	6

Chapter 2

$F(X)$, $F_{\Bbbk}(X)$	free algebra on a set X	8
$\Bbbk\langle X \mid R = 0 \rangle$	algebra with generators X and relations R	9
$A_1(\Bbbk)$	first Weyl algebra, $\Bbbk\langle x, y \mid yx - xy = 1 \rangle$	9
$\underline{\mathrm{Set}}$, $\underline{\mathrm{Alg}}$	categories of sets and of algebras	10
e, f, h	standard basis of $\mathfrak{sl}_2$	10

Chapter 3

$A\langle X \rangle$	algebra obtained from A by adjoining a generator X	12
$A\langle X \mid S = 0 \rangle$	algebra obtained from A by adjoining X subject to the relations $S = 0$	14
${}_A\underline{\mathrm{Mod}}$	category of left A-modules	14

Chapter 4

$A[x; \sigma, \delta]$	skew polynomial ring (Ore extension), with $xa = \sigma(a)x + \delta(a)$	15
$A[x; \sigma]$	skew polynomial ring $A[x; \sigma, 0]$, with $xa = \sigma(a)x$	17
Res_A^B	restriction functor from B-modules to A-modules, for a subalgebra $A \subseteq B$	18

Chapter 5

$M[X]$	the module $\bigoplus_{k \geq 0} MX^k$ over $A[X]$, for an A-module M	20
$R(\sigma, t)$	generalized Weyl algebra with base algebra R, automorphism σ and central element t	22

Chapter 6

$\mathrm{Aut}_{\Bbbk\text{-}\mathrm{Alg}}(R)$	group of algebra automorphisms of R	25
$R[X, X^{-1}; \sigma]$	skew Laurent polynomial ring over R with automorphism σ	26
$(A_m)_{m \in M}$	M-gradation of A; A_m is the component of degree m	27
$A_m A_n$	span of all products ab with $a \in A_m$, $b \in A_n$	27

CHAPTER 7

$E_n(d)$	eigenspace of a linear map d for the eigenvalue n	30
$\overline{A_m}$	image of the component A_m in a graded quotient A/I	31
I_m	degree-m component $A_m \cap I$ of a graded ideal I	31
$C_A(R)$	centralizer of a subset R in an algebra A	33
$Z(A)$	center of an algebra A	33

CHAPTER 8

$A_{(n)} = F_n A$	n-th piece of a filtration F of an algebra A	34
$\operatorname{gr} A = \operatorname{gr}_F A$	associated graded algebra of a filtered algebra (A, F)	35
$\operatorname{gr}_n A$	degree-n component $A_{(n)}/A_{(n-1)}$ of $\operatorname{gr} A$	35
$\bar{a}$	leading term $a + A_{(n-1)} \in \operatorname{gr}_n A$ of $a \in A_{(n)}$	35
$\operatorname{Kdim}(A)$	Krull dimension of A	37
$\operatorname{gl.dim}(A)$	global dimension of A	37

CHAPTER 9

$\mathbb{H}$	rational Cherednik algebra of S_2	38
$\mathbb{H}(W)$	rational Cherednik algebra of a complex reflection group W	39
ad_a	inner derivation $b \mapsto [a, b] = ab - ba$	40

CHAPTER 10

$\sigma = (W_\sigma, f_\sigma)$	element of a reduction system, the rule $W_\sigma \rightsquigarrow f_\sigma$	43
$r_{A\sigma B}$	reduction replacing $AW_\sigma B$ by $Af_\sigma B$	43
$\Bbbk\langle X\rangle_{\text{irr}}$	subspace of irreducible elements for a reduction system	44
$r_S(a)$	common irreducible value of all final reductions of a reduction-unique a	44
R_S	subspace of reduction-unique elements of $\Bbbk\langle X\rangle$	45
$f < A$	every word occurring in f is smaller than the word A	46
$\ell(A)$	length of a word A	46
I_A	span of the $B(W_\sigma - f_\sigma)C$ with $BW_\sigma C < A$	47

CHAPTER 11

c_{ijk}	structure constants $b_i b_j = \sum_k c_{ijk} b_k$ of an algebra in a basis $\{b_i\}$	49
$N_X(A)$	number of occurrences of the letter X in a word A	50
$m(A)$	misordering index of a word A in the generators of a GWA	50
$\mathfrak{so}_3(\mathbb{C})$	Lie algebra $\mathbb{C}^3$ with the cross product	52

CHAPTER 12

RG	group algebra of G over an algebra R	55
$R \rtimes_\sigma G$, $R \rtimes G$	skew group algebra of G acting on R via σ	56
R^G	algebra of invariants of a group G acting on R	57
$R^\times$	group of units of an algebra R	58
$R *_{\sigma,\alpha} G$	crossed product of R and G with crossed product data (σ, α)	58
Ad_a	inner automorphism $r \mapsto ara^{-1}$ for a unit a	58

CHAPTER 13

$V \otimes W$, $V \otimes_\Bbbk W$	tensor product of vector spaces V and W	61
$\operatorname{Bil}(V \times W, U)$	vector space of bilinear maps $V \times W \to U$	61
$F \dashv G$	F is left adjoint to G	64
$\underline{\text{Vect}}$, $\underline{\text{Vect}}_\Bbbk$	category of $\Bbbk$-vector spaces	64

CHAPTER 14

${}_A\underline{\mathrm{Mod}}_B$	category of (A,B)-bimodules	65
$\underline{\mathrm{Mod}}_B$	category of right B-modules	65
${}_AX_B$	X regarded as an (A,B)-bimodule	65
$\mathrm{Bal}(X\times Y, Z)$	$\Bbbk$-module of balanced maps $X\times Y\to Z$	65
$X\otimes_B Y$	tensor product over B of ${}_AX_B$ and ${}_BY_C$, an (A,C)-bimodule	66
$\mathrm{Ind}_{\Bbbk H}^{\Bbbk G}$	induction from $\Bbbk H$-modules to $\Bbbk G$-modules, $\Bbbk G\otimes_{\Bbbk H}-$	68

CHAPTER 15

Ind_R^A	induction along a morphism $R\to A$, $M\mapsto A\otimes_R M$	70
$\Bbbk_\alpha$	one-dimensional $\Bbbk[x]$-module $\Bbbk[x]/(x-\alpha)$	71
$Z^{(n)}$	X^n, 1 or $Y^{\lvert n\rvert}$ in a GWA, according as $n>0$, $n=0$, $n<0$	72
$\widehat{M}$	module $\mathrm{Ind}_R^A M$ induced from R to a GWA $A=R(\sigma,t)$	72

CHAPTER 16

$\mathcal{C}\simeq\mathcal{C}'$	the categories $\mathcal{C}$ and $\mathcal{C}'$ are equivalent	77
$A\overset{\text{Mor.}}{\approx}A'$	A and A' are Morita equivalent	78

CHAPTER 17

$A^{\otimes n}$	n-fold tensor power $A\otimes_\Bbbk\cdots\otimes_\Bbbk A$	80
eAe	corner algebra of A at an idempotent e, with unit e	81

CHAPTER 18

${}^xW$	twist $B_x\otimes_A W$ of an A-module W by $x\in\Gamma$	86
Γ_W	stabilizer of an A-module W in Γ	86
ℓ_S	multiplicity of a simple module S in a module of finite length	86
$V(S)$	S-homogeneous component of a module V	87
B_S	subalgebra $\bigoplus_{x\in\Gamma_S}B_x$ of a Γ-crossed product B	87

CHAPTER 19

$\mathrm{ann}_A(v)$	annihilator of an element v of an A-module (a left ideal)	90
$\mathrm{Ann}_A(M)$	annihilator of an A-module M (a two-sided ideal)	90
$\mathrm{MaxSpec}(R)$	set of maximal ideals of a commutative ring R	90
$M_{\mathfrak{m}}$	weight space of weight $\mathfrak{m}$ in a module M	91

CHAPTER 20

V^n	span of all products of n elements of a subspace V of an algebra	95
$V^{(n)}$	$V^0+V^1+\cdots+V^n$, for a subspace V of an algebra	95
$d_V(n)$	growth function $\dim V^{(n)}$ of a subspace V	95
$\mathrm{GKdim}\,A$	Gelfand–Kirillov dimension of an algebra A	96
$\mathrm{Spec}(A)$	set of prime ideals of a commutative algebra A	97

CHAPTER 21

Φ	set of eventually monotone increasing positive functions $\mathbb{N}\to\mathbb{R}$	98
$f\preceq^* g$	$f(n)\le c\,g(mn)$ for some $c,m\in\mathbb{N}$ and all $n\gg 0$	98
$f\sim g$	equivalence of growth functions: $f\preceq^* g$ and $g\preceq^* f$	98
$\mathscr{G}(f)$	growth (equivalence class) of a function $f\in\Phi$	98
$\mathscr{P}_\gamma$	growth of $n\mapsto n^\gamma$	99
$\mathscr{E}_\varepsilon$	growth of $n\mapsto e^{n^\varepsilon}$	99

$\mathscr{G}(A)$ — growth of a finitely generated algebra A 99

CHAPTER 22

$A_1 \times A_2$ — direct product of algebras, with componentwise multiplication 103

CHAPTER 23

$S^{-1}A$ — localization of A at a multiplicative set S (central, or later Ore) 109
$S(\mathfrak{g})$ — symmetric algebra on a vector space $\mathfrak{g}$ 110

CHAPTER 24

$\Bbbk_q[x, y]$, $\mathcal{O}_q(\mathbb{A}^2_\Bbbk)$ — quantum plane $\Bbbk\langle x, y\rangle/\langle yx - qxy\rangle_{\text{ideal}}$ 112
A_Q, $\Bbbk_Q[x_1, \dots, x_n]$ — quantum affine space with parameter matrix $Q = (q_{ij})$ 113
$\Bbbk_Q[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ — quantum torus with parameter matrix Q 114
$U_q(\mathfrak{sl}_2)$ — quantized enveloping algebra of $\mathfrak{sl}_2$ 115
$\Bbbk[\![t]\!]$ — ring of formal power series in t over $\Bbbk$ 115
$U_\hbar(\mathfrak{sl}_2)$ — $\hbar$-adic quantized enveloping algebra of $\mathfrak{sl}_2$ over $\Bbbk[\![\hbar]\!]$ 115
$s_m(T_1, \dots, T_m)$ — standard polynomial of degree m 116

CHAPTER 25

$\varprojlim A_j$ — limit of a system of objects A_j 117
$\underline{\mathrm{Ab}}$ — category of abelian groups 117
$\widehat{\mathbb{Z}}$ — profinite integers $\varprojlim \mathbb{Z}/n\mathbb{Z}$ 118
$\widehat{R}_I$ — I-adic completion of a ring R 119
$\underline{\mathrm{Rng}}$ — category of rings (with 1) 119
$\mathbb{Z}_p$ — ring of p-adic integers 119
$\mathscr{P}(X)$ — power set of a set X 119
$\langle \mathscr{A} \rangle$ — topology generated by a family $\mathscr{A}$ of subsets 119

CHAPTER 26

$A[\![\hbar]\!]$ — formal power series in $\hbar$ with coefficients in A 122
$C^n(A)$ — space of Hochschild n-cochains on A 124
d^n — Hochschild coboundary operator $C^n(A) \to C^{n+1}(A)$ 124
$Z^n(A)$ — Hochschild n-cocycles, Ker d^n 124
$B^n(A)$ — Hochschild n-coboundaries, im d^{n-1} 124
$H^n(A)$ — n-th Hochschild cohomology group $Z^n(A)/B^n(A)$ 124
$\mathrm{Der}(A)$ — space of derivations of A 124
$\mathrm{Inn}(A)$ — space of inner derivations $[b, -]$ of A 124
ob_n — obstruction to extending a deformation from order n to order $n+1$ . . 125

Part I

Algebras by generators and relations

Chapter 1

Algebras and Modules

This chapter sets up the basic language: associative algebras, their modules (which are the same thing as representations), and a first look at Lie algebras and their universal enveloping algebras, which will be one of our main sources of examples. We end with a couple of open problems, to give a sense of where the subject currently stands.

1.1 Algebras

Throughout the book we work over a field $\Bbbk$. In this first chapter we mostly take $\Bbbk = \mathbb{C}$, to keep the examples concrete. Whenever a later result needs a hypothesis on $\Bbbk$ (typically $\operatorname{char} \Bbbk = 0$), we will say so explicitly.

Definition 1.1. An *(associative, unital) algebra* over $\Bbbk$, or *$\Bbbk$-algebra*, is a vector space A together with

- a *multiplication* $A \otimes A \to A$, $a \otimes b \mapsto ab$, which is associative:
$$(ab)c = a(bc) \qquad \text{for all } a, b, c \in A;$$
- a *unit* $1_A \in A$ satisfying $1_A a = a = a 1_A$ for all $a \in A$.

Writing the multiplication as a linear map out of $A \otimes A$ is just a compact way of saying that it is bilinear: by the universal property of the tensor product (Chapter 13) there is a bijection

$$\left\{ \begin{array}{c} \text{bilinear maps} \\ A \times A \to A \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{c} \text{linear maps} \\ A \otimes A \to A \end{array} \right\}.$$

Remark 1.2. Nothing in Definition 1.1 needs $\Bbbk$ to be a field: over any commutative ring k one gets the notion of a k-algebra by replacing "vector space" with "k-module". For instance, a $\mathbb{Z}$-algebra is just a ring. We will stick to fields.

1.2 Modules and representations

Definition 1.3. Let A be an algebra. A *left A-module* is a vector space V together with an *action* $A \otimes V \to V$, $a \otimes v \mapsto a.v$, such that

$$(ab).v = a.(b.v) \qquad \text{and} \qquad 1_A.v = v$$

for all $a, b \in A$ and $v \in V$.

Right modules are defined in the same way, with A acting on the right. Unless we say otherwise, "module" means left module.

Example 1.4. The smallest possible algebra is $A = \Bbbk = \Bbbk \cdot 1_A$ itself, and a $\Bbbk$-module is nothing but a vector space.

Example 1.5. For any algebra A and $n \geq 1$, the *free module* $V = A^n = \{(a_1, \dots, a_n) \mid a_i \in A\}$ is an A-module with the action

$$a.(a_1, \dots, a_n) = (aa_1, \dots, aa_n).$$

Example 1.6. Let G be a finite group acting on a set X. The *group algebra*

$$A = \mathbb{C}G = \left\{ \sum_{g \in G} \lambda_g g \;\middle|\; \lambda_g \in \mathbb{C} \right\}$$

has G as a basis, and its multiplication extends the multiplication of G bilinearly. Let $V = \mathbb{C}X$ be the vector space with basis X. Extending the action of G on X bilinearly,

$$\Big(\sum_g \lambda_g g\Big).\Big(\sum_x \mu_x x\Big) = \sum_{g,x} \lambda_g \mu_x \, (g.x),$$

makes $\mathbb{C}X$ into a $\mathbb{C}G$-module.

Definition 1.7. A *representation* of an algebra A is a pair (V, ρ), where V is a vector space and $\rho : A \to \mathrm{End}(V)$ is an algebra map.

Modules and representations are two names for the same thing. There is a bijection

$$\left\{ \begin{matrix} \text{representations} \\ (V, \rho) \text{ of } A \end{matrix} \right\} \xrightarrow{\sim} \left\{ \begin{matrix} A\text{-modules} \\ V \end{matrix} \right\}, \qquad (V, \rho) \longmapsto V \text{ with } a.v = \rho(a)v,$$

whose inverse sends a module V to the representation $\rho(a) = (v \mapsto a.v)$. The axioms of a module say precisely that ρ is an algebra map.

Remark 1.8 (Right modules). For an algebra A, let A^{op} be the same vector space with the opposite multiplication $a \cdot_{\mathrm{op}} b = ba$. A right A-module is the same thing as an algebra map

$$\rho : A \to \mathrm{End}(V)^{\mathrm{op}}, \qquad v.a = \rho(a)v,$$

since $v.(ab) = (v.a).b$ says exactly that $\rho(ab) = \rho(b) \circ \rho(a) = \rho(a) \cdot_{\mathrm{op}} \rho(b)$.

For group algebras, the difference between left and right is harmless. The map $g \mapsto g^{-1}$ is an isomorphism of groups $G \cong G^{\mathrm{op}}$, and it extends linearly to an algebra isomorphism $\mathbb{C}G \cong (\mathbb{C}G)^{\mathrm{op}}$. So a right $\mathbb{C}G$-module becomes a left one via $g.v := v.g^{-1}$, and vice versa.

Since modules and representations are the same thing, the subject has two parallel vocabularies, and we will use both freely:

Module term		Representation term
module	$\longleftrightarrow$	representation
simple module	$\longleftrightarrow$	irreducible representation
indecomposable module	$\longleftrightarrow$	indecomposable representation
($V \not\cong U_1 \oplus U_2$ with $U_1, U_2 \neq 0$)		
semisimple module	$\longleftrightarrow$	completely reducible representation
($V = \bigoplus_i V_i$ with each V_i simple)		
A-module map $\varphi : V \to W$	$\longleftrightarrow$	intertwining operator $\varphi : V \to W$
($\varphi(a.v) = a.\varphi(v)$)		

Here a module V is *simple* if $V \neq 0$ and its only submodules are 0 and V.

1.3 Examples of algebras

Here are some of the algebras that will keep us company. We will meet several of them properly later on.

- $\mathrm{End}(V)$, and in particular the matrix algebra $M_n(\mathbb{C}) = \mathrm{End}(\mathbb{C}^n)$;
- group algebras $\mathbb{C}G$ of finite groups G (Example 1.6);
- universal enveloping algebras $U(\mathfrak{g})$ of Lie algebras $\mathfrak{g}$ (Section 1.5 below);
- generalized Weyl algebras (GWAs, Chapter 5) and their relatives, the *twisted* generalized Weyl algebras (TGWAs);
- quantum groups, which are, roughly, deformations of $U(\mathfrak{g})$ depending on a parameter q (we meet some simpler quantum algebras in Chapter 24);
- affine Hecke algebras, which show up in the representation theory of p-adic groups;
- $\vdots$

For most of these algebras, finite-dimensional modules are much better understood than infinite-dimensional ones. We will come back to this at the end of the chapter.

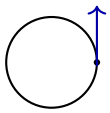
1.4 Lie algebras

Lie algebras arise as linearisations of Lie groups. A *Lie group* is a group that is also a smooth manifold, in such a way that multiplication and inversion are smooth. Its tangent space at the identity carries a natural bracket, which makes it a Lie algebra:

$$\text{Lie group} \;\; \overset{\longleftarrow}{\rightsquigarrow} \;\; \text{Lie algebra.}$$

Surprisingly little information is lost in this passage, and one can go back (the dotted arrow), at least up to coverings.

The simplest example is the circle group $U(1) = \{ z \in \mathbb{C} \mid |z| = 1 \}$. A tangent vector to the circle at 1 is determined by a real number x, so the Lie algebra of $U(1)$ is $\mathbb{R}$ (more precisely, the tangent line $i\mathbb{R}$), with the zero bracket $[x, y] = 0$ for all x, y. This is as it should be, since $U(1)$ is commutative.



$U(1)$

$\longmapsto \quad x \in \mathbb{R}, \quad [x, y] = 0$ for all x, y

Definition 1.9. A *Lie algebra* is a vector space $\mathfrak{g}$ together with a linear map $\mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$, $x \otimes y \mapsto [x, y]$, called the *bracket*, such that for all $x, y, z \in \mathfrak{g}$:

(L1) $[x, x] = 0$;
(L2) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ (the *Jacobi identity*).

Remark 1.10. Expanding $0 = [x + y, x + y]$ using (L1) gives $[x, y] = -[y, x]$, so the bracket is antisymmetric. In characteristic $\neq 2$ the converse also holds: antisymmetry gives $2[x, x] = 0$, hence $[x, x] = 0$. So away from characteristic 2 one can equally well take antisymmetry as the first axiom.

Example 1.11. Any associative algebra A becomes a Lie algebra with the commutator bracket $[x, y] = xy - yx$. Axiom (L1) is clear. For the Jacobi identity, expand all twelve terms: they cancel in pairs.

1.5 Universal enveloping algebras

Recall that a group G gives rise to an associative algebra $\mathbb{C}G$, and that modules over G (that is, representations of G) are the same thing as $\mathbb{C}G$-modules. Lie algebras work in the same way. A *module* over a Lie algebra $\mathfrak{g}$ (a *representation* of $\mathfrak{g}$) is a vector space V with a linear action $\mathfrak{g} \otimes V \to V$ satisfying $[x, y].v = x.(y.v) - y.(x.v)$. To each Lie algebra we attach an associative algebra $U(\mathfrak{g})$, its *(universal) enveloping algebra*, such that

$$\left\{ \begin{array}{c} G\text{-modules} \\ (= \text{representations of } G) \end{array} \right\} \longleftrightarrow \big\{ \mathbb{C}G\text{-modules} \big\},$$

$$\left\{ \begin{array}{c} \mathfrak{g}\text{-modules} \\ (= \text{representations of } \mathfrak{g}) \end{array} \right\} \longleftrightarrow \big\{ U(\mathfrak{g})\text{-modules} \big\}.$$

Here is the construction in the most important example.

Example 1.12 ($U(\mathfrak{gl}_n)$)**.** The matrix algebra $A = M_n(\mathbb{C}) = \{n \times n \text{ matrices}\}$ is an associative algebra, so by Example 1.11 it is also a Lie algebra:

$$\mathfrak{gl}_n = M_n(\mathbb{C}) \quad \text{with} \quad [x, y] = xy - yx \ \text{ for all } n \times n \text{ matrices } x, y,$$

the *general linear Lie algebra*. Since $\operatorname{tr}(xy - yx) = 0$, the traceless matrices

$$\mathfrak{sl}_n = \{ x \in \mathfrak{gl}_n \mid \operatorname{tr}(x) = 0 \}$$

are closed under the bracket. They form the *special linear Lie algebra*.

The enveloping algebra of $\mathfrak{gl}_n$ is

$$U(\mathfrak{gl}_n) = \frac{T(\mathfrak{gl}_n)}{\langle x \otimes y - y \otimes x - [x,y] \mid x, y \in \mathfrak{gl}_n \rangle_{\text{ideal}}}.$$

Here $T(V) = \bigoplus_{j \geq 0} V^{\otimes j}$ is the *tensor algebra* of a vector space V, whose multiplication is concatenation of tensors: $(u_1 \otimes \cdots \otimes u_i)(v_1 \otimes \cdots \otimes v_j) = u_1 \otimes \cdots \otimes u_i \otimes v_1 \otimes \cdots \otimes v_j$. The notation $\langle \cdots \rangle_{\text{ideal}}$ denotes the two-sided ideal generated by the given elements. Throughout the book, a subscript on angle brackets says what kind of object is generated: $\langle R \rangle_{\text{ideal}}$ is the ideal generated by R, $\langle X \rangle_{\text{mon}}$ the monoid generated by X (the free monoid if X is an abstract set), $\langle X \rangle_{\text{grp}}$ the group generated by X, and $\langle X \rangle_{\text{alg}}$ the algebra generated by X (free if X is an abstract set, a subalgebra if X is a subset of a given algebra). (In Chapter 2 we will recognise this as a presentation by generators and relations.)

Writing products in $U(\mathfrak{gl}_n)$ by juxtaposition, every element of $U(\mathfrak{gl}_n)$ is a linear combination of products $x_1 x_2 \cdots x_j$ with $x_1, \ldots, x_j \in \mathfrak{gl}_n$. The relations say exactly that the commutator $xy - yx$ in $U(\mathfrak{gl}_n)$ agrees with the bracket $[x, y]$ of $\mathfrak{gl}_n$. The same construction, with $\mathfrak{gl}_n$ replaced by any Lie algebra $\mathfrak{g}$, defines $U(\mathfrak{g})$.

1.6 Some open problems

To end the chapter, here are two problems that are open (as of 2020) and that indicate how much remains to be understood about infinite-dimensional modules. Both concern a class of modules that sits between the finite-dimensional ones and all modules.

Gelfand–Tsetlin modules. Embedding each $\mathfrak{gl}_k$ into $\mathfrak{gl}_{k+1}$ as the upper-left $k \times k$ block gives a chain of algebras $U(\mathfrak{gl}_1) \subset U(\mathfrak{gl}_2) \subset \cdots \subset U(\mathfrak{gl}_n)$. The subalgebra Γ of $U(\mathfrak{gl}_n)$ generated by the centers of all the $U(\mathfrak{gl}_k)$ is commutative. It is called the *Gelfand–Tsetlin subalgebra*. A $U(\mathfrak{gl}_n)$-module V is a *Gelfand–Tsetlin module* if it is the direct sum of its generalised eigenspaces for Γ, that is, of the subspaces $V(\mathfrak{m}) = \left\{ v \in V \;\middle|\; \mathfrak{m}^k v = 0 \text{ for some } k \right\}$ for maximal ideals $\mathfrak{m}$ of Γ. Every finite-dimensional module is a Gelfand–Tsetlin module; this is the origin of the classical Gelfand–Tsetlin bases. (For the general theory, see Drozd, Futorny and Ovsienko [DFO94].)

Problem 1.13. *Classify all simple $U(\mathfrak{gl}_n)$-modules.*

This is a crazy problem. In the language of the tame/wild dichotomy it is wild, and for $n \geq 3$ nobody expects a complete answer. For $n = 2$ (essentially $\mathfrak{sl}_2$) there is a classification, due to Block [Block81], but even that one is rather implicit. Much of the work in the area therefore goes into classifying simple modules within more manageable classes, such as Gelfand–Tsetlin modules.

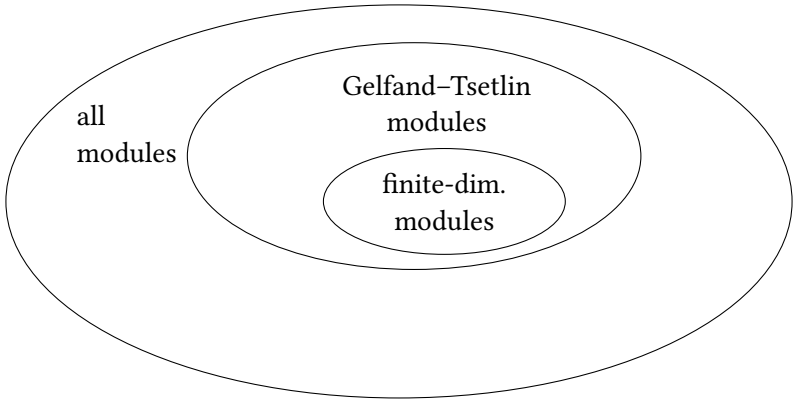
Problem 1.14. *Given two simple Gelfand–Tsetlin $U(\mathfrak{gl}_n)$-modules V and W, understand the tensor product $V \otimes W$, on which $x \in \mathfrak{gl}_n$ acts by*

$$x.(v \otimes w) = (x.v) \otimes w + v \otimes (x.w).$$

For instance, find its composition factors when $V \otimes W$ has finite length.

(This formula defines a $\mathfrak{gl}_n$-module, and hence a $U(\mathfrak{gl}_n)$-module, as you can check directly from the condition $[x, y].v = x.(y.v) - y.(x.v)$.)

The picture to keep in mind is the following:



Modules over $U(\mathfrak{gl}_n)$: finite-dimensional $\subset$ Gelfand–Tsetlin $\subset$ all.

1.7 Exercises

Exercise 1.15. Verify the Jacobi identity for the commutator bracket $[x, y] = xy - yx$ on an associative algebra A (Example 1.11): expand $[x, [y, z]] + [y, [z, x]] + [z, [x, y]]$ into twelve products and check that they cancel in pairs.

Exercise 1.16. Show that

$$e = E_{12}, \qquad f = E_{21}, \qquad h = E_{11} - E_{22}$$

form a basis of $\mathfrak{sl}_2$, with $[h, e] = 2e$, $[h, f] = -2f$ and $[e, f] = h$. More generally, show that $\mathfrak{sl}_n$ is closed under the bracket and has dimension $n^2 - 1$.

Chapter 2

Presentations of Algebras

Most of the algebras in this book are described by generators and relations. This chapter makes that precise, using free algebras and presentations. It also explains the most useful practical consequence: how to construct homomorphisms out of an algebra given by a presentation.

2.1 Free algebras

Let X be any set. A *word* in X is a finite sequence

$$w = x_1 x_2 \cdots x_n, \qquad x_i \in X,\ n \geq 0.$$

The word with $n = 0$ is the *empty word*, which we denote by 1.

Definition 2.1. The *free algebra* on X is the vector space with the words in X as a basis,

$$F(X) = F_{\Bbbk}(X) = \bigoplus_{w \text{ word in } X} \Bbbk w,$$

with multiplication given by juxtaposition (that is, concatenation) of words, extended bilinearly: $(x_1 \cdots x_m)(y_1 \cdots y_n) = x_1 \cdots x_m y_1 \cdots y_n$. The empty word 1 is the unit.

Example 2.2.

$$\begin{aligned} X = \emptyset &\implies F(X) = \Bbbk 1 \cong \Bbbk, \\ X = \{x\} &\implies F(X) = \Bbbk[x], \\ X = \{x, y\} &\implies F(X) = \underbrace{\Bbbk 1}_{\deg 0} \oplus \underbrace{\Bbbk x \oplus \Bbbk y}_{\deg 1} \oplus \underbrace{\Bbbk x^2 \oplus \Bbbk xy \oplus \Bbbk yx \oplus \Bbbk y^2}_{\deg 2} \oplus \cdots \end{aligned}$$

As soon as X has two elements, $F(X)$ is noncommutative: xy and yx are different words.

Remark 2.3. If V is a vector space with basis X, then the tensor algebra $T(V)$ from Example 1.12 is the free algebra $F(X)$: the tensors $x_1 \otimes \cdots \otimes x_n$ with $x_i \in X$ form a basis of $T(V)$, and they multiply by concatenation.

2.2 Presentations

Definition 2.4. Let X be a set and R a subset of $F(X)$. The *algebra with generators X and relations R* is

$$\Bbbk\langle X \mid R = 0\rangle = \frac{F_\Bbbk(X)}{\langle R\rangle_{\text{ideal}}},$$

where $\langle R\rangle_{\text{ideal}}$ is the two-sided ideal generated by R, that is, the span of all products $a\rho b$ with $a, b \in F(X)$ and $\rho \in R$. An algebra of this form is said to be *given by a presentation.*

A word about notation: in $\Bbbk\langle X \mid R = 0\rangle$ the angle brackets enclose a presentation, while the ideal generated by R carries a subscript, $\langle R\rangle_{\text{ideal}}$, following the convention of Chapter 1.

Example 2.5 (The Weyl algebra)**.** Take $X = \{x, y\}$ and $R = \{yx - xy - 1\}$. The *(first) Weyl algebra* is

$$A_1(\Bbbk) = \Bbbk\langle x, y \mid yx - xy - 1 = 0\rangle = \frac{F_\Bbbk(\{x, y\})}{\langle yx - xy - 1\rangle_{\text{ideal}}}.$$

Think of x as "multiplication by x" and of y as $\frac{d}{dx}$. The relation $yx - xy = 1$ is then the product rule $\frac{d}{dx}(xp) - x\frac{d}{dx}p = p$. Exercise 2.12 makes this precise. We use the convention $yx - xy = 1$ (equivalently $\partial x - x\partial = 1$) throughout the book.

Example 2.6 (A random example)**.** Take $X = \{x, y, z\}$ and

$$R = \{xy^2 - 3zy + yz - 4,\ x + y - z^2\}.$$

There is nothing special about this one: any set of noncommutative polynomials will do.

The price of this generality is that it is usually hard to say anything about the algebra a presentation defines. Two warnings:

1) It is possible that $1 \in \langle R\rangle_{\text{ideal}}$! Then the presentation defines the *zero* algebra. (For any algebra A we have $1 = 0$ in A if and only if $\dim A = 0$.) For instance, adding the relation $x = 0$ to the Weyl algebra kills everything, since then $1 = yx - xy = 0$.
2) *The isomorphism problem.* It is hard to know when two presentations define isomorphic algebras, $\Bbbk\langle X \mid R = 0\rangle \cong \Bbbk\langle Y \mid S = 0\rangle$.

2.3 Homomorphisms out of a presented algebra

The reason presentations are so useful is that it is easy to define homomorphisms out of them. Everything rests on the universal property of free algebras.

Proposition 2.7 (Universal property of free algebras)**.** *Let X be a set and A an algebra. Every map of sets $f : X \to A$ extends uniquely to an algebra homomorphism $\tilde{f} : F(X) \to A$.*

Proof. A homomorphism must send the word $x_1 \cdots x_n$ to $f(x_1) \cdots f(x_n)$, and the empty word to 1_A. Since the words form a basis, this forces uniqueness. Conversely, defining $\tilde{f}$ on words by this formula and extending linearly gives a linear map that respects products of words, because those are concatenations. By bilinearity it then respects all products. □

In other words, to construct an algebra homomorphism $F(X) \to A$ it suffices to choose an image for each $x \in X$, and any choice is fine.

Remark 2.8 (Functor language). The assignment $X \rightsquigarrow F(X)$ is a functor $\underline{\mathrm{Set}} \to \underline{\mathrm{Alg}}$. Let $\mathcal{O} : \underline{\mathrm{Alg}} \to \underline{\mathrm{Set}}$ be the forgetful functor, which sends an algebra to its underlying set. Then Proposition 2.7 says that restriction to X gives a bijection

$$\boxed{\mathrm{Hom}_{\underline{\mathrm{Set}}}\big(X, \mathcal{O}(A)\big) \cong \mathrm{Hom}_{\underline{\mathrm{Alg}}}\big(F(X), A\big)}$$

for every set X and every algebra A. In the language of Chapter 5, F is *left adjoint* to the forgetful functor.

To give an example, we need an interesting target algebra. Recall the Lie algebra $\mathfrak{sl}_2(\Bbbk)$ of traceless 2×2 matrices from Example 1.12. It has the basis

$$e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \qquad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

with brackets $[e, f] = h$, $[h, e] = 2e$ and $[h, f] = -2f$. By Remark 2.3, the tensor algebra $T(\mathfrak{sl}_2)$ is the free algebra on $\{e, f, h\}$. The relation $x \otimes y - y \otimes x - [x, y]$ defining $U(\mathfrak{sl}_2)$ is bilinear in x and y, vanishes for $x = y$, and changes sign when x and y are swapped. So it suffices to impose it for the three pairs (e, f), (h, e) and (h, f) of basis elements. This gives the presentation

$$U(\mathfrak{sl}_2(\Bbbk)) = \Bbbk \left\langle e, f, h \;\middle|\; \begin{array}{c} ef - fe - h = 0 \\ he - eh - 2e = 0 \\ hf - fh + 2f = 0 \end{array} \right\rangle.$$

Example 2.9. The assignment

$$\begin{cases} \varphi(x) = ehh = eh^2, \\ \varphi(y) = feh + h, \end{cases}$$

a map of sets $\{x, y\} \to U(\mathfrak{sl}_2(\Bbbk))$, extends uniquely to an algebra homomorphism $\varphi : F_{\Bbbk}(\{x, y\}) \to U(\mathfrak{sl}_2(\Bbbk))$.

For an algebra given by a presentation there is one extra step: we must check that the relations are respected.

Proposition 2.10. *Let X be a set, $R \subseteq F(X)$, and A an algebra. Let $f : X \to A$ be a map of sets whose extension $\tilde{f} : F(X) \to A$ satisfies $\tilde{f}(\rho) = 0$ for all $\rho \in R$. Then there is a unique algebra homomorphism*

$$\bar{f} : \Bbbk\langle X \mid R = 0\rangle \longrightarrow A \qquad \textit{with} \qquad \bar{f}\big(x + \langle R\rangle_{\mathrm{ideal}}\big) = f(x) \;\; \textit{for all } x \in X.$$

Proof. The kernel of $\tilde{f}$ is a two-sided ideal containing R, so it contains $\langle R\rangle_{\mathrm{ideal}}$, the smallest such ideal. Hence $\tilde{f}$ induces a homomorphism $\bar{f}(a + \langle R\rangle_{\mathrm{ideal}}) = \tilde{f}(a)$ on the quotient. It is unique because the cosets of the elements of X generate $\Bbbk\langle X \mid R = 0\rangle$. □

So, to construct an algebra map $\bar{f} : \Bbbk\langle X \mid R = 0\rangle \to A$:

1) pick $f(x) \in A$ for all $x \in X$;
2) get the unique extension to an algebra map $\tilde{f} : F(X) \to A$;
3) check that $R \subseteq \operatorname{Ker} \tilde{f}$;
4) then $\langle R \rangle_{\text{ideal}} \subseteq \operatorname{Ker} \tilde{f}$, and we get the induced algebra map $\bar{f} : F(X)/\langle R \rangle_{\text{ideal}} \to A$.

Example 2.11 (Dual numbers)**.** Let $A = \Bbbk\langle x \mid x^2 = 0\rangle$, so $X = \{x\}$ and $R = \{x^2\}$. We want a map $A \to M_2(\Bbbk)$. Pick

$$f(x) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

This extends to $\tilde{f} : F(\{x\}) = \Bbbk[x] \to M_2(\Bbbk)$, and $x^2 \in \operatorname{Ker} \tilde{f}$ because

$$\tilde{f}(x^2) = \big(\tilde{f}(x)\big)^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

So we get an algebra map $\bar{f} : A \to M_2(\Bbbk)$.

In this case we can also see exactly what A is. Since $\Bbbk[x]$ is commutative, the ideal generated by x^2 is just $x^2\Bbbk[x]$, so

$$A = \Bbbk 1 \oplus \Bbbk \bar{x}, \qquad \bar{x} = x + \langle x^2 \rangle_{\text{ideal}}, \qquad \bar{x}^2 = 0.$$

Moreover $\bar{f}(a + b\bar{x}) = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}$, so $\bar{f}$ is injective, and A is isomorphic to the algebra of upper triangular 2×2 matrices with equal diagonal entries.

2.4 Exercises

Exercise 2.12. Let $A_1(\Bbbk) = \Bbbk\langle x, y \mid yx - xy - 1 = 0\rangle$ be the Weyl algebra. Show that there is a unique algebra map

$$\varphi : A_1(\Bbbk) \longrightarrow \operatorname{End}_{\Bbbk}(\Bbbk[x])$$

satisfying

$$\begin{aligned} \varphi(x) &= L_x, & L_x(p(x)) &= x \cdot p(x), \\ \varphi(y) &= D_x, & D_x(p(x)) &= p'(x) = \frac{dp(x)}{dx}. \end{aligned}$$

Exercise 2.13. Using the presentation of $U(\mathfrak{sl}_2(\Bbbk))$ above, show that there is an algebra map

$$\varphi : U(\mathfrak{sl}_2(\Bbbk)) \longrightarrow \operatorname{End}_{\Bbbk}(\Bbbk[x, y])$$

satisfying

$$\begin{aligned} \varphi(e) &= x\frac{\partial}{\partial y} : p(x, y) \mapsto x\frac{\partial p}{\partial y}, \\ \varphi(f) &= y\frac{\partial}{\partial x} : p(x, y) \mapsto y\frac{\partial p}{\partial x}, \\ \varphi(h) &= x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}. \end{aligned}$$

Chapter 3

Adjoining Generators

Given an algebra A, we often want to enlarge it by throwing in a new element X, possibly subject to some relations. This chapter constructs the algebra $A\langle X\rangle$ ("A adjoin X") and its universal property. Many of the algebras later in the book (Ore extensions, generalized Weyl algebras) arise in exactly this way.

3.1 The construction

Let A be an algebra, and let X be a symbol (an indeterminate, or letter) that is not an element of A. Let $F = F_{\Bbbk}(A \cup \{X\})$ be the free algebra on the set $A \cup \{X\}$. Note that every element of A becomes a *letter* of F. For $a, b \in A$ the word ab (a product in F) is therefore different from the single letter $(ab)_A$, where $(\cdot)_A$ means "compute in A". The relations below identify the two.

Definition 3.1. The *algebra obtained from A by adjoining X* is

$$A\langle X\rangle = \Bbbk\langle A \cup \{X\} \mid R = 0\rangle,$$

where R is the set of relations

$$\begin{cases} 1 - 1_A = 0 & \text{(see Remark 3.3)} \\ a + b - (a+b)_A = 0 & \forall a, b \in A, \\ a \cdot b - (a \cdot b)_A = 0 & \forall a, b \in A, \\ \lambda \cdot a - (\lambda \cdot a)_A = 0 & \forall a \in A,\ \forall \lambda \in \Bbbk. \end{cases}$$

Here 1 is the empty word, the unit of the free algebra. We read $A\langle X\rangle$ as "A adjoin X".

The last three relations say that adding, multiplying and scaling letters from A gives the same result as in A. For instance, the relation $1_A \cdot a - a = 0$ is the multiplicative relation with the pair $(1_A, a)$, so it need not be imposed separately.

Examples 3.2. 1) $A = \Bbbk$: then $\Bbbk\langle X\rangle = F_{\Bbbk}(\{X\}) = \Bbbk[X]$.
2) $A = F_{\Bbbk}(\{Y_1, \dots, Y_n\})$: then $A\langle X\rangle = F_{\Bbbk}(\{Y_1, \dots, Y_n, X\})$.

Both identifications follow from the universal property in the next section (see Remark 3.6).

Using the relations to multiply neighbouring letters from A, every element of $A\langle X\rangle$ is a linear combination of products

$$a_1 X a_2 X \cdots X a_n, \qquad a_i \in A.$$

Such expressions span $A\langle X\rangle$, but they are far from unique. For example, $(a+a')Xb = aXb + a'Xb$.

Remark 3.3 (Why $1 - 1_A$?). Let $A \subset B$ be two algebras with units 1_A and 1_B, where A is closed under the operations of B. It can happen that $1_A \neq 1_B$. For example,

$$A = \begin{bmatrix} \mathbb{k} & 0 \\ 0 & 0 \end{bmatrix} \subset B = \begin{bmatrix} \mathbb{k} & \mathbb{k} \\ \mathbb{k} & \mathbb{k} \end{bmatrix} = M_2(\mathbb{k}), \qquad 1_A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \neq 1_B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

We say that A is a *unital subalgebra* of B if $1_A = 1_B$. If we want A to be a unital subalgebra of $A\langle X\rangle$, we must include the relation

$$1 - 1_A = 0,$$

which identifies the empty word 1, the unit of the free algebra, with the letter 1_A, the unit of A. Without it, 1_A would still act as the identity on letters from A, but not on X, and A would sit inside the quotient much like the corner $\mathbb{k}E_{11}$ sits inside $M_2(\mathbb{k})$.

3.2 The universal property

Theorem 3.4. *Let $\varphi : A \to B$ be an algebra map and let $b \in B$ be any element. Then there is a unique algebra map*

$$\tilde{\varphi} : A\langle X\rangle \longrightarrow B$$

such that $\tilde{\varphi}(X) = b$ and $\tilde{\varphi}|_A = \varphi$, that is, $\tilde{\varphi}(a) = \varphi(a)$ for all letters $a \in A$.

Proof. Extend φ to a map of sets $\dot{\varphi} : A \cup \{X\} \to B$ by

$$\dot{\varphi}(a) = \varphi(a) \quad (a \in A), \qquad \dot{\varphi}(X) = b.$$

By the universal property of free algebras (Proposition 2.7), $\dot{\varphi}$ extends to an algebra map $\Phi : F_{\mathbb{k}}(A \cup \{X\}) \to B$ with $\Phi|_{A\cup\{X\}} = \dot{\varphi}$. We check that Φ kills every relation in R:

$$\begin{aligned} \Phi\big(a_1 + a_2 - (a_1 + a_2)_A\big) &= \varphi(a_1) + \varphi(a_2) - \varphi(a_1 + a_2) = 0, \\ \Phi\big(a_1 \cdot a_2 - (a_1 a_2)_A\big) &= \varphi(a_1)\varphi(a_2) - \varphi(a_1 a_2) = 0, \\ \Phi\big(\lambda \cdot a - (\lambda a)_A\big) &= \lambda\varphi(a) - \varphi(\lambda a) = 0, \\ \Phi(1 - 1_A) &= 1_B - \varphi(1_A) = 0. \end{aligned}$$

Each line holds because φ is an algebra map, and algebra maps preserve units. By Proposition 2.10, Φ induces an algebra map $\tilde{\varphi}$ on the quotient $A\langle X\rangle$, with $\tilde{\varphi}(X) = b$ and $\tilde{\varphi}(a) = \varphi(a)$. It is unique because $A\langle X\rangle$ is generated by (the images of) the letters in $A \cup \{X\}$. □

Corollary 3.5. *The map $A \to A\langle X\rangle$ sending $a \in A$ to (the image of) the letter a is an injective algebra map. So A is a unital subalgebra of $A\langle X\rangle$.*

Proof. The relations in R say precisely that this map is linear, multiplicative and unital. Applying Theorem 3.4 with $B = A$, $\varphi = \mathrm{id}_A$ and $b = 0$ gives an algebra map $A\langle X\rangle \to A$ that sends the letter a back to a. So our map has a left inverse, and in particular it is injective. □

From now on we simply regard A as a subalgebra of $A\langle X\rangle$.

Remark 3.6. The universal property also explains Examples 3.2. For $A = \Bbbk$ there is exactly one algebra map $\Bbbk \to B$, so Theorem 3.4 says that algebra maps $\Bbbk\langle X\rangle \to B$ correspond to elements $b \in B$. By Proposition 2.7, the same is true for $\Bbbk[X] = F_{\Bbbk}(\{X\})$. Two algebras with the same universal property are isomorphic: the maps $X \mapsto X$ in both directions are inverse to each other, by uniqueness. The second example works the same way, since an algebra map out of $F_{\Bbbk}(\{Y_1, \dots, Y_n\})\langle X\rangle$ is a free choice of images for $Y_1, \dots, Y_n$ and X.

3.3 Relations

Definition 3.7. Let S be a subset of $A\langle X\rangle$. The algebra obtained from A by adjoining X *subject to the relations* $S = 0$ is

$$A\langle X \mid S = 0\rangle = \frac{A\langle X\rangle}{\langle S\rangle_{\text{ideal}}}.$$

Maps out of $A\langle X \mid S = 0\rangle$ are constructed just as in Proposition 2.10: choose $\varphi : A \to B$ and $b \in B$, and check that the resulting $\tilde{\varphi}$ kills every element of S.

Example 3.8. Let $A = \Bbbk[x]$. Then

$$\Bbbk[x, y] = \Bbbk[x]\langle y \mid xy - yx = 0\rangle.$$

Example 3.9. The Weyl algebra (Example 2.5) can be obtained by adjoining y to $\Bbbk[x]$:

$$A_1(\Bbbk) = \Bbbk[x]\langle y \mid yx - xy - 1 = 0\rangle.$$

(See Exercise 3.12.)

We end with a question that will occupy us for a while: *what is the relationship between A-modules and $A\langle X\rangle$-modules?*

${}_A\underline{\text{Mod}}$	vs.	${}_{A\langle X\rangle}\underline{\text{Mod}}$
M	$\rightsquigarrow$	?
?	$\leftsquigarrow$	N

One direction is easy: an $A\langle X\rangle$-module N is an A-module by restricting the action to $A \subset A\langle X\rangle$. The other direction, producing an $A\langle X\rangle$-module from an A-module M, is more interesting. We come back to it in Chapter 5.

3.4 Exercises

Exercise 3.10. Show that any algebra is isomorphic to a quotient of a free algebra.

Exercise 3.11. Construct $U(\mathfrak{sl}_2(\Bbbk))$ from $\Bbbk[h]$ by adjoining e and f in two steps, subject to relations. (Compare with the presentation of $U(\mathfrak{sl}_2(\Bbbk))$ in Chapter 2.)

Exercise 3.12. Show that $\Bbbk[x]\langle y \mid yx - xy - 1 = 0\rangle$ is isomorphic to the Weyl algebra $A_1(\Bbbk) = \Bbbk\langle x, y \mid yx - xy - 1 = 0\rangle$ of Example 2.5, by constructing algebra maps in both directions.

Chapter 4

Skew Polynomial Rings and Ore Extensions

A polynomial ring $A[x]$ is obtained by adjoining to A a variable that commutes with everything. In a skew polynomial ring, x is instead allowed to "twist" past elements of A according to a rule $xa = \sigma(a)x + \delta(a)$. We ask when this produces a ring that is as nice as a polynomial ring, and find the answer: exactly when σ is an endomorphism and δ is a σ-derivation.

4.1 The definition

Our reference for this chapter is McConnell–Robson [MR01, Chapter 1, §§2–4].

Let A be an algebra over $\Bbbk$, and let $\sigma, \delta : A \to A$ be functions. For the moment they are arbitrary maps of sets. Using the notation of Chapter 3, define

$$A[x;\sigma,\delta] = A\Big\langle x \;\Big|\; xa - \sigma(a)x - \delta(a) = 0 \text{ for all } a \in A\Big\rangle = \frac{A\langle x\rangle}{\big\langle xa - (\sigma(a)x + \delta(a)) \;\big|\; a \in A\big\rangle_{\text{ideal}}}.$$

So x can be moved past any $a \in A$, at the cost of replacing a by $\sigma(a)$ and adding a correction term $\delta(a)$.

Example 4.1. If $\sigma = \mathrm{Id}_A$ and $\delta = 0$, the relations say that x commutes with A, and

$$A[x;\mathrm{Id}_A,0] \cong A[x]$$

is the usual polynomial ring.

4.2 When is it a good ring?

The property we would like $A[x;\sigma,\delta]$ to have, like an honest polynomial ring, is the following.

Condition. $A[x;\sigma,\delta]$ is free as a left A-module on the set $\{x^n\}_{n=0}^{\infty}$:

$$A[x;\sigma,\delta] = \bigoplus_{n=0}^{\infty} Ax^n.$$

For arbitrary σ and δ this can fail badly.

Example 4.2. Suppose $\sigma(1_A) = 0$ and $\delta(1_A) = 0$. Then

$$x = x \cdot 1_A = \sigma(1_A)x + \delta(1_A) = 0x + 0 = 0,$$

so the Condition fails here.

So suppose the Condition holds, and let us see what it forces on σ and δ. The idea is always the same: compute some product in two ways, bring both results into the form $(\cdots)x + (\cdots)1$, and compare coefficients. This is legitimate because $\{1, x\} \subset \{x^n\}$ is linearly independent over A on the left.

Additivity. For $a_1, a_2 \in A$ we have, on the one hand,

$$x(a_1 + a_2) = \sigma(a_1 + a_2)x + \delta(a_1 + a_2),$$

and on the other hand, applying the relation to a_1 and a_2 separately,

$$x(a_1 + a_2) = xa_1 + xa_2 = \big(\sigma(a_1) + \sigma(a_2)\big)x + \delta(a_1) + \delta(a_2).$$

Subtracting,

$$\Big(\sigma(a_1 + a_2) - \big(\sigma(a_1) + \sigma(a_2)\big)\Big)x + \Big(\delta(a_1 + a_2) - \big(\delta(a_1) + \delta(a_2)\big)\Big)1 = 0,$$

and both coefficients must vanish. So σ and δ are *additive*.

Linearity. Let $\lambda \in \Bbbk$ and $a \in A$. Scalars commute with everything in an algebra, so

$$x(\lambda a) = \sigma(\lambda a)x + \delta(\lambda a) \qquad \text{and} \qquad x(\lambda a) = \lambda(xa) = \lambda\sigma(a)x + \lambda\delta(a).$$

Comparing coefficients gives $\sigma(\lambda a) = \lambda\sigma(a)$ and $\delta(\lambda a) = \lambda\delta(a)$. So σ and δ are $\Bbbk$-linear maps.

Multiplicativity. For all $a_1, a_2 \in A$,

$$x(a_1 a_2) = \sigma(a_1 a_2)x + \delta(a_1 a_2),$$

while

$$\begin{aligned}(xa_1)a_2 &= \big(\sigma(a_1)x + \delta(a_1)\big)a_2 = \sigma(a_1)(xa_2) + \delta(a_1)a_2 \\ &= \sigma(a_1)\big(\sigma(a_2)x + \delta(a_2)\big) + \delta(a_1)a_2 \\ &= \sigma(a_1)\sigma(a_2)\,x + \big(\sigma(a_1)\delta(a_2) + \delta(a_1)a_2\big).\end{aligned}$$

Comparing the coefficients of x gives $\sigma(a_1 a_2) = \sigma(a_1)\sigma(a_2)$, so σ is multiplicative. Comparing the coefficients of 1 gives

$$\delta(a_1 a_2) = \sigma(a_1)\delta(a_2) + \delta(a_1)a_2,$$

a twisted version of the Leibniz rule.

The unit. Finally, 1_A is the unit of $A[x; \sigma, \delta]$, so

$$x = 1_A x + 0 \qquad \text{and} \qquad x = x1_A = \sigma(1_A)x + \delta(1_A).$$

Comparing coefficients gives $\sigma(1_A) = 1_A$ and $\delta(1_A) = 0$.

Altogether, σ must be an algebra map, and δ must be a linear map satisfying the twisted Leibniz rule. This deserves a name.

Definition 4.3. Let $\sigma : A \to A$ be an algebra map. A *σ-derivation* is a linear map $\delta : A \to A$ satisfying

$$\delta(a_1 a_2) = \sigma(a_1)\delta(a_2) + \delta(a_1)a_2 \qquad \text{for all } a_1, a_2 \in A.$$

Taking $a_1 = a_2 = 1_A$ shows that $\delta(1_A) = 0$ automatically. An Id_A-derivation is simply called a *derivation.*

Theorem 4.4. *$A[x; \sigma, \delta]$ is free as a left A-module on $\{x^n\}_{n=0}^{\infty}$ if and only if*

1) $\sigma : A \to A$ is an algebra endomorphism, and
2) $\delta : A \to A$ is a σ-derivation.

Proof. ($\Rightarrow$) We have just done this.

($\Leftarrow$) This direction is less obvious. See [MR01, Chapter 1, §2], or work it out yourself in Exercise 4.9. □

Definition 4.5. When the conditions of Theorem 4.4 hold, $A[x; \sigma, \delta]$ is called a *skew polynomial ring*, or an *Ore extension* of A.

4.3 Examples

Example 4.6 ($\delta = 0$)**.** When $\delta = 0$ we write

$$A[x; \sigma] = A[x; \sigma, 0] = \left\{ \sum_{n=0}^{\infty} a_n x^n \;\middle|\; a_n \in A, \text{ almost all } a_n = 0 \right\},$$

with the multiplication determined by

$$xa = \sigma(a)x \qquad \text{for all } a \in A.$$

For instance, let $A = \Bbbk[t]$. By the universal property of $\Bbbk[t] = F_{\Bbbk}(\{t\})$ (Proposition 2.7), there is a unique algebra endomorphism σ of $\Bbbk[t]$ with $\sigma(t) = t^2 + 1$. It is determined by its value on t; for example,

$$\sigma(t^3) = \sigma(t)^3 = (t^2 + 1)^3.$$

In $A[x; \sigma]$ we then have

$$\boxed{xt = (t^2 + 1)x}.$$

Note that this σ is not surjective. That is fine: Theorem 4.4 only asks for an endomorphism.

Example 4.7 (The Weyl algebra again)**.** Let $A = \Bbbk[t]$ and $\sigma = \mathrm{Id}_A$, and let δ be the derivation of $\Bbbk[t]$ with $\delta(t) = 1$. (It exists and equals $\frac{d}{dt}$; see Remark 4.8 below.) The Leibniz rule computes δ on powers of t, for example

$$\delta(t^2) = t\delta(t) + \delta(t)t = t \cdot 1 + 1 \cdot t = 2t.$$

In $\Bbbk[t][x; \mathrm{Id}, \frac{d}{dt}]$ the defining relation for $a = t$ reads

$$x \cdot t = t \cdot x + 1.$$

By Exercise 4.11, this one relation implies all the others. Comparing with Exercise 3.12, we find

$$\Bbbk[t]\Big[x; \mathrm{Id}, \frac{d}{dt}\Big] \cong A_1(\Bbbk),$$

the Weyl algebra, with t playing the role of the generator x of Example 2.5 and x playing the role of $y = \partial$.

Remark 4.8 (Derivations of $\Bbbk[t]$). Let $p(t) \in \Bbbk[t]$. There is exactly one derivation δ of $\Bbbk[t]$ with $\delta(t) = p(t)$, namely

$$\delta = p(t)\frac{d}{dt} : \ q(t) \mapsto p(t)q'(t).$$

Indeed:

1) $p(t)\frac{d}{dt}$ is a derivation, by the product rule, and on t it takes the value $p(t)$;
2) a derivation is determined by its value on t. By the Leibniz rule and induction, $\delta(t^n) = \sum_{i=0}^{n-1} t^i\, \delta(t)\, t^{n-1-i}$, and δ is linear.

4.4 A question for next time

At the end of Chapter 3 we asked how A-modules and $A\langle X\rangle$-modules are related. In the language of functors, the question becomes: *show that the forgetful (restriction) functor*

$$\mathrm{Res}_A^{A\langle X\rangle} : \ {}_{A\langle X\rangle}\underline{\mathrm{Mod}} \longrightarrow {}_A\underline{\mathrm{Mod}}$$

has a left adjoint, and describe it. That is, we are looking for a construction $M \mapsto ?(M)$ with natural bijections

$$\mathrm{Hom}_{A\langle X\rangle}\big(\,?(M), N\big) \cong \mathrm{Hom}_A\big(M, \mathrm{Res}_A^{A\langle X\rangle} N\big) \qquad \text{for } M \in {}_A\underline{\mathrm{Mod}},\ N \in {}_{A\langle X\rangle}\underline{\mathrm{Mod}}.$$

Chapter 5 begins with the answer.

4.5 Exercises

Exercise 4.9. Prove the direction ($\Leftarrow$) of Theorem 4.4. Assume that σ is an algebra endomorphism and δ a σ-derivation.

(a) Show that $\sum_n Ax^n$ is closed under left multiplication by elements of A and by x, and conclude that it is all of $A[x; \sigma, \delta]$.
(b) On the free left A-module $M = \bigoplus_{n\geq 0} Ae_n$, define $\Bbbk$-linear operators $L_a(b\ e_n) = ab\ e_n$ for $a \in A$, and $X(b\ e_n) = \sigma(b)e_{n+1} + \delta(b)e_n$. Check that $XL_a = L_{\sigma(a)}X + L_{\delta(a)}$. Use Theorem 3.4 to get an algebra map $A[x; \sigma, \delta] \to \mathrm{End}_{\Bbbk}(M)$.
(c) Apply the image of $\sum_n a_n x^n$ to e_0, and conclude that the x^n are linearly independent over A on the left.

Exercise 4.10. Show that any function $\delta' : \{y_1, \dots, y_n\} \to F_{\Bbbk}(\{y_1, \dots, y_n\})$ extends to a unique derivation $\delta : \ F_{\Bbbk}(\{y_1, \dots, y_n\}) \to F_{\Bbbk}(\{y_1, \dots, y_n\})$.

Exercise 4.11. Let σ be an algebra endomorphism of A and δ a σ-derivation, and suppose A is generated as an algebra by a subset Y. Show that

$$A[x;\sigma,\delta] = A\Big\langle x \;\Big|\; xa - \sigma(a)x - \delta(a) = 0 \text{ for all } a \in Y \Big\rangle.$$

In other words, the relations for the generators imply all the others. (Hint: if the relation holds for a and for b, show that it holds for ab, $a+b$ and λa.)

Chapter 5

Adjunctions, Derivations, and GWAs

This chapter has three threads. First we answer the question from the end of the last two chapters, at least for polynomial rings, which leads us to adjoint functors. Then we set up derivations and learn how to construct them on algebras given by a presentation. Finally we meet the generalized Weyl algebras (GWAs), which will be one of our main examples for the rest of the book.

5.1 Adjoint functors and restriction

We have seen one adjunction already: the free algebra functor is left adjoint to the forgetful functor (Remark 2.8). Here is the general definition, stated informally.

Definition 5.1. Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors. We say that F is *left adjoint* to G (and G is *right adjoint* to F) if there are bijections

$$\mathrm{Hom}_{\mathcal{D}}\big(F(M), N\big) \cong \mathrm{Hom}_{\mathcal{C}}\big(M, G(N)\big)$$

for all objects M of $\mathcal{C}$ and N of $\mathcal{D}$. These bijections should be *natural*, that is, compatible with composing by morphisms $M' \to M$ and $N \to N'$.

Now let A be an algebra, and $i : A \hookrightarrow A\langle X\rangle$ the inclusion (Corollary 3.5). A representation $\rho : A\langle X\rangle \to \mathrm{End}(V)$ restricts to a representation of A:

$$A \xhookrightarrow{i} A\langle X\rangle \xrightarrow{\rho} \mathrm{End}(V).$$

This gives the *restriction functor*

$$\mathrm{Res}_A^{A\langle X\rangle} : {}_{A\langle X\rangle}\underline{\mathrm{Mod}} \longrightarrow {}_A\underline{\mathrm{Mod}}, \qquad (V, \rho) \longmapsto (V, \rho \circ i),$$

and the question from Chapter 4 asks for a left adjoint:

$$\mathrm{Hom}_A\big(M, \mathrm{Res}_A^{A\langle X\rangle} N\big) \cong \mathrm{Hom}_{A\langle X\rangle}(\,?M, N), \qquad \varphi \longmapsto \tilde{\varphi}.$$

A first try. Given an A-module M, let $M[X] = \bigoplus_{k\geq 0} MX^k$ (formal expressions $\sum m_k X^k$). Let A act on the coefficients and let X act by $X.(mX^k) = mX^{k+1}$, so in particular $X.m = mX$. But then, for $a \in A$ and $m \in M$,

$$\begin{aligned}(Xa).m &= X.(a.m) = (a.m)X,\\ (aX).m &= a.(X.m) = a.(mX) = (a.m)X,\end{aligned}$$

so Xa and aX act in the same way. In other words, $M[X]$ is only a module over the quotient of $A\langle X\rangle$ in which X commutes with A, namely the polynomial ring $A[X]$.

For $A\langle X\rangle$ we need something bigger, in which X and the elements of A do not commute. A natural guess is to build $M\langle X\rangle$ out of expressions like $m_1Xm_2X\cdots Xm_n$. That doesn't quite make sense, since elements of M cannot be multiplied, but it points in the right direction. What works is expressions $a_1Xa_2X\cdots a_nXm$ with $a_i \in A$ and $m \in M$, subject to the obvious rules for moving elements of A onto m. This is the *induced module* $A\langle X\rangle \otimes_A M$, which we will construct properly when we discuss tensor products over algebras and induced modules in Chapters 14 and 15, and describe it explicitly in Chapter 17.

Meanwhile, our first try does answer the question for $A[X] = A[X; \mathrm{Id}_A, 0]$.

Proposition 5.2. *Let M be an A-module, and make $M[X] = \bigoplus_{k\geq 0} MX^k$ into an $A[X]$-module by $a.(mX^k) = (a.m)X^k$ and $X.(mX^k) = mX^{k+1}$. Then $M \mapsto M[X]$ is left adjoint to $\mathrm{Res}_A^{A[X]}$:*

$$\mathrm{Hom}_A\big(M, \mathrm{Res}_A^{A[X]} N\big) \cong \mathrm{Hom}_{A[X]}\big(M[X], N\big), \qquad \varphi \longmapsto \tilde{\varphi}, \quad \tilde{\varphi}\Big(\sum_k m_k X^k\Big) = \sum_k X^k.\varphi(m_k).$$

Proof. The two actions on $M[X]$ commute, so $M[X]$ is indeed an $A[X]$-module. The map $\tilde{\varphi}$ clearly commutes with X. It also commutes with $a \in A$, because X is central in $A[X]$: $\tilde{\varphi}(a.(mX^k)) = X^k.\varphi(a.m) = X^k a.\varphi(m) = aX^k.\varphi(m)$. Conversely, an $A[X]$-module map $\psi : M[X] \to N$ is determined by its restriction to $M = MX^0$, since $\psi(mX^k) = \psi(X^k.m) = X^k.\psi(m)$. So $\psi \mapsto \psi|_M$ is inverse to $\varphi \mapsto \tilde{\varphi}$. □

5.2 Derivations

Recall from Exercise 4.10 that any map $\delta : \Omega \to F_{\Bbbk}(\Omega)$ from a set of generators Ω into the free algebra extends uniquely to a derivation $\delta : F_{\Bbbk}(\Omega) \to F_{\Bbbk}(\Omega)$. We would like to do the same for an algebra $A = F_{\Bbbk}(\Omega)/I$ given by a presentation. Prescribe $\delta : \Omega \to F_{\Bbbk}(\Omega)/I$; when does it extend to a derivation of A? To answer this it helps to allow derivations with values in a bimodule. Recall that an (A, A)-*bimodule* is a vector space M with a left and a right A-action that commute: $(a.m).b = a.(m.b)$.

Definition 5.3. Let A be an algebra and M an (A, A)-bimodule. A linear map $\delta : A \to M$ is a *derivation* if

$$\delta(ab) = \delta(a)\cdot b + a\cdot\delta(b) \qquad \text{for all } a, b \in A.$$

For $M = A$ this is the same as an Id_A-derivation in the sense of Definition 4.3.

Lemma 5.4. *Let $\delta : A \to M$ be a derivation and $\varphi : M \to N$ a map of (A, A)-bimodules. Then $\varphi \circ \delta$ is a derivation.*

Proof.

$$(\varphi\circ\delta)(ab) = \varphi\big(\delta(a)\cdot b + a\cdot\delta(b)\big) = (\varphi\circ\delta)(a)\cdot b + a\cdot(\varphi\circ\delta)(b).$$ □

Example 5.5. Let δ be a derivation of $F_{\Bbbk}(\Omega)$. The quotient $F_{\Bbbk}(\Omega)/I$ is an $(F_{\Bbbk}(\Omega), F_{\Bbbk}(\Omega))$-bimodule, and the projection $\pi : F_{\Bbbk}(\Omega) \to F_{\Bbbk}(\Omega)/I$ is a bimodule map. So

$$\pi\circ\delta : F_{\Bbbk}(\Omega) \longrightarrow F_{\Bbbk}(\Omega) \longrightarrow F_{\Bbbk}(\Omega)/I$$

is a derivation.

This derivation descends to the quotient precisely when it kills I, and to check that, it is enough to look at the relations.

Proposition 5.6. *Let $A = F_{\Bbbk}(\Omega)/I$ with $I = \langle R \rangle_{\text{ideal}}$, and let δ be a derivation of $F_{\Bbbk}(\Omega)$ such that $\delta(\rho) \in I$ for every relation $\rho \in R$. Then $\delta(I) \subseteq I$, and*

$$\bar{\delta} : A \to A, \qquad \bar{\delta}(a + I) = \delta(a) + I,$$

is a well-defined derivation of A.

Proof. For $a, b \in F_{\Bbbk}(\Omega)$ and $\rho \in R$, the Leibniz rule gives

$$\delta(a\rho b) = \delta(a)\rho b + a\,\delta(\rho)\,b + a\rho\,\delta(b),$$

and all three terms lie in I. Since δ is linear and I is spanned by such elements $a\rho b$, we get $\delta(I) \subseteq I$. So $\bar{\delta}$ is well defined, and it inherits linearity and the Leibniz rule from δ. □

In practice, then, to construct a derivation of a presented algebra: choose its values on the generators, extend to the free algebra, and check that each relation is sent into I. We will use this in Chapter 7.

5.3 Generalized Weyl algebras

Generalized Weyl algebras were introduced in the early 1990s by V. Bavula, A. Rosenberg and D. Jordan; see for example [Bavula92]. The data is:

- an algebra R, "the base algebra";
- an algebra automorphism $\sigma : R \to R$;
- a central element $t \in R$.

Definition 5.7. The *generalized Weyl algebra* (GWA) associated to (R, σ, t) is the algebra $R\langle X, Y\rangle/I$ obtained by adjoining X and Y to R, where I is generated by

$$Xr - \sigma(r)X, \qquad Yr - \sigma^{-1}(r)Y \qquad \text{for all } r \in R,$$
$$YX - t, \qquad XY - \sigma(t).$$

We denote it by $R(\sigma, t)$.

Replacing r by $\sigma(r)$, the second family of relations can equivalently be written as $rY = Y\sigma(r)$.

Remark 5.8 (Why is t central?). The relations force t to commute with R anyway. In $R(\sigma, t)$ we have, for every $r \in R$,

$$tr = YXr = Y\sigma(r)X = rYX = rt.$$

So if t were not central in R, the map $R \to R(\sigma, t)$ could not be injective. That is why t is required to be central. In most examples R is in fact commutative.

Example 5.9. Let $R = \Bbbk[u]$ be the polynomial algebra, and let σ be the automorphism of $\Bbbk[u]$ determined by $\sigma(u) = u - 1$ (its inverse is $u \mapsto u + 1$), so that

$$\sigma\big(p(u)\big) = p(u - 1).$$

Take $t = u$.

Proposition 5.10. *Assume* $\operatorname{char} \Bbbk = 0$. *With R, σ, t as in Example 5.9,*

$$R(\sigma, t) \cong A_1(\Bbbk) = \frac{\Bbbk\langle x, \partial\rangle}{\langle \partial x - x\partial - 1\rangle_{\text{ideal}}}.$$

Proof. Define a map of sets on the generators of $A_1(\Bbbk)$ by

$$x \longmapsto X, \qquad \partial \longmapsto Y,$$

and extend it to an algebra map $\tilde{\varphi} : \Bbbk\langle x, \partial\rangle \to R(\sigma, t)$. It kills the relation:

$$\tilde{\varphi}(\partial x - x\partial - 1) = YX - XY - 1 = t - \sigma(t) - 1 = u - (u - 1) - 1 = 0.$$

So we get an algebra map $\Phi : A_1(\Bbbk) \to R(\sigma, t)$ (Proposition 2.10).

Φ is *surjective*: $R(\sigma, t)$ is generated by $R = \Bbbk[u]$ together with X and Y, so it is generated by u, X, Y, and all three lie in the image:

$$X = \Phi(x), \qquad Y = \Phi(\partial), \qquad u = t = YX = \Phi(\partial x).$$

Φ is *injective* by the following corollary of the simplicity of $A_1(\Bbbk)$ (Exercise 5.12). Here we use that $R(\sigma, t) \neq 0$. This is not obvious from the presentation, but it follows for instance from the basis of $R(\sigma, t)$ that we will find in Chapter 11. □

Corollary 5.11. *Assume* $\operatorname{char} \Bbbk = 0$. *Any nonzero algebra map $\psi : A_1(\Bbbk) \to B$ is injective.*

Proof. $\operatorname{Ker} \psi$ is a two-sided ideal, and it is proper because $\psi(1) = 1 \neq 0$. Since $A_1(\Bbbk)$ is simple, $\operatorname{Ker} \psi = 0$. □

In fact the isomorphism of Proposition 5.10 holds in any characteristic; see Exercise 5.13.

5.4 Exercises

Exercise 5.12. Assume $\operatorname{char} \Bbbk = 0$. Prove that $A_1(\Bbbk)$ is a *simple* algebra: its only two-sided ideals are 0 and $A_1(\Bbbk)$. (Hint: take an ideal $J \neq 0$ and use $[\partial, x^n] = nx^{n-1}$.) What goes wrong in characteristic $p > 0$?

Exercise 5.13. With R, σ, t as in Example 5.9, find an explicit algebra map $\Psi : R(\sigma, t) \to A_1(\Bbbk)$ that is inverse to the map Φ of Proposition 5.10. Conclude that $R(\sigma, t) \cong A_1(\Bbbk)$ over a field of any characteristic. (Hint: where must $u = YX$ go?)

Part II

Gradings, filtrations and first examples

Chapter 6

More about GWAs

We look at generalized Weyl algebras more closely. Two important examples are the enveloping algebra $U(\mathfrak{sl}_2)$ and skew Laurent polynomial rings. We then show that every GWA is a quotient of an iterated skew polynomial ring, which gives a quick proof that GWAs over Noetherian rings are Noetherian. Finally we introduce gradings, which are the subject of the next chapter.

6.1 Standing assumptions and examples

Recall the generalized Weyl algebra $A = R(\sigma, t)$ from Definition 5.7:

$$A = R(\sigma, t) = \frac{R\langle X, Y\rangle}{\left\langle \begin{array}{c} YX - t,\ XY - \sigma(t), \\ Xr - \sigma(r)X,\ Yr - \sigma^{-1}(r)Y\ \ (r \in R) \end{array} \right\rangle_{\text{ideal}}}.$$

In this chapter and the next we make the following *standing assumptions*:

- R is an *integral domain* over $\Bbbk$, that is, a commutative $\Bbbk$-algebra with $rs = 0 \Rightarrow r = 0$ or $s = 0$;
- $t \in R$ and $t \neq 0$ (central automatically, since R is commutative);
- $\sigma \in \mathrm{Aut}_{\Bbbk\text{-Alg}}(R)$ is an algebra automorphism.

Example 6.1. $\Bbbk[u]\big(\sigma(u) = u - 1,\ t = u\big) \cong A_1(\Bbbk)$, the Weyl algebra (Proposition 5.10 and Exercise 5.13).

Proposition 6.2 ($U(\mathfrak{sl}_2)$ is a GWA)**.** *Let $R = \mathbb{C}[h, C]$ be the polynomial algebra in two variables, let $\sigma : R \to R$ be the automorphism with*

$$\sigma(h) = h - 2, \qquad \sigma(C) = C,$$

and let

$$t = \tfrac{1}{2}C + (\alpha h^2 + \beta h + \gamma) \qquad \text{with } \alpha = -\tfrac{1}{4},\ \beta = -\tfrac{1}{2},\ \gamma = 0.$$

Then $R(\sigma, t) \cong U(\mathfrak{sl}_2(\mathbb{C}))$, where the variable C corresponds to the Casimir element

$$C = ef + fe + \tfrac{1}{2}h^2 \in U(\mathfrak{sl}_2(\mathbb{C})).$$

Proof. We define algebra maps in both directions and check that they are mutually inverse. First, using the presentation of $U(\mathfrak{sl}_2(\mathbb{C}))$ from Chapter 2, send

$$e \longmapsto X, \qquad f \longmapsto Y, \qquad h \longmapsto h.$$

A short computation gives $\sigma(t) = \frac{1}{2}C - \frac{1}{4}h^2 + \frac{1}{2}h$, so $\sigma(t) - t = h$. Hence the relations are respected:

$$\begin{aligned} ef - fe - h &\longmapsto XY - YX - h = \sigma(t) - t - h = 0, \\ he - eh - 2e &\longmapsto hX - Xh - 2X = hX - (h-2)X - 2X = 0, \\ hf - fh + 2f &\longmapsto hY - Yh + 2Y = hY - (h+2)Y + 2Y = 0. \end{aligned}$$

Conversely, the Casimir element is central in $U(\mathfrak{sl}_2(\mathbb{C}))$ (Exercise 6.9), so $h \mapsto h$, $C \mapsto C$ defines an algebra map $R \to U(\mathfrak{sl}_2(\mathbb{C}))$. Extend it by $X \mapsto e$, $Y \mapsto f$ (Theorem 3.4). As in Exercise 4.11, it suffices to check the relations $Xr = \sigma(r)X$ and $Yr = \sigma^{-1}(r)Y$ for $r = h$ and $r = C$. These read $eh = (h-2)e$, $fh = (h+2)f$, $eC = Ce$ and $fC = Cf$, which hold in $U(\mathfrak{sl}_2)$. For the remaining relations, use $ef = fe + h$ to write $C = 2fe + h + \frac{1}{2}h^2$, so that

$$fe = \tfrac{1}{2}C - \tfrac{1}{4}h^2 - \tfrac{1}{2}h = t, \qquad ef = fe + h = \tfrac{1}{2}C - \tfrac{1}{4}h^2 + \tfrac{1}{2}h = \sigma(t).$$

Both composites fix the generators. For C this uses $XY + YX + \frac{1}{2}h^2 = \sigma(t) + t + \frac{1}{2}h^2 = C$. So both composites are the identity. □

Example 6.3 (Skew Laurent polynomial rings)**.** Take $t = 1$. Then

$$XY = \sigma(1) = 1, \qquad YX = 1,$$

so $Y = X^{-1}$. The *skew Laurent polynomial ring* $R[X, X^{-1}; \sigma]$ is obtained by adjoining X and X^{-1} to R, subject to $Xr = \sigma(r)X$ and $XX^{-1} = X^{-1}X = 1$. Multiplying $X\sigma^{-1}(r) = rX$ by X^{-1} on both sides gives $X^{-1}r = \sigma^{-1}(r)X^{-1}$, so the two presentations agree and

$$R(\sigma, 1) \cong R[X, X^{-1}; \sigma] = \left\{ \sum_{n \in \mathbb{Z}} r_n X^n \;\middle|\; r_n \in R, \text{ almost all } r_n = 0 \right\},$$

with

$$X^n r = \sigma^n(r) X^n \qquad \text{for all } n \in \mathbb{Z}.$$

So skew Laurent polynomial rings are GWAs.

6.2 Connection to iterated skew polynomial rings

Theorem 6.4.

$$R(\sigma, t) \cong R[Y; \sigma^{-1}][X; \hat{\sigma}, \delta] \Big/ \langle YX - t \rangle_{\text{ideal}},$$

where $\hat{\sigma}$ is the algebra automorphism of $R[Y; \sigma^{-1}]$ and δ the $\hat{\sigma}$-derivation determined by

$$\begin{cases} \hat{\sigma}(r) = \sigma(r) & \forall r \in R, \\ \hat{\sigma}(Y) = Y, \end{cases} \qquad \begin{cases} \delta(r) = 0 & \forall r \in R, \\ \delta(Y) = \sigma(t) - t. \end{cases}$$

In the iterated skew polynomial ring we therefore have

$$Xr = \sigma(r)X, \qquad XY = \underbrace{Y}_{\hat{\sigma}(Y)} X + \underbrace{\sigma(t) - t}_{\delta(Y)}.$$

Proof. That $\hat{\sigma}$ and δ exist is Exercise 6.10. By Exercise 4.11, applied to the generating set $R \cup \{Y\}$ of $R[Y; \sigma^{-1}]$, the iterated ring is $R\langle X, Y\rangle$ modulo the relations

$$Yr - \sigma^{-1}(r)Y, \qquad Xr - \sigma(r)X, \qquad XY - YX - (\sigma(t) - t) \qquad (r \in R).$$

Adding the relation $YX - t$ turns the last one into $XY - \sigma(t)$. Conversely, in $R(\sigma, t)$ we have $XY - YX = \sigma(t) - t$. So both sides are $R\langle X, Y\rangle$ modulo the same ideal. □

Corollary 6.5. *If R is Noetherian, then so is any GWA $R(\sigma, t)$.*

Proof. By the skew Hilbert basis theorem [MR01, Chapter 1, §2], if B is Noetherian and τ is an automorphism of B, then $B[x; \tau, \delta]$ is Noetherian. Apply it twice: first to $R[Y; \sigma^{-1}]$, then to $R[Y; \sigma^{-1}][X; \hat{\sigma}, \delta]$. Here $\hat{\sigma}$ is an automorphism, with inverse given by $r \mapsto \sigma^{-1}(r)$, $Y \mapsto Y$. Finally, quotients of Noetherian rings are Noetherian, so Theorem 6.4 finishes the proof. □

Compare this with the Hilbert basis theorem:

$$R \text{ commutative Noetherian} \implies R[x] \text{ Noetherian.}$$

6.3 Graded algebras

Let A be an algebra and $(M, \cdot, 1)$ a monoid.

Definition 6.6. An *M-gradation* (or *M-grading*) on A is a family of vector subspaces $(A_m)_{m\in M}$ such that

1) $A = \bigoplus_{m\in M} A_m$;
2) $A_m A_n \subseteq A_{mn}$, where $A_m A_n = \operatorname{span}\{\, ab \mid a \in A_m,\ b \in A_n \,\}$;
3) $1_A \in A_{1_M}$.

Example 6.7. Let $A = \Bbbk[x]$ be the polynomial ring and $M = (\mathbb{N}, +, 0)$. Then $(A_n = \Bbbk x^n)_{n\in\mathbb{N}}$ is an $\mathbb{N}$-gradation of A, since $x^m x^n = x^{m+n}$.

Example 6.8. Let $A = R(\sigma, t)$ be a GWA and $M = (\mathbb{Z}, +, 0)$. Put

$$A_n = \begin{cases} R\,X^n, & n > 0, \\ R, & n = 0, \\ R\,Y^{|n|}, & n < 0. \end{cases}$$

Then $(A_n)_{n\in\mathbb{Z}}$ is a $\mathbb{Z}$-gradation on A (Exercise 6.11).

6.4 Exercises

Exercise 6.9. Show that the Casimir element $C = ef + fe + \frac{1}{2}h^2$ is central in $U(\mathfrak{sl}_2(\mathbb{k}))$. (It suffices to check that it commutes with e, f and h.)

Exercise 6.10. Show that $\hat{\sigma}$ and δ in Theorem 6.4 are well defined: $\hat{\sigma}$ is an algebra automorphism of $R[Y;\sigma^{-1}]$, and δ is a $\hat{\sigma}$-derivation. (Hint: they must respect the relation $Yr = \sigma^{-1}(r)Y$. For $\hat{\sigma}$: $\hat{\sigma}(Y)\hat{\sigma}(r) = Y\sigma(r) = rY = \hat{\sigma}(\sigma^{-1}(r))\hat{\sigma}(Y)$. For δ: $\delta(Yr) = \delta(Y)r = (\sigma(t) - t)r$ and $\delta(\sigma^{-1}(r)Y) = r(\sigma(t) - t)$, which agree because R is commutative.)

Exercise 6.11. Prove that $(A_n)_{n\in\mathbb{Z}}$ in Example 6.8 is a $\mathbb{Z}$-gradation on $A = R(\sigma, t)$. (Hint: the tools are in Chapter 7. Grade $R\langle X, Y\rangle$ by $\deg r = 0$, $\deg X = 1$, $\deg Y = -1$, observe that the defining relations are homogeneous, and use the relations to show that the image of the degree-n part is A_n.)

Chapter 7

Degree Derivations and Graded Ideals

Gradings and derivations are closely linked: in characteristic 0, a $\mathbb{Z}$-grading is the same thing as a derivation that acts diagonalizably with integer eigenvalues. We use this to grade the Weyl algebra. We then study ideals that are compatible with a grading, so that quotients of graded algebras are again graded. We end with a remark explaining why a GWA needs both of the relations $YX = t$ and $XY = \sigma(t)$.

7.1 Degree derivations

In this section we assume $\operatorname{char} \Bbbk = 0$, so that $\mathbb{Z} \subset \Bbbk$.

Let $\{A_n\}_{n\in\mathbb{Z}}$ be a $\mathbb{Z}$-gradation on an algebra A (Definition 6.6). Define a linear map $d : A \to A$ by

$$d|_{A_n} = n \cdot \mathrm{Id}_{A_n}.$$

For example,

$$d(a_{-2} + a_3 + a_5) = -2a_{-2} + 3a_3 + 5a_5 \qquad (a_i \in A_i).$$

Proposition 7.1. *d is a derivation of A.*

Proof. We want $d(ab) = d(a)b + ad(b)$ for all $a, b \in A$. By linearity, it suffices to check this for homogeneous elements, i.e. for $a, b \in \bigcup_{n\in\mathbb{Z}} A_n$. Say $a \in A_n$ and $b \in A_m$. Then $ab \in A_{n+m}$, so

$$d(ab) = (n+m)ab, \qquad \text{while} \qquad d(a)b + ad(b) = nab + mab. \qquad \square$$

Definition 7.2. d is called the *degree derivation* of A with respect to $\{A_n\}_{n\in\mathbb{Z}}$.

Definition 7.3. An *M-graded algebra* is an algebra A together with an M-gradation $\{A_m\}_{m\in M}$ on A.

Conversely, suppose that

1) A is an algebra, and
2) $d : A \to A$ is a derivation that is diagonalizable with integer eigenvalues.

Define

$$A_n = E_n(d) = \{\, a \in A \mid d(a) = na \,\}, \qquad n \in \mathbb{Z}.$$

Proposition 7.4. *$\{A_n\}_{n\in\mathbb{Z}}$ is a $\mathbb{Z}$-gradation on A.*

Proof. By 2), $A = \bigoplus_{n\in\mathbb{Z}} A_n$. By the Leibniz rule, if $d(a) = ma$ and $d(b) = nb$ then $d(ab) = (m+n)ab$, so $A_m A_n \subseteq A_{m+n}$. Lastly, $d(1) = d(1\cdot 1) = 2d(1)$ gives $d(1) = 0$, so $1_A \in A_0$. □

In characteristic 0 the two constructions are inverse to each other: the degree derivation of the grading $\{E_n(d)\}$ is d, and the eigenspaces of a degree derivation are the A_n. To use this, recall that assigning a degree to each generator makes the free algebra $F(X)$ graded: a word has the sum of the degrees of its letters as its degree.

Example 7.5 (Grading the Weyl algebra)**.** The Weyl algebra has a $\mathbb{Z}$-gradation determined by $\deg(x) = 1$ and $\deg(\partial) = -1$. Write

$$A_1(\Bbbk) = \Bbbk\langle x, \partial\rangle / I, \qquad I = \langle \underbrace{\partial x - x\partial - 1}_{\in \Bbbk\langle x,\partial\rangle_0} \rangle_{\text{ideal}}.$$

Define $d : \{x, \partial\} \to \Bbbk\langle x, \partial\rangle$ by

$$d(x) = x, \qquad d(\partial) = -\partial,$$

and extend it to a derivation d of $\Bbbk\langle x, \partial\rangle$ (Exercise 4.10). By Proposition 5.6, d descends to $A_1(\Bbbk)$ as soon as $d(\partial x - x\partial - 1) \in I$. In fact this element is 0:

$$d(\partial x) = d(\partial)x + \partial d(x) = -\partial x + \partial x = 0, \qquad d(x\partial) = x\partial - x\partial = 0, \qquad d(1) = 0.$$

So we get a well-defined derivation $d : A_1(\Bbbk) \to A_1(\Bbbk)$. By the Leibniz rule, every word w in x and ∂ is an eigenvector:

$$d(w) = \big(\#\{x\text{'s in } w\} - \#\{\partial\text{'s in } w\}\big)\, w.$$

Since the words span $A_1(\Bbbk)$, d is diagonalizable with integer eigenvalues. Proposition 7.4 then gives the grading, with $x \in A_1(\Bbbk)_1$ and $\partial \in A_1(\Bbbk)_{-1}$.

There is also a route that works in any characteristic. The relation is homogeneous of degree 0, so the ideal I is graded (Proposition 7.8), and Lemma 7.9 below shows that the quotient is graded.

7.2 Graded ideals and quotients

Let A be an algebra, M a monoid, and $\{A_m\}_{m\in M}$ an M-gradation on A. We write general monoids multiplicatively, and $\mathbb{Z}$ and $\mathbb{N}$ additively.

Definition 7.6. A (left, right or two-sided) ideal $I \subseteq A$ is *graded* if

$$I = \bigoplus_{m\in M} (A_m \cap I).$$

Note that the sum on the right is always direct, and the inclusion $\supseteq$ always holds. So the condition is really that $I \subseteq \sum_m (A_m \cap I)$.

Example 7.7. Let $A = \Bbbk[x] = \bigoplus_{n\in\mathbb{N}} \Bbbk x^n$ with its usual $\mathbb{N}$-gradation, and

$$I = \langle x-1\rangle_{\text{ideal}} = \{\, p(x)(x-1) \mid p(x) \in A \,\}.$$

A nonzero multiple of $x-1$ vanishes at $x=1$, while λx^m does not unless $\lambda = 0$. So $A_m \cap I = 0$ for all $m \in \mathbb{N}$. Since $I \neq 0$, the ideal I is *not* graded.

Exercise 7.16 gives a useful criterion for an ideal to be graded. In practice, most graded ideals arise in the following way. Recall that an element is *homogeneous* if it lies in $\bigcup_m A_m$.

Proposition 7.8. *Let $I = \langle a_1, \dots, a_k\rangle_{\text{ideal}} = Aa_1A + \cdots + Aa_kA$ be the two-sided ideal generated by homogeneous elements $a_i \in A_{m_i}$. Then I is graded:*

$$I = \bigoplus_{m\in M} (A_m \cap I).$$

The same holds for one-sided ideals, and for any set of homogeneous generators.

Proof. Since $A = \bigoplus_m A_m$, bilinearity gives

$$Aa_iA = \operatorname{span}\{\, b\,a_i\,c \mid b, c \in A\,\} = \sum_{k,\ell\in M} A_k\; a_i\; A_\ell.$$

So I is spanned by elements $b\,a_i\,c$ with b and c homogeneous. Each such element is homogeneous, of degree $k\,m_i\,\ell$, and lies in I. Hence $I \subseteq \sum_m (A_m \cap I)$, and the reverse inclusion is automatic. □

Lemma 7.9. *Let I be a graded two-sided ideal of A. Then A/I is an M-graded algebra:*

$$A/I = \bigoplus_{m\in M} (A/I)_m, \qquad (A/I)_m = \{\, a + I \mid a \in A_m \,\} =: \overline{A_m}.$$

Proof. Since $A = \sum_m A_m$, we get $A/I = \sum_m \overline{A_m}$. To see that the sum is direct, suppose $\sum_m (a_m + I) = 0$ with $a_m \in A_m$. Then $\sum_m a_m \in I = \bigoplus_m (A_m \cap I)$, say $\sum_m a_m = \sum_m i_m$ with $i_m \in A_m \cap I$. Comparing homogeneous components gives $a_m = i_m \in I$ for every m. Finally, $\overline{A_m}\,\overline{A_n} \subseteq \overline{A_mA_n} \subseteq \overline{A_{mn}}$ and $1 + I \in \overline{A_1}$. □

Remark 7.10. Two comments on the lemma.

1) $(A/I)_m$ is a set of cosets. It should not be confused with the subset

$$A_m + I = \{\, a + i \mid a \in A_m,\ i \in I \,\}$$

of A.

2) If I is only a left (or right) graded ideal, the same argument gives a direct sum decomposition $A/I = \bigoplus_m \overline{A_m}$ of vector spaces. But A/I is then not an algebra, so it does not make sense to call this an algebra grading.

Remark 7.11. If $I \subseteq A$ is graded, put $I_m = A_m \cap I$, so that $I = \bigoplus_{m\in M} I_m$.

Claim 7.12. *As vector spaces,*

$$\overline{A_m} = \{\, a + I \mid a \in A_m \,\} \cong A_m/I_m, \qquad a + I \longmapsto a + I_m.$$

Proof. The map is well defined: if $a, b \in A_m$ and $a + I = b + I$, then $a - b \in I$ and $a - b \in A_m$, so $a - b \in I_m$. It is clearly surjective, and it is injective because $I_m \subseteq I$. □

Corollary 7.13. *For a graded two-sided ideal I,*

$$A/I \cong \bigoplus_{m \in M} A_m/I_m, \qquad \text{with} \qquad (a_m + I_m)(b_n + I_n) = a_m b_n + I_{mn}.$$

With these tools you can now do Exercise 6.11 on the grading of a GWA.

7.3 A remark on GWAs

In the definition of a GWA (Definition 5.7) we imposed both $YX = t$ and $XY = \sigma(t)$. Is one of them redundant? Under a mild hypothesis, yes.

Proposition 7.14. *Let A be an algebra containing a subalgebra R, let σ be an automorphism of R, and let $X, Y \in A$ and $t \in R$ satisfy*

$$YX = t, \qquad Xr = \sigma(r)X \quad \text{for all } r \in R.$$

If X is not a right zero-divisor in A (that is, $aX = 0 \Rightarrow a = 0$ for $a \in A$), then $XY = \sigma(t)$.

Proof. By associativity, $X(YX) = (XY)X$. The left side equals $Xt = \sigma(t)X$. Hence $(XY - \sigma(t))X = 0$, and so $XY = \sigma(t)$. □

The hypothesis on X cannot be dropped.

Example 7.15 (Shift operators)**.** Let V be the space of all sequences $(v_0, v_1, v_2, \dots)$ in $\Bbbk$, let $A = \mathrm{End}(V)$, $R = \Bbbk$, $\sigma = \mathrm{id}$ and $t = 1$. Let X be the right shift and Y the left shift:

$$X(v_0, v_1, \dots) = (0, v_0, v_1, \dots), \qquad Y(v_0, v_1, \dots) = (v_1, v_2, \dots).$$

Then $YX = 1$, but $XY(v_0, v_1, \dots) = (0, v_1, v_2, \dots)$, so $XY \neq 1 = \sigma(t)$. (Here X is a right zero-divisor: the projection a onto the first coordinate satisfies $aX = 0$.) This is why both relations are part of the definition of a GWA.

7.4 Exercises

Exercise 7.16. Let A be an algebra with an M-gradation $\{A_m\}_{m \in M}$, and $I \subseteq A$ a (left, right or two-sided) ideal. Show that the following are equivalent:

1) I is graded;
2) whenever $a_{m_1} + a_{m_2} + \cdots + a_{m_k} \in I$ with $a_{m_i} \in A_{m_i}$ and $m_i \neq m_j$ for $i \neq j$, we have $a_{m_i} \in I$ for all $i = 1, \dots, k$.

Exercise 7.17. Let $A = R(\sigma, t)$ be a GWA, with the standing assumptions of Chapter 6.

1) Describe the centralizer of R in A,

$$C_A(R) = \{\, a \in A \mid ar = ra \text{ for all } r \in R \,\}.$$

2) Describe the center $Z(A) = \{\, a \in A \mid ab = ba \text{ for all } b \in A \,\}$
   a) assuming that σ has infinite order;
   b*) without any assumption on σ.

Chapter 8

Filtered Algebras

Many natural algebras, like the Weyl algebra, are not graded in the most obvious way, because their defining relations are not homogeneous. They are, however, filtered*, and every filtered algebra has an associated graded algebra, which is often much simpler (for the Weyl algebra it is a polynomial ring). Good properties of the associated graded algebra, such as being Noetherian or a domain, then lift back to the original algebra.*

8.1 Filtrations

Definition 8.1. A *filtration* F on an algebra A is a family of subspaces $F = (F_n A)_{n\in\mathbb{N}}$ such that, writing $A_{(n)} = F_n A$,

1) $A_{(0)} \subseteq A_{(1)} \subseteq A_{(2)} \subseteq \cdots$;
2) $A = \bigcup_{n\geq 0} A_{(n)}$;
3) $A_{(m)}A_{(n)} \subseteq A_{(m+n)}$;
4) $1_A \in A_{(0)}$.

An algebra together with a filtration is a *filtered algebra*. We also put $A_{(-1)} = 0$.

Example 8.2. $A_{(n)} = A$ for all n is the *trivial filtration*.

Example 8.3. $\Bbbk[x]_{(n)} = \Bbbk 1 + \Bbbk x + \cdots + \Bbbk x^n$, the polynomials of degree at most n.

Example 8.4 (Filtrations from generators)**.** Let A be any algebra, $X = \{a_i\}_{i\in I}$ any generating set, and $d : I \to \mathbb{N}$ any "degree function", so that $\deg a_i = d(i)$. Define

$$A_{(n)} = \operatorname{span}\left\{ a_{i_1} a_{i_2} \cdots a_{i_k} \;\middle|\; k \geq 0,\ i_j \in I,\ \sum_{j=1}^{k} d(i_j) \leq n \right\}.$$

This is a filtration on A. Note the inequality "$\leq n$", not "$= n$": we want the $A_{(n)}$ to grow with n. The empty product ($k = 0$) is 1, so $1 \in A_{(0)}$.

Example 8.5 (The Weyl algebra)**.** Let $A = A_1(\Bbbk) = \Bbbk\langle \partial, x\rangle / \langle \partial x - x\partial - 1\rangle_{\text{ideal}}$, $X = \{x, \partial\}$ and $d(x) = d(\partial) = 1$. Then

$$A_{(0)} = \Bbbk 1, \qquad A_{(1)} = \Bbbk x + \Bbbk \partial + \Bbbk 1,$$

$$A_{(2)} = \Bbbk x\partial + \Bbbk x^2 + \Bbbk \partial^2 + \Bbbk \underbrace{\partial x}_{= x\partial + 1} + A_{(1)}$$

$$= \Bbbk x^2 + \Bbbk x\partial + \Bbbk \partial^2 + A_{(1)},$$

and in general

$$A_{(n)} = \bigoplus_{k+\ell \le n} \Bbbk x^k \partial^\ell.$$

The sum is direct because the monomials $x^k\partial^\ell$ are linearly independent. Indeed, by Example 4.7 we have $A_1(\Bbbk) \cong \Bbbk[x][\partial; \mathrm{Id}, \frac{d}{dx}]$, which is a free left $\Bbbk[x]$-module on the powers of ∂ (Theorem 4.4).

8.2 Why filtrations?

Why not just use a grading? Take the free algebra $A = \Bbbk\langle x, \partial\rangle$. Declaring $\deg x = 1 = \deg \partial$ defines an $\mathbb{N}$-gradation on A, and

$$x\partial - \partial x \in A_2, \qquad \text{but} \qquad \partial x - x\partial - 1 \notin \bigcup_{n\ge 0} A_n.$$

The defining relation of the Weyl algebra is not homogeneous for this grading. So Proposition 7.8 does not apply, and $A_1(\Bbbk)$ does not inherit the grading. (It does have the grading of Example 7.5, with $\deg \partial = -1$, but that one is not an $\mathbb{N}$-grading.) What survives of $\deg x = \deg \partial = 1$ is exactly the filtration of Example 8.5.

Example 8.6. Let $X = \{e, f, h\} \subset U(\mathfrak{sl}_2)$ and $d(e) = d(f) = d(h) = 1$. This gives a filtration of $U(\mathfrak{sl}_2)$, described in Exercise 8.13.

8.3 The associated graded algebra

Let (A, F) be a filtered algebra, with $A_{(n)} = F_n A$. Define the vector space

$$\operatorname{gr} A = \operatorname{gr}_F A = \bigoplus_{n=0}^{\infty} \operatorname{gr}_n A, \qquad \operatorname{gr}_n A = A_{(n)}/A_{(n-1)}.$$

Nonzero elements of $\operatorname{gr}_n A$ are called *leading terms of degree n*. For $a \in A_{(n)}$ we write $\bar{a} = a + A_{(n-1)} \in \operatorname{gr}_n A$ (with n understood from context). Define a product on $\operatorname{gr} A$ by

$$(a_n + A_{(n-1)})(b_m + A_{(m-1)}) = \overbrace{a_n b_m + A_{(n+m-1)}}^{\in \operatorname{gr}_{n+m} A}, \qquad a_n \in A_{(n)},\ b_m \in A_{(m)},$$

extended bilinearly.

Proposition 8.7. *This product is well defined, and makes* $\operatorname{gr} A$ *into an* $\mathbb{N}$*-graded algebra, the* associated graded algebra *of* (A, F).

Proof. If we change a_n by an element of $A_{(n-1)}$, the product $a_n b_m$ changes by an element of $A_{(n-1)}A_{(m)} \subseteq A_{(n+m-1)}$. The same holds for b_m, so the product is well defined. Associativity is inherited from A, the unit is $\bar{1} \in \operatorname{gr}_0 A$, and $\operatorname{gr}_n A \cdot \operatorname{gr}_m A \subseteq \operatorname{gr}_{n+m} A$ by construction. □

Example 8.8. Let $A = A_1(\Bbbk)$ with $d(x) = d(\partial) = 1$. Then

$$\operatorname{gr}_1 A = A_{(1)}/A_{(0)} = \Bbbk\bar{x} \oplus \Bbbk\bar{\partial}, \qquad \operatorname{gr}_n A = A_{(n)}/A_{(n-1)} = \bigoplus_{k+\ell=n} \Bbbk \bar{x}^k \bar{\partial}^\ell.$$

The generators of $\operatorname{gr} A$ commute:

$$\begin{aligned} \bar{\partial}\bar{x} - \bar{x}\bar{\partial} &= \partial x - x\partial + A_{(1)} \\ &= 1 + A_{(1)} \\ &= 0 + A_{(1)} \qquad \text{since } 1 \in A_{(0)} \subset A_{(1)}. \end{aligned}$$

So there is a surjective algebra map $\Bbbk[X, D] \to \operatorname{gr} A$, $X \mapsto \bar{x}$, $D \mapsto \bar{\partial}$. It sends the monomials of degree n to the basis $\{\bar{x}^k \bar{\partial}^\ell\}_{k+\ell=n}$ of $\operatorname{gr}_n A$, so it is also injective. Hence

$$\boxed{\operatorname{gr} A_1(\Bbbk) \cong \Bbbk[\bar{x}, \bar{\partial}]},$$

a polynomial algebra.

8.4 Applications

Theorem 8.9 ([MR01, Theorem 1.6.9])**.** *Let A be a filtered algebra. If* $\operatorname{gr} A$ *is left (respectively right) Noetherian, then so is A.*

Recall that an algebra A is a *domain* (or *entire*) if

$$ab = 0 \implies a = 0 \text{ or } b = 0.$$

Theorem 8.10. *Let A be a filtered algebra. If* $\operatorname{gr} A$ *is a domain, then so is A.*

Proof. Let $a, b \in A$ be nonzero. Since the filtration is exhaustive and $A_{(-1)} = 0$, there are n and m with $a \in A_{(n)} \setminus A_{(n-1)}$ and $b \in A_{(m)} \setminus A_{(m-1)}$. Then the leading terms $\bar{a} \in \operatorname{gr}_n A$ and $\bar{b} \in \operatorname{gr}_m A$ are nonzero, so $\bar{a}\bar{b} = ab + A_{(n+m-1)} \neq 0$. In particular $ab \neq 0$. □

Corollary 8.11. *$A_1(\Bbbk)$ is left and right Noetherian, and it is a domain.*

Proof. By Example 8.8, $\operatorname{gr} A_1(\Bbbk) \cong \Bbbk[\bar{x}, \bar{\partial}]$. This is Noetherian by Hilbert's basis theorem, and it is a domain. Now apply Theorems 8.9 and 8.10. □

8.5 A preview: dimensions

Part II of [MR01] is about dimensions of rings, three of which we mention here:

Krull dimension	$\operatorname{Kdim}(A)$
Gelfand–Kirillov dimension	$\operatorname{GKdim}(A)$
global dimension	$\operatorname{gl.dim}(A)$

For a filtered algebra A, and under suitable hypotheses on the filtration (see [MR01, Part II] and [KL00]), these compare with the dimensions of $\operatorname{gr} A$ as follows:

i) $\operatorname{Kdim}(A) \leq \operatorname{Kdim}(\operatorname{gr} A)$;
ii) $\operatorname{GKdim}(A) \geq \operatorname{GKdim}(\operatorname{gr} A)$;
iii) $\operatorname{gl.dim}(A) \leq \operatorname{gl.dim}(\operatorname{gr} A)$ [MR01, Theorem 7.6.18].

We return to the Gelfand–Kirillov dimension in Part VI.

Corollary 8.12. $\operatorname{gl.dim}(A_1(\mathbb{C})) \leq \operatorname{gl.dim}(\mathbb{C}[\bar{x}, \bar{\partial}]) = 2$.

In fact the inequality is strict: $\operatorname{gl.dim}(A_1(\mathbb{C})) = 1$. More generally, the n-th Weyl algebra over a field of characteristic 0 has global dimension n, a theorem of Roos.

8.6 Exercises

Exercise 8.13. Let $U(\mathfrak{sl}_2)$ have the filtration with $d(e) = d(f) = d(h) = 1$. Show that

$$U(\mathfrak{sl}_2)_{(N)} = \operatorname{span}\left\{ h^k f^\ell e^m \;\middle|\; k, \ell, m \geq 0,\ k + \ell + m \leq N \right\}.$$

(For instance, $ef = fe + h$ lies in the right-hand side.)

Exercise 8.14. With the same filtration on $U(\mathfrak{sl}_2)$, show that

1) $\operatorname{gr} U(\mathfrak{sl}_2)$ is generated by $\bar{e}, \bar{f}, \bar{h} \in \operatorname{gr}_1 U(\mathfrak{sl}_2)$;
2) $\operatorname{gr} U(\mathfrak{sl}_2)$ is commutative.

Conclude that $U(\mathfrak{sl}_2)$ is left and right Noetherian. (In fact $\operatorname{gr} U(\mathfrak{sl}_2) \cong \Bbbk[\bar{e}, \bar{f}, \bar{h}]$ is a polynomial algebra; this is the Poincaré–Birkhoff–Witt theorem, see e.g. [Humphreys72, §17].)

Chapter 9

The Rational Cherednik Algebra

We meet a small but important example: the rational Cherednik algebra of the group S_2. It acts on polynomials by Dunkl operators, it is graded, and the grading is internal*: it is given by the commutator with a single element. Along the way we meet inner derivations.*

9.1 The algebra $\mathbb{H}$

Definition 9.1. The *rational Cherednik algebra* (of the group S_2) is

$$\mathbb{H} = \mathbb{C}\langle x, t, s\rangle / \langle \text{Rels}\rangle_{\text{ideal}}, \qquad \text{Rels}: \begin{cases} tx - xt = 1 + s, \\ sx + xs = 0, \\ st + ts = 0, \\ s^2 = 1. \end{cases}$$

It is a "doubly degenerate" version of the double affine Hecke algebra (DAHA).

Proposition 9.2. *There is an algebra map $\rho: \mathbb{H} \to \mathrm{End}(\mathbb{C}[x])$ with*

$$\begin{aligned} x &\longmapsto \big(\textit{multiplication by } x\big), \\ s &\longmapsto \big(p(x) \mapsto p(-x)\big), \\ t &\longmapsto \Big(p(x) \mapsto \frac{dp}{dx} + \frac{p(x) - p(-x)}{2x}\Big). \end{aligned}$$

The operator $\rho(t)$ is a *Dunkl operator*. It does map polynomials to polynomials: $p(x) - p(-x)$ is an odd polynomial, hence divisible by x.

Proof. Define the map on generators $\{x, s, t\} \to \mathrm{End}(\mathbb{C}[x])$ by the formulas above, and extend it to an algebra map $\rho': \mathbb{C}\langle x, s, t\rangle \to \mathrm{End}(\mathbb{C}[x])$. By Proposition 2.10, we must check that the

relations go to 0. For the first one,

$$\begin{aligned}\rho'\big(tx - xt - (1+s)\big)\big(p(x)\big) &= t\big(xp(x)\big) - x\cdot t\big(p(x)\big) - \big(p(x) + p(-x)\big)\\ &= \frac{d}{dx}\big(xp(x)\big) + \frac{xp(x) + xp(-x)}{2x}\\ &\qquad - x\Big(\frac{dp}{dx} + \frac{p(x) - p(-x)}{2x}\Big) - \big(p(x) + p(-x)\big)\\ &= p(x) + xp'(x) + \tfrac{1}{2}\big(p(x) + p(-x)\big)\\ &\qquad - xp'(x) - \tfrac{1}{2}\big(p(x) - p(-x)\big) - \big(p(x) + p(-x)\big)\\ &= p(x) + p(-x) - \big(p(x) + p(-x)\big) = 0.\end{aligned}$$

The other three relations are left as Exercise 9.6. □

Remark 9.3 (Parameters). The rational Cherednik algebra usually depends on parameters. If we replace the coefficient $\frac{1}{2}$ in the Dunkl operator by a parameter c, so that $t \mapsto \big(p \mapsto p' + c\,\frac{p(x)-p(-x)}{x}\big)$, the same computation gives $tx - xt = 1 + 2cs$. So $\mathbb{H}$ is the case $c = \frac{1}{2}$. Note also that t is a *generator* here, not the parameter that is often called t in the literature.

Remark 9.4. $\mathbb{H}$ is really $\mathbb{H}(S_2)$, where S_2 is the symmetric group. There is a whole family

$$\mathbb{H}(S_n) = \frac{\mathbb{C}\langle x_1, \dots, x_{n-1}, t_1, \dots, t_{n-1}, s_1, \dots, s_{n-1}\rangle}{\langle \text{relations}\rangle_{\text{ideal}}},$$

and more generally a rational Cherednik algebra $\mathbb{H}(W)$ for every complex reflection group W. For a survey, see Rouquier [Rouquier05]; see also the lecture notes of Etingof and Ma [EM10].

9.2 The grading

Recall the relations, and assign the degrees $\deg t = -1$, $\deg x = 1$, $\deg s = 0$ to the generators. Then every relation is homogeneous:

$$\begin{cases} tx - xt - (1+s) = 0 & \text{(degree 0)},\\ sx + xs = 0 & \text{(degree 1)},\\ st + ts = 0 & \text{(degree } -1\text{)},\\ s^2 - 1 = 0 & \text{(degree 0)}.\end{cases}$$

So the ideal they generate is graded (Proposition 7.8), and $\mathbb{H}$ is a $\mathbb{Z}$-graded algebra (Lemma 7.9).

In fact more is true: $\mathbb{H}$ is *internally graded*, meaning that there is an element $d \in \mathbb{H}$ such that

$$\mathbb{H}_n = \{\, u \in \mathbb{H} \mid [d, u] = nu \,\} \qquad \text{for all } n \in \mathbb{Z}.$$

Compare this with the Weyl algebra, where $[x\partial, x^n] = nx^n$.

Aside: inner derivations. Let A be any algebra and $a \in A$. Then

$$\mathrm{ad}_a : A \longrightarrow A, \qquad b \longmapsto [a, b] = ab - ba,$$

is a derivation:

$$[a, bc] = [a, b]c + b[a, c].$$

Derivations of this form are called *inner*. Recall (Proposition 7.1) that a $\mathbb{Z}$-graded algebra $B = \bigoplus_n B_n$ has a degree derivation D, with $D|_{B_n} = n \cdot \mathrm{Id}_{B_n}$. A $\mathbb{Z}$-graded algebra is internally graded exactly when its degree derivation is inner, $D = \mathrm{ad}_d$.

For the Weyl algebra with the grading of Example 7.5, $D = \mathrm{ad}_{x\partial}$. What about $\mathbb{H}$? By analogy, let us try $D = \mathrm{ad}_d$ with $d = xt$:

$$\begin{aligned}
[xt, x] &= x[t, x] = x(1 + s) \neq 1 \cdot x, \\
[xt, t] &= [x, t]t = -(1 + s)t \neq -1 \cdot t, \\
[xt, s] &= xts - sxt = x\,s(-t) - sxt = s(-x)(-t) - sxt = 0 = 0 \cdot s.
\end{aligned}$$

That doesn't work. Instead, try the symmetrized version

$$d = \boxed{\tfrac{1}{2}(xt + tx)}.$$

Now

$$\begin{array}{rrcll}
 & [xt, x] & = & x[t, x] & = x(1 + s) \\
+ & [tx, x] & = & [t, x]x & = (1 + s)x \\
\hline
 & [xt + tx, x] & = & x + xs + x + sx = 2x, &
\end{array}$$

using $xs + sx = 0$. Dividing by 2 gives $[d, x] = x$. Similarly, $[tx, s] = 0$ (by the same computation as for $[xt, s]$), so $[d, s] = 0$. Finally,

$$[d, t] = \tfrac{1}{2}\big([x, t]t + t[x, t]\big) = -\tfrac{1}{2}\big((1 + s)t + t(1 + s)\big) = -\tfrac{1}{2}(2t + st + ts) = -t.$$

Proposition 9.5. *$\mathbb{H}$ is internally graded, with $d = \frac{1}{2}(xt + tx)$: $\mathbb{H}_n = \{\, u \in \mathbb{H} \mid [d, u] = nu \,\}$.*

Proof. Both ad_d and the degree derivation D are derivations of $\mathbb{H}$. By the computations above they agree on the generators x, t, s. By the Leibniz rule, a derivation is determined by its values on generators, so $\mathrm{ad}_d = D$. The eigenspaces of D are the $\mathbb{H}_n$. □

Bonus fact: the Fourier automorphism. There is an automorphism $\theta : \mathbb{H} \to \mathbb{H}$ with

$$\theta(x) = t, \qquad \theta(t) = -x, \qquad \theta(s) = s.$$

Indeed, θ sends the relations to $\pm$ relations: $tx - xt - (1+s) \mapsto (-x)t - t(-x) - (1+s) = tx - xt - (1+s)$, $sx + xs \mapsto st + ts$, $st + ts \mapsto -(sx + xs)$, and $s^2 - 1 \mapsto s^2 - 1$. It is invertible because $\theta^4 = \mathrm{id}$. Since $\theta(d) = -d$, applying θ to $[d, x] = x$ gives $[-d, t] = t$. This is another way to see that $[d, t] = -t$.

9.3 Further reading

The next part of the book is about the Diamond Lemma. Before reading it, have a look at Bergman's paper [Bergman78]: Section 1, (2.1), and Section 3. Section 3 uses the Diamond Lemma to prove the Poincaré–Birkhoff–Witt theorem, which gives a basis for $U(\mathfrak{g})$.

9.4 Exercises

Exercise 9.6. Complete the proof of Proposition 9.2: check that the operators $\rho(x)$, $\rho(s)$, $\rho(t)$ satisfy the remaining relations $sx + xs = 0$, $st + ts = 0$ and $s^2 = 1$.

Exercise 9.7. Let θ be the Fourier automorphism of $\mathbb{H}$. Compute θ^2 on the generators, and show that $\theta^2(u) = sus^{-1}$ for all $u \in \mathbb{H}$, so that θ^2 is conjugation by s. Deduce that θ has order 4.

Part III

The Diamond Lemma

Chapter 10

The Diamond Lemma

When an algebra is given by generators and relations, we usually want a basis. The Diamond Lemma gives a practical way to find one. Orient each relation into a rewriting rule; if every place where two rules compete (an "ambiguity") can be resolved, then the words that cannot be rewritten form a basis. This chapter sets up the language, states the lemma precisely and explains the proof. The next chapter is devoted to applications.

The Diamond Lemma in the form we use is due to G. Bergman [Bergman78]. We follow his notation, so it is a good idea to have the paper at hand.

10.1 Reductions

We work over our field $\Bbbk$. (Everything in this chapter works equally well over any commutative ring, which is the generality of Bergman's paper.) Let X be a set, let

$$\langle X\rangle_{\mathrm{mon}} = \{\text{words in } X\}$$

be the free monoid on X, and let $\Bbbk\langle X\rangle = F_{\Bbbk}(X)$ be the free algebra (Chapter 2), which has $\langle X\rangle_{\mathrm{mon}}$ as a basis.

Definition 10.1. A *reduction system* is a subset $S \subseteq \langle X\rangle_{\mathrm{mon}} \times \Bbbk\langle X\rangle$. For $\sigma \in S$ we write $\sigma = (W_\sigma, f_\sigma)$, and think of σ as a rewriting rule $W_\sigma \rightsquigarrow f_\sigma$.

Definition 10.2. For $A, B \in \langle X\rangle_{\mathrm{mon}}$ and $\sigma \in S$, the *reduction* $r_{A\sigma B}$ is the $\Bbbk$-linear map $\Bbbk\langle X\rangle \to \Bbbk\langle X\rangle$ defined on words by

$$r_{A\sigma B}(C) = \begin{cases} A f_\sigma B, & C = A W_\sigma B, \\ C, & C \neq A W_\sigma B. \end{cases}$$

Example 10.3. Let $\sigma = (xy,\ yx + 1)$. Then $r_{y\sigma x^2}(y^2xyx^3) = y^2xyx^3$, since $y^2xyx^3 \neq y\,xy\,x^2$; but $r_{y^2\sigma x^3}(y^2xyx^3) = y^2(yx+1)x^3$.

Definition 10.4. Let $a \in \Bbbk\langle X\rangle$.

- The reduction $r_{A\sigma B}$ *acts trivially* on a if the coefficient of the word $AW_\sigma B$ in a is zero (so that $r_{A\sigma B}(a) = a$).
- a is *irreducible* (with respect to S) if every reduction $r_{A\sigma B}$ acts trivially on a.

The irreducible elements form a subspace $\Bbbk\langle X\rangle_{\text{irr}}$. It is spanned by the *irreducible words*, the words that contain no W_σ as a subword.

Example 10.5. Let $X = \{x, y\}$ and $S = \{\sigma\}$ with $\sigma = (yx,\ xy + 1)$. Then $r_{1\sigma 1}(yx) = xy + 1$, and

$$r_{x\sigma x^2}\left(yx^3 + 3xyx^3 + yx\right) = yx^3 + 3x(xy + 1)x^2 + yx,$$

since only the middle word equals $x\ W_\sigma\ x^2$. (The first word yx^3 would need $r_{1\sigma x^2}$ instead.) An element of $\Bbbk\langle X\rangle$ is irreducible if and only if it is a linear combination of words that do not contain yx as a subword, so

$$\Bbbk\langle X\rangle_{\text{irr}} = \operatorname{span}_{\Bbbk}\{\, x^m y^n \mid m, n \geq 0 \,\}.$$

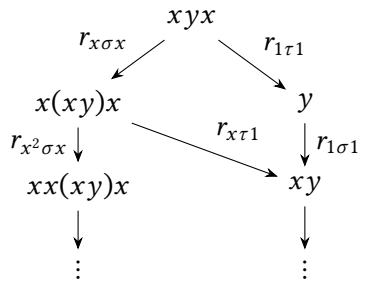
10.2 Termination and uniqueness

Definition 10.6. Let $a \in \Bbbk\langle X\rangle$.

- A sequence $r_1, r_2, \ldots, r_n$ of reductions is *final for* a if $r_n \cdots r_2 r_1(a) \in \Bbbk\langle X\rangle_{\text{irr}}$.
- a is *reduction-finite* if for every infinite sequence of reductions $r_1, r_2, \ldots$ there is an N such that r_i acts trivially on $r_{i-1} \cdots r_1(a)$ for all $i \geq N$.

Informally, a is reduction-finite if you "can't reduce it forever". In that case, any maximal sequence $r_1, r_2, \ldots$ such that each r_i acts nontrivially on $r_{i-1} \cdots r_1(a)$ must be finite, and hence final for a.

Example 10.7. Let $X = \{x, y\}$ and $S = \{\sigma = (y, xy),\ \tau = (xyx, y)\}$. Then xyx can be reduced forever:



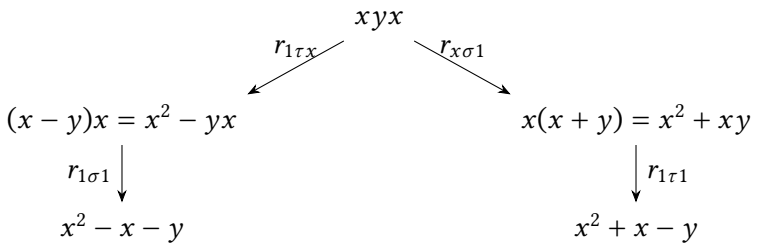
The rule $y \rightsquigarrow xy$ reproduces the letter it removes, so xyx is not reduction-finite.

Example 10.8. Here is a different way to fail. Let $X = \{x, y\}$ and $S = \{\sigma :\ xy \rightsquigarrow 1,\ \tau :\ xy \rightsquigarrow x^2y^2\}$. Then xy is not reduction-finite, because the rule τ makes words grow:

$$xy \xrightarrow{r_{1\tau 1}} x^2y^2 \xrightarrow{r_{x\tau y}} x^3y^3 \xrightarrow{r_{x^2\tau y^2}} \cdots$$

Definition 10.9. $a \in \Bbbk\langle X\rangle$ is *reduction-unique* if it is reduction-finite and its images under all final sequences of reductions are the same. This common value is denoted $r_S(a)$.

Example 10.10. Let $X = \{x, y\}$ and $S = \{\sigma = (yx,\ x + y),\ \tau = (xy,\ x - y)\}$. Think of the algebra $B = \Bbbk\langle x, y \mid yx = x + y,\ xy = x - y\rangle$. The word xyx reduces to two different irreducible elements:



So xyx is not reduction-unique. (Both results are equal to xyx in B, so $2x = 0$ in B: this presentation defines a much smaller algebra than it seems to.)

The next two lemmas are the basic tools for working with reduction-unique elements.

Lemma 10.11. *The reduction-unique elements form a subspace $R_S \subseteq \Bbbk\langle X\rangle$, and $r_S : R_S \to \Bbbk\langle X\rangle_{\mathrm{irr}}$ is a linear map.*

Proof. Let $a, b \in R_S$ and $\alpha \in \Bbbk$. First, $\alpha a + b$ is reduction-finite. Given an infinite sequence of reductions, it eventually acts trivially on the images of a and of b. A reduction that acts trivially on both $r_{i-1}\cdots r_1(a)$ and $r_{i-1}\cdots r_1(b)$ also acts trivially on their combination $r_{i-1}\cdots r_1(\alpha a + b)$.

Now let $r = r_n \cdots r_1$ be any final sequence for $\alpha a + b$. Since $a \in R_S$, we can extend r by further reductions r' so that $r'r(a) = r_S(a)$. Then, since $b \in R_S$, we can extend again by r'' so that $r''r'r(b) = r_S(b)$. The element $r(\alpha a + b)$ is already irreducible, so $r''r'$ acts trivially on it, and $r''r'r(a) = r_S(a)$ is irreducible too. Hence

$$r(\alpha a + b) = r''r'r(\alpha a + b) = \alpha\, r''r'r(a) + r''r'r(b) = \alpha\, r_S(a) + r_S(b).$$

So every final sequence for $\alpha a + b$ gives the same result, $\alpha r_S(a) + r_S(b)$. □

Lemma 10.12. *Let $a, b, c \in \Bbbk\langle X\rangle$ be such that ABC is reduction-unique for all words A, B, C occurring (with nonzero coefficient) in a, b, c respectively. Then for every sequence of reductions r, the element $a \cdot r(b) \cdot c$ is reduction-unique, and*

$$r_S\big(a \cdot r(b) \cdot c\big) = r_S(abc).$$

Proof. By Lemma 10.11, both sides are linear in a, b and c, so we may assume that $a = A$, $b = B$ and $c = C$ are words. By induction on the length of r, it suffices to treat a single reduction $r = r_{D\sigma E}$. Then

$$A\, r_{D\sigma E}(B)\, C = r_{AD\sigma EC}(ABC),$$

which is a reduction of the reduction-unique element ABC. Any final sequence for it, preceded by $r_{AD\sigma EC}$, is a final sequence for ABC. So it is reduction-unique, with $r_S\big(r_{AD\sigma EC}(ABC)\big) = r_S(ABC)$. □

10.3 Ambiguities

Reductions can compete in two ways.

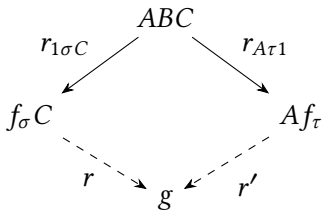
Definition 10.13. An *overlap ambiguity* is a 5-tuple (σ, τ, A, B, C) with $\sigma, \tau \in S$ and $A, B, C \in \langle X \rangle_{\mathrm{mon}} \setminus \{1\}$ such that

$$W_\sigma = AB, \qquad W_\tau = BC.$$

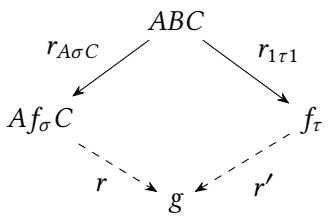
An *inclusion ambiguity* is a 5-tuple (σ, τ, A, B, C) with $\sigma \neq \tau \in S$ and $A, B, C \in \langle X \rangle_{\mathrm{mon}}$ such that

$$W_\sigma = B, \qquad W_\tau = ABC.$$

In both cases the word ABC can be reduced in two ways:



overlap



inclusion

The ambiguity is *resolvable* if the diamond can be completed: there are compositions of reductions r and r' such that

$$r(f_\sigma C) = r'(Af_\tau) \quad \text{(overlap)}, \qquad \text{resp.} \qquad r(Af_\sigma C) = r'(f_\tau) \quad \text{(inclusion)}.$$

10.4 Semigroup orders

To control termination, we compare words.

Definition 10.14. A *semigroup partial order* on $\langle X \rangle_{\mathrm{mon}}$ is a partial order $\leq$ such that

$$A < B \implies CAD < CBD \qquad \text{for all } C, D \in \langle X \rangle_{\mathrm{mon}}.$$

It is *compatible with S* if, for every $\sigma \in S$, f_σ is a linear combination of words $< W_\sigma$. It satisfies the *descending chain condition* (DCC) if there is no infinite strictly decreasing chain $A_1 > A_2 > A_3 > \cdots$.

For $f \in \Bbbk\langle X \rangle$ and a word A we write $f < A$ if every word occurring in f is $< A$. So compatibility says $f_\sigma < W_\sigma$.

Example 10.15. Let $\ell(A)$ denote the length (number of letters) of a word A, and put $A < B$ if $\ell(A) < \ell(B)$. This is a semigroup partial order with DCC, but not a total order: for instance xyz and yxz are incomparable.

Example 10.16 (Deg-lex order)**.** Fix a total order on X, say $x < y < z$. Put $A < B$ if $\ell(A) < \ell(B)$, or $\ell(A) = \ell(B)$ and A comes before B in the lexicographic order. Then, for example, $xyz < yxz$. This is a semigroup total order, and it has DCC if X is well ordered (for instance, finite).

Remark 10.17. If $\leq$ is compatible with S and has DCC, then every element of $\Bbbk\langle X \rangle$ is reduction-finite. Indeed, a nontrivial reduction replaces one word occurring in an element by a combination of strictly smaller words, and with DCC this cannot go on forever. (To make this precise, compare the finite sets of words occurring in successive elements in the multiset order; see [Bergman78, §1].)

From now on, let $\le$ be a semigroup partial order on $\langle X\rangle_{\mathrm{mon}}$ compatible with S. Put

$$I = \operatorname{span}_{\Bbbk}\{\, A(W_\sigma - f_\sigma)B \mid A, B \in \langle X\rangle_{\mathrm{mon}},\ \sigma \in S\,\} = \langle\, W_\sigma - f_\sigma \mid \sigma \in S\rangle_{\mathrm{ideal}},$$

the two-sided ideal generated by the relations $W_\sigma = f_\sigma$, and for each word A

$$\begin{aligned} I_A &= \operatorname{span}_{\Bbbk}\{\, B(W_\sigma - f_\sigma)C \mid B, C \in \langle X\rangle_{\mathrm{mon}},\ \sigma \in S,\ BW_\sigma C < A\,\} \\ &= \operatorname{span}_{\Bbbk}\{\, B - r_1(B) \mid B \in \langle X\rangle_{\mathrm{mon}},\ B < A,\ r_1 \text{ a single reduction}\,\}. \end{aligned}$$

The two descriptions agree because $B - r_{D\sigma E}(B)$ is 0 unless $B = DW_\sigma E$, in which case it equals $D(W_\sigma - f_\sigma)E$.

Lemma 10.18. *Let A be a word and $f \in \Bbbk\langle X\rangle$ with $f < A$. Then $r(f) < A$ and $f - r(f) \in I_A$ for every sequence of reductions r.*

Proof. For a single reduction $r_1 = r_{D\sigma E}$ we have $f - r_1(f) = \lambda\, D(W_\sigma - f_\sigma)E$, where λ is the coefficient of $DW_\sigma E$ in f. If $\lambda \neq 0$ then $DW_\sigma E < A$, so this lies in I_A. Also $r_1(f) < A$, since $Df_\sigma E < DW_\sigma E$ by compatibility and the semigroup property. For a sequence $r = r_n \cdots r_1$, telescope: $f - r(f) = \sum_i \big(r_{i-1}\cdots r_1(f) - r_i r_{i-1}\cdots r_1(f)\big)$. □

Definition 10.19. An ambiguity (σ, τ, A, B, C) is *resolvable relative to* $\le$ if

$$f_\sigma C - Af_\tau \in I_{ABC} \quad \text{(overlap)}, \qquad \text{resp.} \qquad Af_\sigma C - f_\tau \in I_{ABC} \quad \text{(inclusion)}.$$

Proposition 10.20. *A resolvable ambiguity is resolvable relative to $\le$.*

Proof. Consider an overlap ambiguity. Since $\le$ is compatible with S and is a semigroup order,

$$f_\sigma C < W_\sigma C = ABC, \qquad Af_\tau < AW_\tau = ABC.$$

If the ambiguity is resolvable, say $r(f_\sigma C) = r'(Af_\tau) = g$, then by Lemma 10.18 both $f_\sigma C - g$ and $Af_\tau - g$ lie in I_{ABC}, and hence so does their difference $f_\sigma C - Af_\tau$. The inclusion case is the same, with $Af_\sigma C$ and f_τ in place of $f_\sigma C$ and Af_τ. □

10.5 The Diamond Lemma

Theorem 10.21 (Diamond Lemma; [Bergman78, Theorem 1.2])**.** *Let S be a reduction system for $\Bbbk\langle X\rangle$, and let $\le$ be a semigroup partial order on $\langle X\rangle_{\mathrm{mon}}$ that is compatible with S and has DCC. Let $I = \langle W_\sigma - f_\sigma \mid \sigma \in S\rangle_{\mathrm{ideal}}$. The following are equivalent:*

(a) all ambiguities of S are resolvable;
(a′) all ambiguities of S are resolvable relative to $\le$;
(b) every element of $\Bbbk\langle X\rangle$ is reduction-unique;
(c) the composite

$$\Bbbk\langle X\rangle_{\mathrm{irr}} \hookrightarrow \Bbbk\langle X\rangle \twoheadrightarrow \Bbbk\langle X\rangle/I$$

is a bijection.

When these hold, $\Bbbk\langle X\rangle/I$ can be identified with $\Bbbk\langle X\rangle_{\mathrm{irr}}$, with the multiplication

$$a \cdot b = r_S(ab) \qquad \text{for } a, b \in \Bbbk\langle X\rangle_{\mathrm{irr}} \quad (\text{the product } ab \text{ taken in } \Bbbk\langle X\rangle).$$

In particular, the irreducible words form a basis of $\Bbbk\langle X\rangle/I$.

Proof. *(a)⇒(a′)* is Proposition 10.20.

(a′)⇒(b): sketch. By Remark 10.17 all elements are reduction-finite, and by Lemma 10.11 it suffices to show that every word W is reduction-unique. By DCC we may assume that all words $< W$ are reduction-unique. Then r_S is defined and linear on all $f < W$, and Lemma 10.12 shows that r_S vanishes on I_W. Any final sequence for W starts with a single reduction $W \mapsto Df_\sigma E$, after which we are dealing with an element $< W$, which is reduction-unique. So it suffices to show that $r_S(Df_\sigma E)$ does not depend on the first reduction. Take two first reductions. If they act on disjoint subwords of W, both results have the common reduction obtained by applying the other rule as well, and Lemma 10.12 gives equal values of r_S. If they act on overlapping subwords, or one inside the other, then their difference has the form $D'(f_\sigma C - Af_\tau)E'$ (or the inclusion analogue). By (a′) and the semigroup property this lies in I_W, so r_S kills it. For the details see Bergman's proof of [Bergman78, Theorem 1.2].

(b)⇒(c). By Lemma 10.11, $r_S : \Bbbk\langle X\rangle \to \Bbbk\langle X\rangle_{\mathrm{irr}}$ is linear. It is the identity on irreducible elements, and every reduction changes an element by an element of I, so $a - r_S(a) \in I$ for all a. Moreover r_S kills each generator $A(W_\sigma - f_\sigma)B$ of I, because $r_S(AW_\sigma B) = r_S(Af_\sigma B)$ by Lemma 10.12. Hence $\Bbbk\langle X\rangle = \Bbbk\langle X\rangle_{\mathrm{irr}} \oplus I$, which is (c). Since $ab - r_S(ab) \in I$, the multiplication is as claimed.

(c)⇒(a). Given an ambiguity, reduce $f_\sigma C$ and Af_τ (resp. $Af_\sigma C$ and f_τ) to irreducible elements g and g'; this is possible by Remark 10.17. Both are congruent to ABC modulo I, so $g - g' \in \Bbbk\langle X\rangle_{\mathrm{irr}} \cap I = 0$ by (c). So the ambiguity is resolvable. □

In practice we use the theorem as follows. Orient the relations of a presentation into a reduction system, find a compatible semigroup order with DCC (usually a variant of deg-lex), list the ambiguities, and check that each one resolves. The irreducible words then form a basis. The next chapter does this for the Weyl algebra, for generalized Weyl algebras, and for the enveloping algebra of $\mathfrak{so}_3$.

10.6 Exercises

Exercise 10.22. Prove the claims of Example 10.16: the deg-lex order is a semigroup total order on $\langle X\rangle_{\mathrm{mon}}$, and it has DCC when X is finite. Show that the plain lexicographic order (with $x < y$, and without first comparing lengths) does not have DCC:

$$y > xy > x^2y > x^3y > \cdots.$$

Exercise 10.23. Let $B = \Bbbk\langle x, y \mid yx = x + y,\ xy = x - y\rangle$ be the algebra of Example 10.10. Show that the ambiguity xyx, which does not resolve, forces $2x = 0$ in B. Deduce that $B \cong \Bbbk$ when $\operatorname{char}\Bbbk \neq 2$.

Chapter 11

Applications of the Diamond Lemma

We put the Diamond Lemma (Theorem 10.21) to work. We find bases for the Weyl algebra, for every generalized Weyl algebra, and for the enveloping algebra of $\mathfrak{so}_3$. The last example shows where the Jacobi identity enters into the Poincaré–Birkhoff–Witt theorem.

Recall the recipe from the end of Chapter 10: orient the relations into a reduction system, find a compatible semigroup order with DCC, list the ambiguities, and resolve each of them. The irreducible words then form a basis.

11.1 A warm-up: the Weyl algebra

Example 11.1. Let $X = \{x, y\}$ and $S = \{\sigma = (yx,\ xy + 1)\}$, as in Example 10.5. The deg-lex order with $x < y$ (Example 10.16) is compatible with S, since $xy < yx$ and $1 < yx$, and it has DCC. There are no ambiguities at all: the word yx does not overlap itself ($yx = AB$, $yx = BC$ with $B \neq 1$ would force $y = x$), and there is only one rule. By the Diamond Lemma and Example 10.5,

$$\{\, x^m y^n \mid m, n \geq 0 \,\} \text{ is a basis for } \Bbbk\langle x, y \mid yx = xy + 1\rangle = A_1(\Bbbk).$$

11.2 Generalized Weyl algebras

Let $A = R(\sigma, t)$ be a GWA (Definition 5.7): R is an algebra, $\sigma \in \mathrm{Aut}_{\Bbbk\text{-Alg}}(R)$, and t is central in R. (We do not need the standing assumptions of Chapter 6 here.)

Let $\mathbb{B} = \{b_i\}_{i \in \Lambda}$ be a $\Bbbk$-basis for R, chosen so that $b_0 = 1_R$ for some index $0 \in \Lambda$. Write

$$b_i b_j = \sum_k c_{ijk}\, b_k, \qquad t = \sum_i t_i\, b_i, \qquad \sigma(b_i) = \sum_j s_{ij}\, b_j, \qquad \sigma^{-1}(b_i) = \sum_j s'_{ij}\, b_j,$$

with all coefficients in $\Bbbk$. The c_{ijk} are the *structure constants* of R with respect to $\mathbb{B}$.

Let $\mathcal{X} = \{b_i\}_{i \neq 0} \cup \{X, Y\}$. We do *not* include b_0 among the generators: in all the formulas below, b_0 stands for the empty word 1. Then

$$A = \Bbbk\langle \mathcal{X} \rangle \Big/ \langle \mathrm{Rels} \rangle_{\mathrm{ideal}},$$

where the relations are

$$Xb_i = \sigma(b_i)X = \Big(\sum_j s_{ij} b_j\Big)X, \qquad Yb_i = \sigma^{-1}(b_i)Y = \Big(\sum_j s'_{ij} b_j\Big)Y,$$
$$YX = t = \sum_i t_i b_i, \qquad XY = \sigma(t) = \sum_j \Big(\sum_i t_i s_{ij}\Big) b_j,$$
$$b_i b_j = \sum_k c_{ijk} b_k,$$

for $i, j \neq 0$. Indeed, the last family of relations presents R (its multiplication table), with $b_0 = 1$ as the unit. The relations $Xr = \sigma(r)X$ and $Yr = \sigma^{-1}(r)Y$ are linear in r, so it suffices to impose them on basis elements.

Orient the relations into a reduction system S on $\Bbbk\langle\mathcal{X}\rangle$ (again with $b_0 = 1$):

$$S: \quad \begin{cases} Xb_i \rightsquigarrow \sum_j s_{ij}\, b_j X, & Yb_i \rightsquigarrow \sum_j s'_{ij}\, b_j Y, \\ YX \rightsquigarrow \sum_i t_i\, b_i, & XY \rightsquigarrow \sum_j \Big(\sum_i t_i s_{ij}\Big) b_j, \\ b_i b_j \rightsquigarrow \sum_k c_{ijk}\, b_k. \end{cases}$$

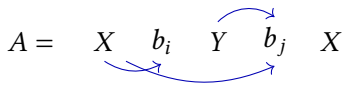
In words: move X and Y to the right past elements of R, replace YX and XY by elements of R, and multiply elements of R.

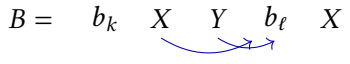
The order. For a word $A \in \langle\mathcal{X}\rangle_{\mathrm{mon}}$, let $\ell(A)$ be its length, let $N_X(A)$ and $N_Y(A)$ be the numbers of occurrences of X and Y in A, and let the *misordering index* $m(A)$ be the number of pairs of positions $p < q$ in A such that the letter at position p is X or Y and the letter at position q is one of the b_i.

Definition 11.2. For words $A, B \in \langle\mathcal{X}\rangle_{\mathrm{mon}}$, put $A < B$ if either

1) $\ell(A) < \ell(B)$, or
2) $\ell(A) = \ell(B)$, $N_X(A) = N_X(B)$, $N_Y(A) = N_Y(B)$ and $m(A) < m(B)$.

Example 11.3. For the words $A = Xb_iYb_jX$ and $B = b_kXYb_\ell X$ we have $N_X(A) = N_X(B) = 2$ and $N_Y(A) = N_Y(B) = 1$, and





so $m(A) = 3$ and $m(B) = 2$, and therefore $B < A$.

Lemma 11.4. *The relation $\leq$ of Definition 11.2 is a semigroup partial order on $\langle\mathcal{X}\rangle_{\mathrm{mon}}$. It is compatible with S and has DCC.*

Proof. It is clearly a partial order. For the semigroup property, let $A < B$ and let C, D be words. If $\ell(A) < \ell(B)$, then $\ell(CAD) < \ell(CBD)$. Otherwise A and B have the same length and the same numbers of X's, Y's and b's. Then $m(CAD)$ is $m(C) + m(A) + m(D)$ plus cross terms that only depend on these numbers, so $m(CAD) - m(CBD) = m(A) - m(B) < 0$. For compatibility, check each rule. The rules for YX, XY and b_ib_j shorten words. The rule $Xb_i \rightsquigarrow \sum_j s_{ij} b_j X$ either shortens the word (the term with $b_0 = 1$) or keeps the length and the number of X's and Y's while lowering m by one, and the same holds for Y. Finally, along a strictly decreasing chain the pair $(\ell, m) \in \mathbb{N} \times \mathbb{N}$ strictly decreases in the lexicographic order, so there is no infinite such chain. □

The ambiguities. All the words W_σ have length 2 and are distinct, so there are no inclusion ambiguities, and the overlap ambiguities are the words of length 3 in which both the first two and the last two letters can be reduced:

$$XYb_i, \quad YXb_i, \quad Xb_ib_j, \quad Yb_ib_j, \quad YXY, \quad XYX, \quad b_ib_jb_k.$$

Along the way we freely use that a product of two elements of R, written in the basis $\mathbb{B}$, reduces in one step to its value in R. We resolve three of the ambiguities here; the remaining four are Exercise 11.9.

The ambiguity XYb_i. On the one hand $(XY)b_i \rightsquigarrow \sigma(t)\, b_i$, which then reduces to the element $\sigma(t)b_i$ of R. On the other hand,

$$X(Yb_i) \rightsquigarrow X\sigma^{-1}(b_i)Y = \sum_j s'_{ij}\, Xb_jY \rightsquigarrow \sum_j s'_{ij}\, \sigma(b_j)XY = b_i\, XY \rightsquigarrow b_i\, \sigma(t),$$

where we used $\sum_j s'_{ij}\sigma(b_j) = \sigma(\sigma^{-1}(b_i)) = b_i$. Since t is central and σ is an automorphism, $\sigma(t)$ is central, so $b_i\sigma(t) = \sigma(t)b_i$, and both sides reduce to the same element of R.

The ambiguity Xb_ib_j. Reducing in the two possible ways,

$$(Xb_i)b_j \rightsquigarrow \sigma(b_i)Xb_j \rightsquigarrow \sigma(b_i)\sigma(b_j)X, \qquad X(b_ib_j) \rightsquigarrow \sum_k c_{ijk}\, Xb_k \rightsquigarrow \sum_k c_{ijk}\, \sigma(b_k)X = \sigma(b_ib_j)X.$$

These agree after multiplying out in R, because σ is multiplicative: $\sigma(b_i)\sigma(b_j) = \sigma(b_ib_j)$. (The term with $b_k = b_0 = 1$ is consistent, since $\sigma(1) = 1$.)

The ambiguity YXY.

$$(YX)Y \rightsquigarrow tY, \qquad Y(XY) \rightsquigarrow Y\sigma(t) \rightsquigarrow \sigma^{-1}(\sigma(t))Y = tY.$$

Theorem 11.5. *Let $A = R(\sigma, t)$ be a GWA, and let $\{b_i\}$ be a basis of R with $b_0 = 1$. Then*

$$\langle \mathcal{X} \rangle_{\mathrm{mon}_{\mathrm{irr}}} = \{b_i\}_i \cup \{b_iX^n\}_{i,\, n\geq 1} \cup \{b_iY^n\}_{i,\, n\geq 1}$$

is a $\Bbbk$-basis for A.

Proof. By Lemma 11.4 and Exercise 11.9, the Diamond Lemma applies. A word is irreducible exactly when it contains none of $Xb_i, Yb_i, YX, XY, b_ib_j$ as a subword. So it contains at most one b_i, which must come first, followed by a word in X and Y with no XY or YX, i.e. by a power of X or a power of Y. □

Corollary 11.6. *$R(\sigma, t)$ is a free left R-module with basis $\{1\} \cup \{X^n\}_{n\geq 1} \cup \{Y^n\}_{n\geq 1}$. In particular:*

1) $R \to R(\sigma, t)$ is injective, so $R(\sigma, t)$ is an R-ring;
2) $R(\sigma, t) \neq 0$ (as was used in the proof of Proposition 5.10);
3) $R(\sigma, t) = \bigoplus_{n\in\mathbb{Z}} A_n$ with A_n as in Example 6.8. This gives the directness of the sum in Exercise 6.11.

11.3 Enveloping algebras and the Jacobi identity

Example 11.7. Let

$$A = \mathbb{C}\left\langle x, y, z \;\middle|\; \begin{array}{l} xy - yx = z \\ yz - zy = x \\ zx - xz = y \end{array} \right\rangle.$$

This is the enveloping algebra $U(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ with basis x, y, z and brackets

$$[x, y] = z, \qquad [y, z] = x, \qquad [z, x] = y,$$

that is, $\mathbb{C}^3$ with the cross product, which is $\mathfrak{so}_3(\mathbb{C})$. (As for $U(\mathfrak{sl}_2)$ in Chapter 2, it suffices to impose the defining relations of $U(\mathfrak{g})$ on pairs of basis elements.)

Use the deg-lex order with $x < y < z$, and orient the relations as

$$yx \rightsquigarrow xy + [y, x], \qquad zy \rightsquigarrow yz + [z, y], \qquad zx \rightsquigarrow xz + [z, x],$$

where $[a, b]$ now stands for the value of the bracket in $\mathfrak{g}$: $[y, x] = -z$, $[z, y] = -x$, $[z, x] = y$. The only ambiguity is zyx. Reducing it in the two possible ways gives

$$\begin{array}{ccc} & zyx & \\ \swarrow & & \searrow \\ (yz + [z, y])x & & z(xy + [y, x]) \\ = yzx + [z, y]x & & = zxy + z[y, x] \\ \downarrow & & \downarrow \\ yxz + y[z, x] + [z, y]x & & xzy + [z, x]y + z[y, x] \\ \downarrow & & \downarrow \\ xyz + [y, x]z + y[z, x] + [z, y]x & & xyz + x[z, y] + [z, x]y + z[y, x] \end{array}$$

Inserting the values of the brackets, both sides become

$$xyz - z^2 + y^2 - x^2,$$

which is irreducible. So the ambiguity resolves.

The general pattern is visible in the last line. Subtracting the two sides, the difference is

$$\big([y, x]z - z[y, x]\big) + \big(y[z, x] - [z, x]y\big) + \big([z, y]x - x[z, y]\big),$$

a sum of commutators of elements of $\mathfrak{g}$. One more round of reductions turns each such commutator $ab - ba$ into the bracket $[a, b]$ of $\mathfrak{g}$. So the ambiguity resolves if and only if

$$[z, [y, x]] + [[z, x], y] + [x, [z, y]] = 0 \quad \text{in } \mathfrak{g},$$

which is the *Jacobi identity* for x, y, z. The same computation works for an arbitrary Lie algebra $\mathfrak{g}$ with an ordered basis: every ambiguity $e_k e_j e_i$ with $k > j > i$ resolves precisely because of the Jacobi identity. This proves the Poincaré–Birkhoff–Witt theorem: the ordered monomials in a basis of $\mathfrak{g}$ form a basis of $U(\mathfrak{g})$. See [Bergman78, §3].

Corollary 11.8. *The ordered monomials*

$$\left\{ x^a y^b z^c \;\middle|\; a, b, c \geq 0 \right\}$$

form a basis of $U(\mathfrak{so}_3(\mathbb{C}))$.

11.4 Exercises

Exercise 11.9. Resolve the remaining four ambiguities YXb_i, Yb_ib_j, XYX and $b_ib_jb_k$ of the reduction system for $R(\sigma, t)$. Which properties of R, σ and t does each one use?

Exercise 11.10. Find a basis for the rational Cherednik algebra $\mathbb{H}$ of Definition 9.1.

Part IV

Group actions, tensor products and Morita theory

Chapter 12

Group-Graded Algebras

Many algebras come with a grading by a group. We start with group algebras, then let the group act on the coefficients (skew group algebras), and finally also twist the multiplication of the group elements by a cocycle (crossed products). A theorem characterizes crossed products among all group-graded algebras: they are exactly those with an invertible element in every degree.

12.1 Group gradings

Let A be a $\Bbbk$-algebra and G a group with identity element e. A *G-gradation* of A is an M-gradation in the sense of Definition 6.6 with $M = G$, that is, a family of subspaces $\{A_g\}_{g\in G}$ with

$$A = \bigoplus_{g\in G} A_g, \qquad 1_A \in A_e, \qquad A_g A_h \subseteq A_{gh}.$$

The subalgebra A_e is called the *identity component*.

Example 12.1 (Group algebras)**.** The *group algebra*

$$\Bbbk G = \left\{ \sum_{g\in G} \lambda_g\, g \;\middle|\; \lambda_g \in \Bbbk,\ \text{almost all } \lambda_g = 0 \right\},$$

with multiplication extending that of G bilinearly, is G-graded with $(\Bbbk G)_g = \Bbbk g$. Indeed, $1_{\Bbbk G} = 1_\Bbbk \cdot e = e \in \Bbbk e = (\Bbbk G)_e$, and

$$(\Bbbk G)_g (\Bbbk G)_h = \operatorname{span}\{\, \alpha g \cdot \beta h \mid \alpha, \beta \in \Bbbk \,\} = \Bbbk\, gh.$$

Example 12.2 (Group algebras over R)**.** Let R be any $\Bbbk$-algebra. The *group algebra over R* (or *group ring over R*) is

$$RG = \left\{ \sum_{g\in G} r_g\, g \;\middle|\; r_g \in R,\ \text{almost all } r_g = 0 \right\},$$

with multiplication $(rg)(sh) = (rs)(gh)$, extended bi-additively. It is G-graded with $(RG)_g = Rg$. In RG the elements of G commute with those of R.

12.2 Skew group algebras

Now let the group act on R. Let R be a $\Bbbk$-algebra and

$$\sigma : G \longrightarrow \mathrm{Aut}_{\Bbbk\text{-Alg}}(R), \qquad g \mapsto \sigma_g,$$

a group homomorphism.

Definition 12.3. The *skew group algebra* is

$$R \rtimes_\sigma G = \left\{ \sum_{g\in G} r_g\, g \;\middle|\; r_g \in R, \text{ almost all } r_g = 0 \right\},$$

with multiplication

$$(rg)(sh) = \big(r\,\sigma_g(s)\big)\, gh,$$

extended bi-additively (= $\mathbb{Z}$-bilinearly). In words: to move g past s, replace s by $\sigma_g(s)$.

Remark 12.4. Several other notations are in use for the same algebra: $R *_\sigma G$, $R * G$, $R\#G$ and $R\#\Bbbk G$. We write $R \rtimes_\sigma G$, or $R \rtimes G$ when σ is understood.

For general elements, the multiplication reads

$$\Big(\sum_{g\in G} r_g\, g\Big)\Big(\sum_{h\in G} s_h\, h\Big) = \sum_{g,h\in G} r_g\,\sigma_g(s_h)\, gh = \sum_{k\in G}\Bigg(\sum_{\substack{g,h\in G\\ gh=k}} r_g\,\sigma_g(s_h)\Bigg)k.$$

The multiplication is associative. On homogeneous elements,

$$\begin{aligned}(rg)\big((sh)(tk)\big) &= (rg)\big(s\,\sigma_h(t)\,hk\big) = r\,\sigma_g(s)\,\sigma_g\sigma_h(t)\,ghk,\\ \big((rg)(sh)\big)(tk) &= \big(r\,\sigma_g(s)\,gh\big)(tk) = r\,\sigma_g(s)\,\sigma_{gh}(t)\,ghk,\end{aligned}$$

using that σ_g is multiplicative. These agree because σ is a homomorphism: $\sigma_g\sigma_h = \sigma_{gh}$. The unit is $1_R\, e$, and $R \rtimes_\sigma G$ is G-graded with $(R \rtimes_\sigma G)_g = Rg$.

Alternatively: by a presentation. We can also obtain $R \rtimes_\sigma G$ from R by adjoining the elements of G as new symbols (Chapter 3), subject to the relations

$$g \cdot h = (gh)_G, \qquad e = 1, \qquad g\,r = \sigma_g(r)\,g \qquad (g, h \in G,\ r \in R),$$

where $(gh)_G$ is the product in G. The Diamond Lemma shows that this algebra is spanned freely by the elements rg (Exercise 12.10), so it agrees with the algebra of Definition 12.3, and

$$\begin{aligned}R \rtimes_\sigma G &= \bigoplus_{g\in G} Rg \quad \text{as left } R\text{-modules}\\ &= \bigoplus_{g\in G} gR \quad \text{as right } R\text{-modules},\end{aligned}$$

the second because $gR = \sigma_g(R)\,g = Rg$.

The multiplication rule and the commutation relation say the same thing.

Proposition 12.5. *Let B be an algebra containing R as a unital subalgebra and containing elements $g \in B$, one for each $g \in G$, that multiply as in G and with $e = 1_R$. The following are equivalent:*

1) $gr = \sigma_g(r)\, g$ for all $g \in G$ and $r \in R$;
2) $(rg)(sh) = \big(r\, \sigma_g(s)\big)\, gh$ for all $g, h \in G$ and $r, s \in R$.

Proof. 1) $\Rightarrow$ 2): by associativity and 1),

$$(rg)(sh) = r\,(gs)\,h = r\,\big(\sigma_g(s)\, g\big)\, h = \big(r\, \sigma_g(s)\big)\, gh.$$

2) $\Rightarrow$ 1): take $r = 1_R$ and $h = e$ in 2). This gives $gs = \sigma_g(s)\, g$. □

12.3 An example: polynomials and the symmetric group

Example 12.6. Let

$$R = \mathbb{C}[x_1, x_2, \dots, x_n],$$

the polynomial algebra (the free commutative algebra on $\{x_1, \dots, x_n\}$), and let $G = S_n$. For a permutation $\pi \in S_n$, the map $x_i \mapsto x_{\pi(i)}$ on generators extends to a $\mathbb{C}$-algebra endomorphism φ_π of R. Since $\varphi_\pi \varphi_{\pi^{-1}} = \mathrm{Id}_R$, each φ_π is an automorphism, and

$$\varphi : S_n \longrightarrow \mathrm{Aut}_{\mathbb{C}}(R), \qquad \pi \longmapsto \varphi_\pi,$$

is a group homomorphism, since $\varphi_\pi \varphi_\rho(x_i) = x_{\pi\rho(i)} = \varphi_{\pi\rho}(x_i)$. We get the skew group algebra

$$R \rtimes_\varphi S_n = \bigoplus_{\pi \in S_n} R\pi,$$

in which, for example,

$$(12) \cdot (x_1 + x_2^2) = (x_2 + x_1^2) \cdot (12).$$

A preview. The *algebra of invariants*

$$R^{S_n} = \{\, r \in R \mid \varphi_\pi(r) = r \text{ for all } \pi \in S_n \,\}$$

is the algebra of symmetric polynomials. For example, when $n = 3$, $x_1^2 x_2 x_3 + x_1 x_2^2 x_3 + x_1 x_2 x_3^2 \in R^{S_3}$. It is closely related to the skew group algebra. Let $\varepsilon = \frac{1}{n!} \sum_{\pi \in S_n} \pi \in R \rtimes S_n$. Then ε is an idempotent, and

$$R^{S_n} \cong \varepsilon\,(R \rtimes S_n)\,\varepsilon, \qquad f \mapsto f\varepsilon.$$

In Chapter 17 we will see that such a "corner" $\varepsilon A \varepsilon$ is Morita equivalent to A when $A\varepsilon A = A$. That condition fails here, and indeed $R \rtimes S_n$ and R^{S_n} are *not* Morita equivalent for $n \geq 2$. Every permutation fixes the origin of $\mathbb{C}^n$, and $R \rtimes S_n$ has one simple module on which all x_i act by zero for each irreducible representation of S_n, whereas R^{S_n} has only one simple module on which all symmetric polynomials without constant term act by zero.

For more on the relation between skew group rings and invariants, see Cohen [Cohen82].

12.4 Crossed products

In a skew group algebra the elements g multiply exactly as in G. In a crossed product we allow the product of two group elements to pick up a unit of R.

Let R be a $\Bbbk$-algebra and G a group, and suppose we are given maps

$$\sigma : G \longrightarrow \mathrm{Aut}_{\Bbbk\text{-Alg}}(R), \qquad \alpha : G \times G \longrightarrow R^{\times},$$

where $R^{\times}$ is the group of units (invertible elements) of R. Note that σ is *not* assumed to be a homomorphism. Let $\{u_g\}_{g\in G}$ be a set of symbols indexed by G, and define a product on

$$R *_{\sigma,\alpha} G = \bigoplus_{g\in G} R\,u_g = \left\{ \sum_{g\in G} r_g\,u_g \;\middle|\; r_g \in R,\ \text{almost all } r_g = 0 \right\}$$

by

$$(ru_g)(su_h) = r\,\sigma_g(s)\,\alpha(g,h)\,u_{gh}, \qquad \text{so that} \qquad \boxed{u_g u_h = \alpha(g,h)\,u_{gh}}, \quad u_g s = \sigma_g(s) u_g.$$

What do σ and α need to satisfy for this to be an associative algebra with unit u_e? Computing $u_g u_h u_k$ in two ways,

$$\begin{aligned} u_g(u_h u_k) &= u_g\,\alpha(h,k)\,u_{hk} = \sigma_g\big(\alpha(h,k)\big)\,u_g u_{hk} = \sigma_g\big(\alpha(h,k)\big)\,\alpha(g,hk)\,u_{ghk}, \\ (u_g u_h)u_k &= \alpha(g,h)\,u_{gh}u_k = \alpha(g,h)\,\alpha(gh,k)\,u_{ghk}. \end{aligned}$$

Similarly, computing $u_g u_h r$ in two ways gives $\sigma_g\sigma_h(r)\,\alpha(g,h) = \alpha(g,h)\,\sigma_{gh}(r)$, and u_e is a unit exactly when $\sigma_e = \mathrm{id}$ and $\alpha(e,g) = \alpha(g,e) = 1$.

Definition 12.7. The pair (σ,α) is *crossed product data* if, for all $g,h,k \in G$,

(C1) $\sigma_e = \mathrm{id}_R$ and $\alpha(e,g) = \alpha(g,e) = 1$;
(C2) $\sigma_g\sigma_h = \mathrm{Ad}_{\alpha(g,h)} \circ \sigma_{gh}$, where $\mathrm{Ad}_a(r) = ara^{-1}$;
(C3) $\boxed{\sigma_g\big(\alpha(h,k)\big)\,\alpha(g,hk) = \alpha(g,h)\,\alpha(gh,k)}$ (the *σ-twisted cocycle identity*).

In that case $R *_{\sigma,\alpha} G$ is an associative algebra with unit u_e (Exercise 12.11), called a *crossed product*. It is G-graded, with $(R *_{\sigma,\alpha} G)_g = Ru_g$ and identity component $R = Ru_e$.

If $\alpha \equiv 1$, then (C2) says that σ is a homomorphism, and the crossed product is the skew group algebra $R \rtimes_\sigma G$. If moreover σ is trivial, it is the group algebra RG.

Lemma 12.8. *Let A be G-graded, and let $u \in A_g$ be invertible. Then $u^{-1} \in A_{g^{-1}}$.*

Proof. Write $u^{-1} = \sum_h v_h$ with $v_h \in A_h$. Then $1 = uu^{-1} = \sum_h uv_h$ with $uv_h \in A_{gh}$. Since $1 \in A_e$, we get $uv_h = 0$ for $h \neq g^{-1}$. Multiplying by u^{-1} gives $v_h = 0$ for $h \neq g^{-1}$, so $u^{-1} = v_{g^{-1}}$. □

Theorem 12.9. *The following are equivalent for a G-graded algebra $A = \bigoplus_{g\in G} A_g$:*

1) $A \cong A_e *_{\sigma,\alpha} G$ as G-graded algebras, for some crossed product data (σ,α);
2) for every $g \in G$ there is a unit $u_g \in A_g \cap A^{\times}$.

Proof (sketch). Assume 2), and choose $u_e = 1$. Since $u_g u_h \in A_{gh}$ and $u_{gh}^{-1} \in A_{(gh)^{-1}}$ by Lemma 12.8, we have

$$\alpha(g,h) := u_g u_h u_{gh}^{-1} \in (A_e)^\times, \qquad \text{so that} \qquad u_g u_h = \alpha(g,h)\, u_{gh}.$$

Here $\alpha(g,h)$ is a unit, and it lies in A_e because $u_g u_h \in A_{gh}$ and $u_{gh}^{-1} \in A_{(gh)^{-1}}$; its inverse $u_{gh} u_h^{-1} u_g^{-1}$ lies in A_e for the same reason. Similarly, $\sigma_g(r) := u_g\, r\, u_g^{-1}$ maps A_e into $A_g A_e A_{g^{-1}} \subseteq A_e$, and so does its inverse $r \mapsto u_g^{-1} r u_g$. So σ_g is an automorphism of A_e, with $u_g r = \sigma_g(r) u_g$.

Now (C2) comes from computing $u_g u_h r$ in two ways, for $r \in A_e$:

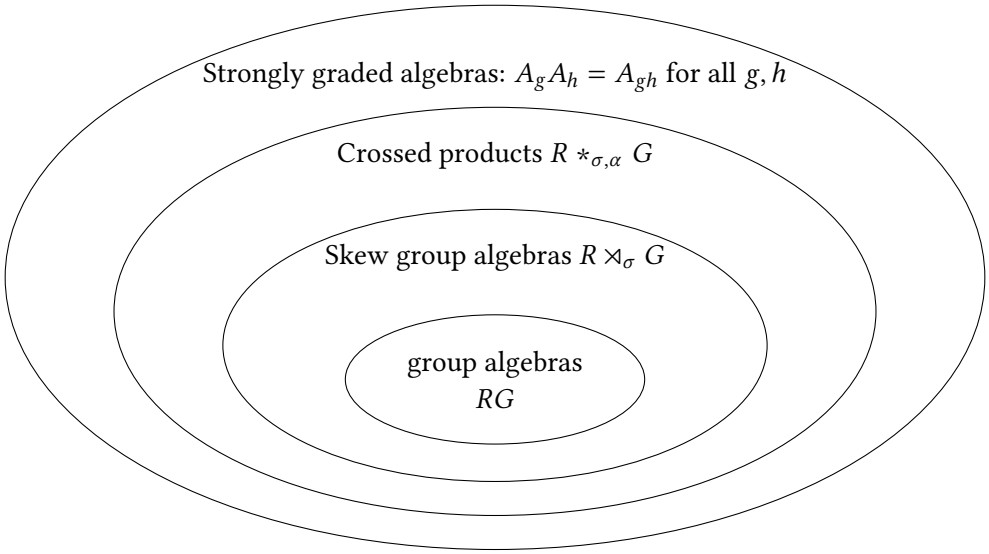
$$\begin{aligned} u_g u_h r &= u_g\, \sigma_h(r)\, u_h = \sigma_g\sigma_h(r)\, u_g u_h, \\ u_g u_h r &= \alpha(g,h)\, u_{gh}\, r = \alpha(g,h)\, \sigma_{gh}(r)\, u_{gh} = \alpha(g,h)\, \sigma_{gh}(r)\, \alpha(g,h)^{-1} \underbrace{\alpha(g,h)\, u_{gh}}_{=u_g u_h}. \end{aligned}$$

Since $u_g u_h$ is invertible, we can cancel it:

$$\boxed{\sigma_g\sigma_h(r) = \alpha(g,h)\, \sigma_{gh}(r)\, \alpha(g,h)^{-1}},$$

which is (C2). Finishing the proof is Exercise 12.12. □

A G-graded algebra is *strongly graded* if $A_g A_h = A_{gh}$ for all $g, h \in G$. The classes of algebras in this chapter fit together as follows:



(Exercise 12.13 shows that crossed products are strongly graded.)

12.5 Exercises

Exercise 12.10. Use the Diamond Lemma to show that the algebra obtained from R by adjoining the elements of G, subject to $g \cdot h = (gh)_G$, $e = 1$ and $gr = \sigma_g(r)g$, has the elements $b_i\, g$ as a basis, where $\{b_i\}$ is a basis of R. Conclude that it is isomorphic to $R \rtimes_\sigma G$. (Model your argument on the GWA example in Chapter 11: take $b_0 = 1$, use the letters $g \neq e$, and list the ambiguities ghk, ghb_i, gb_ib_j, $b_ib_jb_k$.)

Exercise 12.11. Show that conditions (C1)–(C3) imply that $R *_{\sigma,\alpha} G$ is associative with unit u_e.

Exercise 12.12. Finish the proof of Theorem 12.9. Show that $A_g = A_e u_g$, that the (σ, α) defined in the proof satisfy (C1) and (C3), and that $A \cong A_e *_{\sigma,\alpha} G$. Then prove 1) $\Rightarrow$ 2).

Exercise 12.13. Show that crossed products are strongly graded.

Chapter 13

Tensor Products I

We have been writing multiplications as maps $A \otimes A \to A$ since Chapter 1. Now we set up tensor products of vector spaces properly. We give a concrete definition using bases, then the universal property, which characterizes the tensor product up to unique isomorphism, and a construction that needs no bases. We end with the tensor–Hom adjunction.

13.1 A concrete definition

Let V and W be vector spaces over $\mathbb{k}$, and pick bases $\{v_i\}_{i\in I}$ of V and $\{w_j\}_{j\in J}$ of W.

Definition 13.1 (First version)**.** The *tensor product* $V \otimes W = V \otimes_{\mathbb{k}} W$ is the vector space with basis $\{v_i \otimes w_j\}_{(i,j)\in I\times J}$. For $v = \sum_i \lambda_i v_i \in V$ and $w = \sum_j \mu_j w_j \in W$ we define

$$v \otimes w = \sum_{(i,j)\in I\times J} \lambda_i \mu_j \, v_i \otimes w_j.$$

For vector spaces V, W, U, let $\mathrm{Bil}(V \times W, U)$ denote the set (in fact the vector space) of bilinear maps $V \times W \to U$.

Proposition 13.2. *1) The map*

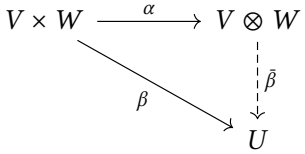
$$\alpha : \ V \times W \longrightarrow V \otimes W, \qquad (v, w) \longmapsto v \otimes w,$$

is bilinear.

2) If U is any vector space and $\beta : \ V \times W \to U$ is bilinear, then the linear map

$$\bar{\beta} : \ V \otimes W \longrightarrow U, \qquad \bar{\beta}(v_i \otimes w_j) = \beta(v_i, w_j),$$

makes the diagram



commute, and it is the only linear map that does so.

Consequently there is a bijection

$$\mathrm{Bil}(V \times W, U) \longrightarrow \mathrm{Hom}_{\Bbbk}(V \otimes W, U), \qquad \beta \longmapsto \bar{\beta},$$

with inverse $\gamma \mapsto \gamma \circ \alpha$.

Proof. 1) The coefficients $\lambda_i \mu_j$ of $v \otimes w$ are linear in (λ_i) for fixed (μ_j), and vice versa.
2) For $v = \sum_i \lambda_i v_i$ and $w = \sum_j \mu_j w_j$, bilinearity of β gives

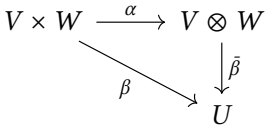
$$\bar{\beta} \circ \alpha(v, w) = \sum_{i,j} \lambda_i \mu_j \, \bar{\beta}(v_i \otimes w_j) = \sum_{i,j} \lambda_i \mu_j \, \beta(v_i, w_j) = \beta(v, w).$$

Uniqueness holds because any linear map making the diagram commute must send the basis vector $v_i \otimes w_j = \alpha(v_i, w_j)$ to $\beta(v_i, w_j)$. □

13.2 The universal property

Part 2) of Proposition 13.2 does not mention bases, and it is the property we really want.

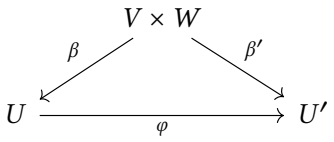
Definition 13.3 (Second version)**.** Let V and W be vector spaces. A *tensor product* of V and W is a pair $(V \otimes W, \alpha)$, where $V \otimes W$ is a vector space and $\alpha : V \times W \to V \otimes W$ is a bilinear map, *such that*, for every pair (U, β) of a vector space U and a bilinear map $\beta : V \times W \to U$, there is a unique linear map $\bar{\beta} : V \otimes W \to U$ making the diagram



commute.

Remark 13.4 (Tensor products as initial objects). Fix V and W, and consider the category $\mathcal{B}(V, W)$ whose

- objects are pairs (U, β), with U a vector space and $\beta : V \times W \to U$ bilinear, and
- morphisms $\varphi : (U, \beta) \to (U', \beta')$ are linear maps $\varphi : U \to U'$ making the diagram



commute.

Then a tensor product $(V \otimes W, \alpha)$ is exactly an *initial object* of $\mathcal{B}(V, W)$: an object with exactly one morphism to every other object.

Initial objects are unique up to unique isomorphism. If (T, α) and (T', α') are both initial, the unique morphisms $T \to T'$ and $T' \to T$ compose to morphisms $T \to T$ and $T' \to T'$, which must be the identities by uniqueness. So *the tensor product is unique up to unique isomorphism.* In particular, the first version (Definition 13.1), which satisfies the universal property by Proposition 13.2, does not depend on the choice of bases, up to unique isomorphism.

Existence without bases. Let F be the vector space with basis $V \times W$. (So the pair (v, w) is now also a basis vector of F; this double use is standard.) Let

$$S = \operatorname{span}_{\Bbbk} \left\{ \begin{array}{l} (\lambda_1 v_1 + \lambda_2 v_2, w) - \big(\lambda_1 (v_1, w) + \lambda_2 (v_2, w)\big), \\ (v, \lambda_1 w_1 + \lambda_2 w_2) - \big(\lambda_1 (v, w_1) + \lambda_2 (v, w_2)\big) \end{array} \middle| \begin{array}{c} v, v_1, v_2 \in V, \ \ w, w_1, w_2 \in W, \\ \lambda_1, \lambda_2 \in \Bbbk \end{array} \right\},$$

and define

$$V \otimes W = F/S, \qquad \alpha : \ V \times W \to V \otimes W, \quad (v, w) \mapsto (v, w) + S.$$

The elements of S are exactly what is needed to make α bilinear.

Proposition 13.5. *$(F/S, \alpha)$ is a tensor product of V and W.*

Proof. Let $\beta : \ V \times W \to U$ be bilinear. Being a function on the basis $V \times W$ of F, it extends to a linear map $\hat{\beta} : \ F \to U$. Since β is bilinear, $S \subseteq \operatorname{Ker} \hat{\beta}$, so $\hat{\beta}$ induces a linear map $\bar{\beta} : \ V \otimes W \to U$ with $\bar{\beta} \circ \alpha = \beta$. It is unique because the elements $\alpha(v, w)$ span F/S. □

13.3 Examples

Example 13.6. Let A be an algebra, with multiplication map

$$m : \ A \times A \to A, \qquad (a, b) \mapsto ab.$$

Since m is bilinear, it induces a *linear* map

$$\bar{m} : \ A \otimes A \longrightarrow A.$$

Conversely, any linear map $A \otimes A \to A$ pulls back along $\alpha : \ A \times A \to A \otimes A$ to a bilinear map $A \times A \to A$. This is the bijection we used in Chapter 1:

$$\boxed{\operatorname{Bil}(V \times W, U) \cong \operatorname{Hom}_{\Bbbk}(V \otimes W, U)}$$

Example 13.7. Compare direct sums and tensor products:

$$\Bbbk^n \oplus \Bbbk^m \cong \Bbbk^{n+m}, \qquad \Bbbk^n \otimes_{\Bbbk} \Bbbk^m \cong \Bbbk^{nm}.$$

More generally, $\dim(V \otimes_{\Bbbk} W) = (\dim V)(\dim W)$.

13.4 The tensor–Hom adjunction

A bilinear map can be viewed as a linear map with values in linear maps:

$$\operatorname{Bil}(V \times W, U) \cong \operatorname{Hom}_{\Bbbk}\big(V, \operatorname{Hom}_{\Bbbk}(W, U)\big), \qquad \Phi \longmapsto \big(v \mapsto (w \mapsto \Phi(v, w))\big).$$

Combining this with the universal property,

$$\boxed{\operatorname{Hom}_{\Bbbk}\big(V, \operatorname{Hom}_{\Bbbk}(W, U)\big) \cong \operatorname{Hom}_{\Bbbk}(V \otimes W, U)}$$

for all V and U (and these bijections are natural). In the language of Definition 5.1: $- \otimes W$ *is left adjoint to* $\mathrm{Hom}_{\Bbbk}(W, -)$,

$$- \otimes W \dashv \mathrm{Hom}_{\Bbbk}(W, -), \qquad \underline{\mathrm{Vect}} \underset{\mathrm{Hom}_{\Bbbk}(W,-)}{\overset{-\otimes W}{\perp}} \underline{\mathrm{Vect}}$$

This is called $(\otimes, \mathrm{Hom})$-*adjointness*. A simple but useful special case of the definitions is $\Bbbk \otimes_{\Bbbk} V \cong V$, via $\lambda \otimes v \mapsto \lambda v$.

Remark 13.8 (Modules again). Let A be an algebra. In Definition 1.3 we wrote the action on an A-module V as a linear map

$$A \otimes_{\Bbbk} V \longrightarrow V, \qquad a \otimes v \longmapsto a.v,$$

satisfying $(ab).v = a.(b.v)$ and $1.v = v$. The tensor product here is over $\Bbbk$. In the next chapter we define tensor products over an algebra, which let us build new modules out of old ones.

Remark 13.9 (Bonus: monoids in monoidal categories). A *monoidal category* is, roughly, a category with a "product" of objects that is associative and has a unit object, up to coherent isomorphism. A *monoid* in it is an object M with an associative multiplication $M \otimes M \to M$ and a unit $\mathbf{1} \to M$. Some examples:

- a monoid in $(\underline{\mathrm{Set}}, \times, *)$, where $*$ is a one-point set, is an ordinary monoid;
- a monoid in $(\underline{\mathrm{Vect}}, \otimes_{\Bbbk}, \Bbbk)$ is a $\Bbbk$-algebra;
- if H is a Hopf algebra, a monoid in $({}_H\underline{\mathrm{Mod}}, \otimes_{\Bbbk}, \Bbbk)$ is an H-*module algebra*: an algebra on which H acts in such a way that the multiplication and the unit are H-module maps.

For Hopf algebras and their actions, see Montgomery's book [Montgomery93].

13.5 Exercises

Exercise 13.10. Identify $\Bbbk^n \otimes \Bbbk^m$ with the space of $n \times m$ matrices so that $v \otimes w \mapsto vw^T$. Show that the elements of the form $v \otimes w$ correspond exactly to the matrices of rank at most 1. Deduce that $v \otimes w = 0$ only if $v = 0$ or $w = 0$, and that not every element of $V \otimes W$ has the form $v \otimes w$.

Exercise 13.11. Using only the universal property (Definition 13.3), construct natural isomorphisms

$$\Bbbk \otimes V \cong V, \qquad V \otimes W \cong W \otimes V, \qquad (U \otimes V) \otimes W \cong U \otimes (V \otimes W).$$

Chapter 14

Tensor Products II

We extend tensor products from vector spaces to bimodules. Given an (A, B)-bimodule X and a (B, C)-bimodule Y, the tensor product $X \otimes_B Y$ is an (A, C)-bimodule, characterized by a universal property involving "balanced" maps. The tensor–Hom adjunction carries over. As an application we get induced modules and Frobenius reciprocity.

14.1 Balanced maps

Let A, B, C be $\Bbbk$-algebras. (Everything in this chapter works equally well when $\Bbbk$ is any commutative ring, for example $\mathbb{Z}$, in which case the algebras are just rings.) Recall that an (A, B)-*bimodule* is a vector space X with a left A-action and a right B-action that commute, $(ax)b = a(xb)$. We write ${}_A\underline{\mathrm{Mod}}_B$ for the category of (A, B)-bimodules, ${}_A\underline{\mathrm{Mod}}$ for left A-modules and $\underline{\mathrm{Mod}}_B$ for right B-modules. Fix

$$X = {}_A X_B \in {}_A\underline{\mathrm{Mod}}_B, \qquad Y = {}_B Y_C \in {}_B\underline{\mathrm{Mod}}_C, \qquad Z = {}_A Z_C \in {}_A\underline{\mathrm{Mod}}_C.$$

Definition 14.1. A map $f : X \times Y \to Z$ is *balanced* if

1) $f(a_1 x_1 + a_2 x_2,\ y) = a_1 f(x_1, y) + a_2 f(x_2, y)$,
2) $f(x,\ y_1 c_1 + y_2 c_2) = f(x, y_1)c_1 + f(x, y_2)c_2$, and
3) $f(xb, y) = f(x, by)$ (the *B-balancing condition*)

for all $a_i \in A$, $c_i \in C$, $b \in B$, $x, x_i \in X$, $y, y_i \in Y$. We write $\mathrm{Bal}(X \times Y, Z)$ for the $\Bbbk$-module of balanced maps.

Note that $\mathrm{Bal}(X \times Y, Z)$ is not an (A, C)-bimodule in general. The natural candidate would be $(afc)(x, y) = a\, f(x, y)\, c$. This map is right C-linear in y, but it is not left A-linear in x: condition 1) would require

$$(afc)(a_1 x, y) = a_1\,(afc)(x, y) = a_1 a\, f(x, y)\, c,$$

whereas in fact $(afc)(a_1 x, y) = a\, f(a_1 x, y)\, c = a a_1\, f(x, y)\, c$. These differ unless a and a_1 commute (in their action on Z). This is why we only ask for a $\Bbbk$-module.

Remark 14.2. 1) If $f : R \to S$ is a morphism of $\Bbbk$-algebras, then S is an (R, S)-bimodule via $r \cdot x \cdot s = f(r)xs$, and an (S, R)-bimodule via $s \cdot x \cdot r = sxf(r)$.
2) ${}_A\underline{\mathrm{Mod}}_{\Bbbk} = {}_A\underline{\mathrm{Mod}}$: a right $\Bbbk$-action is no extra structure.

A morphism $g : Z \to Z'$ in ${}_A\underline{\mathrm{Mod}}_C$ gives a $\Bbbk$-linear map $\mathrm{Bal}(X\times Y, Z) \to \mathrm{Bal}(X\times Y, Z')$, $f \mapsto g\circ f$. So ${}_AX_B$ and ${}_BY_C$ give a covariant functor

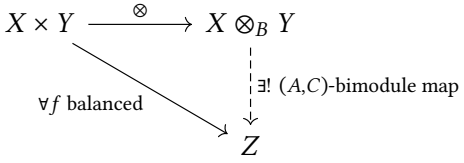
$$ {}_A\underline{\mathrm{Mod}}_C \longrightarrow {}_{\Bbbk}\underline{\mathrm{Mod}}, \qquad Z \longmapsto \mathrm{Bal}(X\times Y, Z). $$

14.2 The tensor product

Definition 14.3. The *tensor product* of ${}_AX_B$ and ${}_BY_C$ is an object $X\otimes_B Y \in {}_A\underline{\mathrm{Mod}}_C$ representing this functor. That is, there are isomorphisms, natural in Z,

$$ \mathrm{Hom}_{{}_A\underline{\mathrm{Mod}}_C}\big(X\otimes_B Y, Z\big) \cong \mathrm{Bal}(X\times Y, Z). $$

Equivalently, there is a balanced map $X\times Y \to X\otimes_B Y$, $(x,y)\mapsto x\otimes y$ (the image of the identity under the isomorphism), through which every balanced map factors uniquely:



Exactly as in Chapter 13, this implies that $X\otimes_B Y$ is unique up to unique isomorphism, if it exists.

Existence. The free (A,C)-bimodule on the set $X\times Y$ is

$$ F = \bigoplus_{(x,y)\in X\times Y} A\cdot(x,y)\cdot C, $$

where each summand is a copy of $A\otimes_{\Bbbk} C$, with the generator (x,y) corresponding to $1\otimes 1$. Define

$$ X\otimes_B Y = F/N, $$

where N is the sub-bimodule generated by the elements

$$ (xb,y)-(x,by), \qquad (a_1x_1+a_2x_2,y)-a_1(x_1,y)-a_2(x_2,y), \qquad (x,y_1c_1+y_2c_2)-(x,y_1)c_1-(x,y_2)c_2, $$

for all choices of the letters. We write $x\otimes y$ for the image of (x,y) in $X\otimes_B Y$. The relations say precisely that $(x,y)\mapsto x\otimes y$ is balanced.

Proposition 14.4. *$X\otimes_B Y = F/N$ is a tensor product of X and Y.*

Proof. Let $f : X\times Y \to Z$ be balanced. Since F is free, f extends to a bimodule map $\hat{f} : F\to Z$. Since f is balanced, $\hat{f}$ kills the generators of N, so it induces a bimodule map $\bar{f} : X\otimes_B Y \to Z$ with $\bar{f}(x\otimes y) = f(x,y)$. It is unique because the elements $x\otimes y$ generate $X\otimes_B Y$ as a bimodule. □

So, informally, $X \otimes_B Y$ consists of sums of elements $x \otimes y$, subject only to the rule that $(x, y) \mapsto x \otimes y$ is balanced. In fact the elements $x \otimes y$ even span $X \otimes_B Y$ over $\Bbbk$, since $a(x \otimes y)c = ax \otimes yc$. Equivalently,

$$X \otimes_B Y \cong (X \otimes_{\Bbbk} Y) \Big/ \operatorname{span}_{\Bbbk} \{ xb \otimes y - x \otimes by \mid x \in X,\ b \in B,\ y \in Y \},$$

with $a(x \otimes y)c = ax \otimes yc$. This description is often the handiest for computations.

Example 14.5 (Over $\mathbb{Z}$)**.** Take $\Bbbk = \mathbb{Z}$ and $A = B = C = \mathbb{Z}$. Then

$$\mathbb{Z}/2\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/3\mathbb{Z} = 0.$$

Indeed, the tensor product is generated by $1 \otimes 1$, and since $1 = 3$ in $\mathbb{Z}/2\mathbb{Z}$ and $3 = 0$ in $\mathbb{Z}/3\mathbb{Z}$,

$$1 \otimes 1 = (1 \cdot 3) \otimes 1 = 1 \otimes (3 \cdot 1) = 1 \otimes 0 = 0.$$

14.3 Properties

Proposition 14.6. *1) (Unit) For ${}_BY_C$ there is an isomorphism of (B, C)-bimodules*

$${}_BB_B \otimes_B {}_BY_C \cong {}_BY_C, \qquad b \otimes y \mapsto by,$$

and similarly ${}_AX_B \otimes_B {}_BB_B \cong {}_AX_B$, $x \otimes b \mapsto xb$.

2) (Associativity) For ${}_CW_D$ there is an isomorphism of (A, D)-bimodules

$${}_AX_B \otimes_B \left({}_BY_C \otimes_C {}_CW_D\right) \cong \left({}_AX_B \otimes_B {}_BY_C\right) \otimes_C {}_CW_D, \qquad x \otimes (y \otimes w) \mapsto (x \otimes y) \otimes w.$$

Proof. 1) The map $(b, y) \mapsto by$ is balanced, so it induces a bimodule map $B \otimes_B Y \to Y$. In the other direction, $y \mapsto 1 \otimes y$ is a bimodule map, since $b'(1 \otimes y)c = b' \otimes yc = 1 \otimes b'yc$. The two are inverse: $b \otimes y = 1 \otimes by$, and $1 \otimes y \mapsto y$. The other isomorphism is the same.

2) This is Exercise 14.9. □

14.4 The tensor–Hom adjunction

For a left A-module map $\varphi \in \operatorname{Hom}_A(X, Z) = \operatorname{Hom}_{{}_A\underline{\mathrm{Mod}}}(X, Z)$ and $b \in B$, $c \in C$, put

$$(b\varphi c)(x) = \varphi(xb)\, c.$$

This is again A-linear, and it makes $\operatorname{Hom}_A(X, Z)$ into a (B, C)-bimodule.

Theorem 14.7. *There is an isomorphism of $\Bbbk$-modules, natural in Y and Z,*

$$\operatorname{Bal}(X \times Y, Z) \cong \operatorname{Hom}_{{}_B\underline{\mathrm{Mod}}_C}\Big(Y, \overbrace{\operatorname{Hom}_A(X, Z)}^{\in {}_B\underline{\mathrm{Mod}}_C}\Big), \qquad \varphi \longmapsto \Phi, \quad \Phi(y)(x) = \varphi(x, y).$$

Proof. Given $\Phi : Y \to \mathrm{Hom}_A(X, Z)$ from the right-hand side, define $\varphi : X \times Y \to Z$ by $\varphi(x, y) = \Phi(y)(x)$. It is B-balanced:

$$\varphi(xb, y) = \Phi(y)(xb) = \big(b \cdot \Phi(y)\big)(x) = \Phi(by)(x) = \varphi(x, by).$$

It is left A-linear in x, since each $\Phi(y)$ is A-linear: $\varphi(ax, y) = \Phi(y)(ax) = a\,\Phi(y)(x) = a\varphi(x, y)$. It is right C-linear in y, since Φ is: $\varphi(x, yc) = (\Phi(y)c)(x) = \Phi(y)(x)c$. Additivity in each variable is clear. So $\varphi \in \mathrm{Bal}(X \times Y, Z)$.

Conversely, given $\varphi \in \mathrm{Bal}(X \times Y, Z)$, define $\Phi : Y \to \mathrm{Hom}_A(X, Z)$ by $\Phi(y)(x) = \varphi(x, y)$. Condition 1) of Definition 14.1 says that $\Phi(y)$ is A-linear. Conditions 3) and 2) say that $\Phi(by) = b\Phi(y)$ and $\Phi(yc) = \Phi(y)c$, so Φ is a (B, C)-bimodule map. The two constructions are visibly inverse to each other, and both commute with composing by bimodule maps $Y' \to Y$ and $Z \to Z'$, which is the naturality. □

Combining Definition 14.3 with Theorem 14.7, we get the general form of the *tensor–Hom adjunction.*

Corollary 14.8. *For $X \in {}_A\underline{\mathrm{Mod}}_B$ there are natural isomorphisms*

$$\mathrm{Hom}_{{}_A\underline{\mathrm{Mod}}_C}\big(X \otimes_B Y, Z\big) \cong \mathrm{Hom}_{{}_B\underline{\mathrm{Mod}}_C}\big(Y, \mathrm{Hom}_A(X, Z)\big).$$

That is, $X \otimes_B -$ is left adjoint to $\mathrm{Hom}_A(X, -)$:

$${}_B\underline{\mathrm{Mod}}_C \underset{\mathrm{Hom}_A(X,-)}{\overset{X\otimes_B -}{\rightleftarrows}} {}_A\underline{\mathrm{Mod}}_C \quad (\perp)$$

Often $C = \Bbbk$, in which case this is an adjunction between ${}_B\underline{\mathrm{Mod}}$ and ${}_A\underline{\mathrm{Mod}}$.

14.5 Induced modules and Frobenius reciprocity

Let $H \leq G$ be a subgroup of a group. Then $\Bbbk H \subseteq \Bbbk G$, and by Remark 14.2, $\Bbbk G$ is a $(\Bbbk G, \Bbbk H)$-bimodule. A left $\Bbbk H$-module Y is the same as a $(\Bbbk H, \Bbbk)$-bimodule. Taking

$$A = \Bbbk G, \qquad B = \Bbbk H, \qquad C = \Bbbk, \qquad X = \Bbbk G,$$

we find that $\Bbbk G \otimes_{\Bbbk H} Y$ is a left $\Bbbk G$-module. This gives a functor

$$\mathrm{Ind}_{\Bbbk H}^{\Bbbk G} = \Bbbk G \otimes_{\Bbbk H} - \; : \; {}_{\Bbbk H}\underline{\mathrm{Mod}} \longrightarrow {}_{\Bbbk G}\underline{\mathrm{Mod}},$$

called *induction.*

For a $\Bbbk G$-module Z, the map $\varphi \mapsto \varphi(1)$ is an isomorphism of $\Bbbk H$-modules $\mathrm{Hom}_{\Bbbk G}(\Bbbk G, Z) \cong \mathrm{Res}_{\Bbbk H}^{\Bbbk G} Z$. Indeed, a $\Bbbk G$-module map out of $\Bbbk G$ is determined by the image of 1, and $(h\varphi)(1) = \varphi(h) = h\varphi(1)$. So Corollary 14.8 becomes

$$\boxed{\mathrm{Hom}_{\Bbbk G}\big(\mathrm{Ind}_{\Bbbk H}^{\Bbbk G} Y,\ Z\big) \cong \mathrm{Hom}_{\Bbbk H}\big(Y, \mathrm{Res}_{\Bbbk H}^{\Bbbk G} Z\big)}$$

for all $\Bbbk H$-modules Y and $\Bbbk G$-modules Z. This is *Frobenius reciprocity*: induction is left adjoint to restriction.

14.6 A look ahead

Let a finite group G act on an algebra A by algebra automorphisms, and let

$$A^G = \{\, a \in A \mid g(a) = a \text{ for all } g \in G \,\} \subseteq A$$

be the invariants. As in the preview after Example 12.6, if $|G|$ is invertible in $\Bbbk$ and $\varepsilon = \frac{1}{|G|} \sum_{g \in G} g \in A \rtimes G$, then ε is an idempotent and $A^G \cong \varepsilon\,(A \rtimes G)\,\varepsilon$. In Chapter 17 we will see that A^G is Morita equivalent to $A \rtimes G$ provided $(A \rtimes G)\,\varepsilon\,(A \rtimes G) = A \rtimes G$. Tensor products over A^G and over $A \rtimes G$ are the main tool. The condition can fail (it fails in Example 12.6), so some such hypothesis is really needed.

14.7 Exercises

Exercise 14.9. Prove the associativity isomorphism of Proposition 14.6 2).

Exercise 14.10. Let A be a $\Bbbk$-algebra, $I \subseteq A$ a two-sided ideal, and M a left A-module. Then A/I is an (A, A)-bimodule and M is an $(A, \Bbbk)$-bimodule. Show that

$$A/I \otimes_A M \cong M/IM \qquad \text{as left } A\text{-modules.}$$

Chapter 15

Some Induced Modules

Induction turns modules over a subalgebra into modules over the big algebra. We compute two examples: the module induced from a one-dimensional $\Bbbk[x]$-module to the Weyl algebra, viewed as a skew polynomial ring, and modules induced from R to a generalized Weyl algebra $R(\sigma, t)$. Seeing the Weyl algebra both ways at the end gives two different decompositions of it.

Let $f : R \to A$ be a morphism of algebras. Then A is an (A, R)-bimodule, with right action $a \cdot r = af(r)$ (Remark 14.2). Induction and restriction are the functors

$$ {}_R\underline{\mathrm{Mod}} \underset{\mathrm{Res}_R^A}{\overset{\mathrm{Ind}_R^A}{\rightleftarrows}} {}_A\underline{\mathrm{Mod}} \qquad \begin{aligned} \mathrm{Ind}_R^A(M) &= A \otimes_R M, \\ \mathrm{Res}_R^A(N) &= \mathrm{Hom}_A(A, N) \cong N, \end{aligned} $$

and $\mathrm{Ind}_R^A \dashv \mathrm{Res}_R^A$: induction is left adjoint to restriction (Corollary 14.8 with $X = A$). The R-module structure on $\mathrm{Hom}_A(A, N)$ comes from the right R-action on A, and the isomorphism $\mathrm{Hom}_A(A, N) \cong N$ is

$$ \varphi \longmapsto \varphi(1_A), \qquad \text{with inverse} \qquad n \longmapsto (a \mapsto an). $$

So $\mathrm{Res}_R^A(N)$ is just N with $r \cdot n = f(r)n$.

15.1 Induced modules over the Weyl algebra

Let R be an algebra, $\sigma \in \mathrm{Aut}_{\Bbbk\text{-Alg}}(R)$, and $\delta : R \to R$ a σ-derivation. Consider the skew polynomial ring (Ore extension, Definition 4.5)

$$ A = R[\partial; \sigma, \delta], \qquad \partial r = \sigma(r)\partial + \delta(r), $$

and the morphism $f : R \to A$, $f(r) = r1_A$ (constant polynomials).

Now specialize to $R = \Bbbk[x]$, $\sigma = \mathrm{Id}_R$, and δ the unique derivation with $\delta(x) = 1$, that is, $\delta = \frac{d}{dx}$. Then $\partial x = x\partial + 1$, and

$$ A = \Bbbk[x]\left[\partial; \mathrm{Id}, \frac{d}{dx}\right] \cong A_1(\Bbbk), $$

the Weyl algebra (compare Example 4.7, where the letters were t and x instead of x and ∂).

Let $\alpha \in \Bbbk$, let $\mathfrak{m}_\alpha = (x - \alpha)$ be the corresponding maximal ideal of $\Bbbk[x]$, and let $\Bbbk_\alpha = \Bbbk[x]/\mathfrak{m}_\alpha$, regarded as a left $\Bbbk[x]$-module. Thus

$$ \Bbbk_\alpha = \Bbbk \cdot 1_\alpha, \qquad \text{where } 1_\alpha := 1 + (x - \alpha), $$

and $p(x).1_\alpha = p(\alpha)1_\alpha$ for all $p(x) \in \Bbbk[x]$.

Goal 15.1. Describe $\mathrm{Ind}_R^A(\Bbbk_\alpha) = A \otimes_R \Bbbk_\alpha = \Bbbk[x]\left[\partial; \mathrm{Id}, \frac{d}{dx}\right] \otimes_{\Bbbk[x]} \Bbbk_\alpha$.

Recall that skew polynomial rings are free as left R-modules (Theorem 4.4). When σ is an automorphism they are also free as right R-modules (Exercise 15.3):

$$A = \bigoplus_{n=0}^{\infty} R\,\partial^n = \bigoplus_{n=0}^{\infty} \partial^n R.$$

We will see that this gives, for every left R-module V, an isomorphism of vector spaces

$$A \otimes_R V \cong \bigoplus_{n=0}^{\infty} \big(\Bbbk \partial^n \otimes_\Bbbk V\big).$$

In more detail, let

$$T = \bigoplus_{n=0}^{\infty} \Bbbk\,\partial^n,$$

a $\Bbbk$-vector space. The map

$$T \times R \longrightarrow A, \qquad (\partial^n, r) \longmapsto \partial^n r \qquad (n \geq 0,\ r \in R),$$

is $\Bbbk$-bilinear, so we get a linear map $T \otimes_\Bbbk R \to A$, $\partial^n \otimes r \mapsto \partial^n r$. Conversely, since $A = \bigoplus_n \partial^n R$ freely, there is a well-defined linear map

$$A \longrightarrow T \otimes_\Bbbk R, \qquad \partial^n r \longmapsto \partial^n \otimes r.$$

The two maps are inverse to each other, and both commute with right multiplication by R. So $T \otimes_\Bbbk R \cong A$ as right R-modules.

Using the unit and associativity isomorphisms of Proposition 14.6, we get isomorphisms of vector spaces

$$\begin{aligned} A \otimes_R \Bbbk_\alpha &\cong (T \otimes_\Bbbk R) \otimes_R \Bbbk_\alpha \\ &\cong T \otimes_\Bbbk (R \otimes_R \Bbbk_\alpha) \\ &\cong T \otimes_\Bbbk \Bbbk_\alpha \qquad (r \otimes 1_\alpha \mapsto r1_\alpha) \\ &\cong T \qquad (\partial^n \otimes 1_\alpha \mapsto \partial^n). \end{aligned}$$

Following an element $\partial^n \otimes 1_\alpha$ of $A \otimes_R \Bbbk_\alpha$ through these maps, it goes to ∂^n. Hence

$$\mathrm{Ind}_R^A(\Bbbk_\alpha) = \bigoplus_{n=0}^{\infty} \Bbbk \cdot (\partial^n \otimes 1_\alpha),$$

with basis $\{\partial^n \otimes 1_\alpha\}_{n=0}^{\infty}$. How do the generators of A act on this basis?

A is generated by x and ∂ as an algebra. Clearly

$$\partial.(\partial^n \otimes 1_\alpha) = \partial^{n+1} \otimes 1_\alpha.$$

For x we need to move x past ∂^n. From $\partial x = x\partial + 1$ we get, by induction on n,

$$\partial^n x = \partial^{n-1}(x\partial + 1) = \big(x\partial^{n-1} + (n-1)\partial^{n-2}\big)\partial + \partial^{n-1} = x\partial^n + n\partial^{n-1},$$

that is, $x\partial^n = \partial^n x - n\partial^{n-1}$. Therefore

$$\begin{aligned} x.(\partial^n \otimes 1_\alpha) &= (x\partial^n) \otimes 1_\alpha = \left(\partial^n x - n\partial^{n-1}\right) \otimes 1_\alpha \\ &= \partial^n \otimes x.1_\alpha - n\partial^{n-1} \otimes 1_\alpha = \alpha\, \partial^n \otimes 1_\alpha - n\partial^{n-1} \otimes 1_\alpha. \end{aligned}$$

So we can think of $\mathrm{Ind}_R^A(\Bbbk_\alpha)$ as the polynomial ring $\Bbbk[\partial]$, on which

$$\partial \text{ acts as multiplication by } \partial, \qquad x \text{ acts as } -\frac{d}{d\partial} + \alpha\ \mathrm{Id}_{\Bbbk[\partial]}.$$

(As a check: $\left[\partial, -\frac{d}{d\partial} + \alpha\right] = 1$, as it should be.) See also Exercise 15.5, which identifies this module as $A_1(\Bbbk)/A_1(\Bbbk)(x-\alpha)$.

15.2 Induced modules over generalized Weyl algebras

Let $A = R(\sigma, t)$ be a GWA (Definition 5.7), with $f : R \to A$, $r \mapsto r1_A$. As a right R-module,

$$A = \left(\bigoplus_{n=1}^{\infty} Y^n R\right) \oplus 1_A R \oplus \left(\bigoplus_{n=1}^{\infty} X^n R\right),$$

that is, A is *free* as a right R-module with basis

$$\{Y^n\}_{n\geq 1} \cup \{1_A\} \cup \{X^n\}_{n\geq 1}$$

(Exercise 15.3). To write this compactly, put

$$Z^{(n)} = \begin{cases} X^n, & n > 0, \\ 1, & n = 0, \\ Y^{|n|}, & n < 0. \end{cases}$$

Let M be a left R-module, for example $M = R/\mathfrak{m}$ for a maximal left ideal $\mathfrak{m}$ (a maximal ideal, when R is commutative). Exactly as in the previous section, with $T = \bigoplus_{n\in\mathbb{Z}} \Bbbk Z^{(n)}$, we get an isomorphism of vector spaces

$$\widehat{M} := \mathrm{Ind}_R^A M = A \otimes_R M \cong \bigoplus_{n\in\mathbb{Z}} \left(\Bbbk Z^{(n)} \otimes_\Bbbk M\right),$$

under which $Z^{(n)} \otimes m \in A \otimes_R M$ corresponds to $Z^{(n)} \otimes m$. So $\widehat{M} = \bigoplus_{n\in\mathbb{Z}} Z^{(n)} \otimes M$, and each summand is a copy of M.

How does R act on a summand? Since $Xr = \sigma(r)X$ and $rY = Y\sigma(r)$, we have $rX^n = X^n\sigma^{-n}(r)$ and $rY^n = Y^n\sigma^n(r)$, so in both cases

$$r.\left(Z^{(n)} \otimes m\right) = Z^{(n)} \otimes \sigma^{-n}(r)m \qquad (n \in \mathbb{Z}).$$

Thus $Z^{(n)} \otimes M$ is M with its R-action twisted by σ^{-n}. Now suppose R is commutative and $M = R/\mathfrak{m}$ for a maximal ideal $\mathfrak{m}$. Then r kills $Z^{(n)} \otimes M$ exactly when $\sigma^{-n}(r) \in \mathfrak{m}$, that is, when $r \in \sigma^n(\mathfrak{m})$.

So every vector of $Z^{(n)} \otimes M$ is a *weight vector* of weight $\sigma^n(\mathfrak{m})$, and the weights of $\widehat{M}$ form the σ-orbit $\{\sigma^n(\mathfrak{m})\}_{n\in\mathbb{Z}}$.

Here, for R commutative and an R-module V, a *weight vector* is a $v \in V$, $v \neq 0$, with $\mathfrak{m}v = 0$ for some maximal ideal $\mathfrak{m}$ of R, its *weight*. The induced modules $\widehat{M}$ turn out to control all simple A-modules that have a weight vector; see Exercise 15.4.

Remark 15.2. Simple weight modules over GWAs have been classified by Bavula [Bavula92]. That is where hypotheses such as "R is an integral domain" and "$t \neq 0$" come in. They are not needed for Exercise 15.4.

15.3 The Weyl algebra two ways

The Weyl algebra is both a skew polynomial ring and a GWA. Let $R = \Bbbk[u]$, $\sigma(u) = u-1$ and $t = u$, as in Example 5.9. By Proposition 5.10 (for $\operatorname{char}\Bbbk = 0$; in any characteristic by Exercise 5.13), there is an isomorphism

$$\begin{array}{ccc} \text{GWA} & & \text{Weyl algebra} \\ \Bbbk[u](\sigma,u) & \xrightarrow{\sim} & A_1(\Bbbk) = \Bbbk\langle x, \partial \mid \partial x - x\partial = 1\rangle \\ X & \longmapsto & x \\ Y & \longmapsto & \partial \\ u & \longmapsto & \partial x. \end{array}$$

On the skew polynomial side, $A_1(\Bbbk) = \bigoplus_n \partial^n \Bbbk[x]$ (Exercise 15.3), so $\{\partial^n x^m\}_{n,m=0}^{\infty}$ is a basis for $A_1(\Bbbk)$. On the GWA side, the right-module decomposition of the previous section, transported through the isomorphism (which sends $R = \Bbbk[u]$ to $\Bbbk[\partial x]$, $Y^n R$ to $\partial^n \Bbbk[\partial x]$ and $X^n R$ to $x^n \Bbbk[\partial x]$), gives a second decomposition:

$$A_1(\Bbbk) = \underbrace{\bigoplus_{n=0}^{\infty} \partial^n\, \Bbbk[x]}_{\text{skew polynomial ring}} = \underbrace{\left(\bigoplus_{n=1}^{\infty} \partial^n\, \Bbbk[\partial x]\right) \oplus \Bbbk[\partial x] \oplus \left(\bigoplus_{n=1}^{\infty} x^n\, \Bbbk[\partial x]\right)}_{\text{GWA}}.$$

The second decomposition is the $\mathbb{Z}$-grading of Example 7.5, where $\deg x = 1$ and $\deg \partial = -1$: the degree-n component is $x^n\Bbbk[\partial x]$ for $n \geq 0$ and $\partial^{|n|}\Bbbk[\partial x]$ for $n < 0$. Indeed, each summand lies in a single homogeneous component, and the summands add up to all of $A_1(\Bbbk)$.

Figure 15.1 shows the basis vector $\partial^n x^m$ as the lattice point (m, n). The degree-k component is spanned by the $\partial^n x^m$ with $m - n = k$, a diagonal.

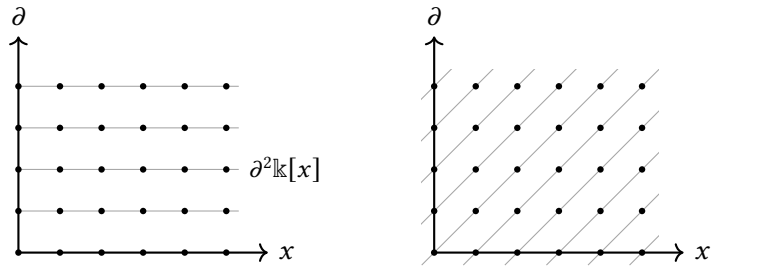
15.4 Exercises

Exercise 15.3. (a) Let σ be an automorphism of R and δ a σ-derivation. Show that $R[\partial; \sigma, \delta]$ is free as a *right* R-module with basis $\{\partial^n\}_{n\geq 0}$. (Hint: show that $\partial^n r - \sigma^n(r)\partial^n$ has lower degree in ∂, and compare leading terms.)

(b) Let $R(\sigma, t)$ be a GWA. Show that it is free as a right R-module with basis

$$\{1\} \cup \{X^n\}_{n\geq 1} \cup \{Y^n\}_{n\geq 1}.$$

(Hint: Corollary 11.6 and $X^n r = \sigma^n(r)X^n$.)



rows: $\partial^n \Bbbk[x]$ diagonals $m - n = k$: $x^k \Bbbk[\partial x]$, $\partial^{|k|} \Bbbk[\partial x]$

Figure 15.1: The basis $\{\partial^n x^m\}$ of $A_1(\Bbbk)$, sliced in two ways: by powers of ∂ (the skew polynomial ring) and by $\mathbb{Z}$-degree (the GWA).

Exercise 15.4. Let $A = R(\sigma, t)$ be a GWA with R commutative.

(a) Let V be a simple left A-module containing a weight vector for R, i.e. some $v \in V$, $v \neq 0$, with $\mathfrak{m}v = 0$ for a maximal ideal $\mathfrak{m}$ of R. Prove that V is a quotient of an induced module $\widehat{M}$ for some M. (Hint: the adjunction Ind $\dashv$ Res, applied to $M = Rv$.)

(b) Show that V is then a weight module, $V = \sum_{\mathfrak{n}} V_{\mathfrak{n}}$ where $V_{\mathfrak{n}} = \{\, w \in V \mid \mathfrak{n}w = 0 \,\}$, and that every weight of V lies in the σ-orbit $\{\sigma^n(\mathfrak{m})\}_{n \in \mathbb{Z}}$.

Exercise 15.5. Let $f : R \to A$ be a morphism of algebras and $I \subseteq R$ a left ideal. Show that

$$\mathrm{Ind}_R^A(R/I) \cong A/Af(I) \qquad \text{as left } A\text{-modules,}$$

via $a \otimes (r + I) \mapsto af(r) + Af(I)$. Deduce that $\mathrm{Ind}_R^A(\Bbbk_\alpha) \cong A_1(\Bbbk)/A_1(\Bbbk)(x - \alpha)$ in the example of this chapter. (Compare the last exercise of Chapter 14.)

Chapter 16

Morita Equivalence

When do two algebras have "the same" module theory? Morita's answer: exactly when there is a pair of bimodules that are inverse to each other under tensor product. We package this as a Morita context, show that a strict Morita context gives an equivalence of module categories (Morita I), and state the converse (Morita II). The basic example is that $M_n(\Bbbk)$ and $\Bbbk$ have equivalent module categories.

16.1 Morita contexts

Definition 16.1. A *Morita context* is a 6-tuple

$$(A, A', M, M', \tau, \mu),$$

where

- A and A' are $\Bbbk$-algebras,
- $M = {}_{A'}M_A$ is an (A', A)-bimodule and $M' = {}_A M'_{A'}$ is an (A, A')-bimodule,
- $\tau : M \otimes_A M' \to A'$ is a morphism of (A', A')-bimodules, and
- $\mu : M' \otimes_{A'} M \to A$ is a morphism of (A, A)-bimodules,

such that

$$\begin{aligned} &(1) && \tau(x \otimes y') \cdot y = x \cdot \mu(y' \otimes y) && \text{for all } x, y \in M,\ y' \in M', \\ &(2) && x' \cdot \tau(x \otimes y') = \mu(x' \otimes x) \cdot y' && \text{for all } x', y' \in M',\ x \in M. \end{aligned}$$

The context is *strict* if μ and τ are isomorphisms.

The letters stand for "trace" and "multiplication". In the basic example below (Example 16.4), μ multiplies a column vector by a row vector, giving a matrix, and τ takes the product of a row and a column, which is the trace of the opposite product. For a corner eAe of an algebra A (Chapter 17), τ lands in eAe and μ is multiplication in A.

Remark 16.2. In these notes the interesting contexts will be the strict ones, but it is convenient to allow non-strict ones. In Chapter 17 we will write down a context for a corner eAe of an algebra A first, and only then ask whether it is strict. Some authors build strictness into the definition; we follow the more common convention, where μ and τ need not be isomorphisms.

Strictness is easier to check than it looks: surjectivity is enough.

Proposition 16.3. *In a Morita context, if μ is surjective then μ is an isomorphism. Similarly for τ.*

Proof. Since μ is surjective, $1_A = \mu\big(\sum_i x_i' \otimes x_i\big)$ for some $x_i' \in M'$, $x_i \in M$. Let $\xi = \sum_j y_j' \otimes y_j \in \operatorname{Ker}\mu$. Using (1), then that the tensor product is balanced over A', then (2):

$$\begin{aligned}\xi &= \sum_{i,j} y_j' \otimes y_j\, \mu(x_i' \otimes x_i) = \sum_{i,j} y_j' \otimes \tau(y_j \otimes x_i')\, x_i = \sum_{i,j} y_j'\, \tau(y_j \otimes x_i') \otimes x_i \\ &= \sum_{i,j} \mu(y_j' \otimes y_j)\, x_i' \otimes x_i = \sum_i \mu(\xi)\, x_i' \otimes x_i = 0.\end{aligned}$$

So μ is injective. The argument for τ is the same with the roles of A and A' exchanged. □

16.2 The Morita algebra

Conditions (1) and (2) are exactly what is needed to multiply "matrices" with entries in A, A', M, M'. Let

$$\begin{pmatrix} A & M' \\ M & A' \end{pmatrix} = \left\{ \begin{pmatrix} a & x' \\ x & a' \end{pmatrix} \;\middle|\; a \in A,\ a' \in A',\ x \in M,\ x' \in M' \right\},$$

with entrywise addition and the multiplication

$$\begin{pmatrix} a & x' \\ x & a' \end{pmatrix} \begin{pmatrix} b & y' \\ y & b' \end{pmatrix} = \begin{pmatrix} ab + \mu(x' \otimes y) & ay' + x'b' \\ xb + a'y & \tau(x \otimes y') + a'b' \end{pmatrix}.$$

Then (1) and (2) together are equivalent to this multiplication being associative (Exercise 16.11). So every Morita context, strict or not, gives an algebra $\begin{pmatrix} A & M' \\ M & A' \end{pmatrix}$, the *Morita algebra* of the context.

Example 16.4. Let $A = M_n(\Bbbk)$ and $A' = \Bbbk$, let $M = (\Bbbk^n)^* = [\Bbbk\ \cdots\ \Bbbk]$ be the row vectors, an (A', A)-bimodule, and let $M' = \Bbbk^n$ be the column vectors, an (A, A')-bimodule. Define

$$\begin{aligned} \mu :\ \Bbbk^n \otimes_{\Bbbk} (\Bbbk^n)^* &\longrightarrow M_n(\Bbbk), \quad x \otimes y \longmapsto xy, \\ \tau :\ (\Bbbk^n)^* \otimes_{M_n(\Bbbk)} \Bbbk^n &\longrightarrow \Bbbk, \qquad x \otimes y \longmapsto xy. \end{aligned}$$

So μ takes the outer product of a column and a row, and τ the inner product of a row and a column. Both are well defined bimodule maps, and conditions (1) and (2) hold by associativity of matrix multiplication.

The context is strict. The map μ is surjective, since $\mu(e_i \otimes e_j^T) = E_{ij}$, and both sides have dimension n^2, so μ is an isomorphism. For τ, we claim that $e_1^T \otimes e_1$ spans the domain. Indeed, using $e_i^T = e_1^T E_{1i}$ and $E_{1i} e_j = \delta_{ij} e_1$,

$$e_i^T \otimes e_j = e_1^T E_{1i} \otimes e_j = e_1^T \otimes E_{1i} e_j = \delta_{ij}\, e_1^T \otimes e_1.$$

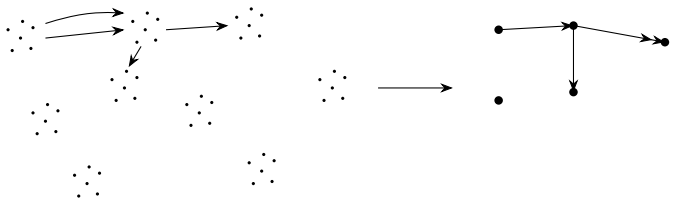
So the domain of τ has dimension at most 1, and $\tau(e_1^T \otimes e_1) = 1 \neq 0$. Hence τ is an isomorphism. (Alternatively, τ is surjective, so Proposition 16.3 applies.)

16.3 Equivalences of categories

Definition 16.5. A functor $F : \mathcal{C} \to \mathcal{C}'$ is an *equivalence of categories* if there is a functor $G : \mathcal{C}' \to \mathcal{C}$ with natural isomorphisms $GF \cong \mathrm{Id}_{\mathcal{C}}$ and $FG \cong \mathrm{Id}_{\mathcal{C}'}$. Such a G is a *quasi-inverse* of F. We write $\mathcal{C} \simeq \mathcal{C}'$.

Equivalently, F is *fully faithful* (bijective on each Hom-set) and *essentially surjective* (every object of $\mathcal{C}'$ is isomorphic to some $F(X)$); see [MacLane98, Theorem IV.4.1].

Here is a picture to keep in mind. Collapse each isomorphism class of objects of $\mathcal{C}$ to a single point, keeping one object from each class. The result is a *skeleton* of $\mathcal{C}$, and $\mathcal{C} \simeq \mathcal{C}'$ exactly when $\mathcal{C}$ and $\mathcal{C}'$ have isomorphic skeletons. So an equivalence may change how many copies of each object there are, but nothing else.



$\mathcal{C}$ skeleton of $\mathcal{C}$

Figure 16.1: Intuition for $\mathcal{C} \simeq \mathcal{C}'$: each cluster of dots is an isomorphism class of objects. Collapsing each class to one point gives a skeleton.

16.4 Morita's theorems

Theorem 16.6 (Morita I)**.** *Let $(A, A', M, M', \tau, \mu)$ be a strict Morita context.*

(i) The functors

$$M \otimes_A - : {}_A\underline{\mathrm{Mod}} \longrightarrow {}_{A'}\underline{\mathrm{Mod}} \qquad \text{and} \qquad M' \otimes_{A'} - : {}_{A'}\underline{\mathrm{Mod}} \longrightarrow {}_A\underline{\mathrm{Mod}}$$

are quasi-inverse equivalences of categories. In particular ${}_A\underline{\mathrm{Mod}} \simeq {}_{A'}\underline{\mathrm{Mod}}$.

(ii) Similarly, $- \otimes_A M'$ and $- \otimes_{A'} M$ give $\underline{\mathrm{Mod}}_A \simeq \underline{\mathrm{Mod}}_{A'}$.

Proof. (i) For a left A-module N, the associativity and unit isomorphisms of Proposition 14.6 and μ give isomorphisms of left A-modules

$$M' \otimes_{A'} (M \otimes_A N) \cong (M' \otimes_{A'} M) \otimes_A N \xrightarrow{\mu \otimes \mathrm{Id}} A \otimes_A N \cong N,$$

$x' \otimes (x \otimes n) \mapsto \mu(x' \otimes x)n$. This is natural in N, so $(M' \otimes_{A'} -) \circ (M \otimes_A -) \cong \mathrm{Id}$. In the same way τ gives $(M \otimes_A -) \circ (M' \otimes_{A'} -) \cong \mathrm{Id}$.

(ii) The same argument, with the tensor factors on the other side. □

(Only the fact that μ and τ are bimodule isomorphisms was used here. Conditions (1) and (2) are needed for the Morita algebra and for Proposition 16.3.)

Example 16.7. Applying Morita I to Example 16.4 gives

$$_{M_n(\Bbbk)}\underline{\mathrm{Mod}} \simeq {}_{\Bbbk}\underline{\mathrm{Mod}} = \underline{\mathrm{Vect}}_{\Bbbk}.$$

In particular, $M_n(\Bbbk)$ has a *unique simple module* up to isomorphism, namely $M' \otimes_{\Bbbk} \Bbbk \cong \Bbbk^n$, the column vectors. The reason is that a module N is simple if and only if $N \neq 0$ and every injective map $N_0 \to N$ with $N_0 \neq 0$ is an isomorphism. This is a property of N in the category (injective maps are the monomorphisms), so an equivalence takes simple modules to simple modules. And $\underline{\mathrm{Vect}}_{\Bbbk}$ has a unique simple object, $\Bbbk$.

The converse of Morita I is also true.

Theorem 16.8 (Morita II)**.** *Let A and A' be $\Bbbk$-algebras, and let $F : {}_A\underline{\mathrm{Mod}} \to {}_{A'}\underline{\mathrm{Mod}}$ be an equivalence with quasi-inverse $G : {}_{A'}\underline{\mathrm{Mod}} \to {}_A\underline{\mathrm{Mod}}$. Then*

(i) there is a strict Morita context $(A, A', M, M', \tau, \mu)$, with $M \cong F(A)$ and $M' \cong G(A')$, and
(ii) there are natural isomorphisms

$$F \cong M \otimes_A -, \qquad G \cong M' \otimes_{A'} -.$$

The same holds for right modules: an equivalence $\underline{\mathrm{Mod}}_A \simeq \underline{\mathrm{Mod}}_{A'}$ comes from a strict Morita context, as in Morita I (ii).

Sketch. Put $M = F(A)$, a left A'-module. For $a \in A$, right multiplication $\rho_a : A \to A$, $b \mapsto ba$, is a map of left A-modules, and we let a act on M from the right by $F(\rho_a)$. Since $\rho_{ab} = \rho_b \circ \rho_a$, this makes M an (A', A)-bimodule. Similarly $M' = G(A')$ is an (A, A')-bimodule. The functors F and $M \otimes_A -$ agree on A, both commute with direct sums and both preserve cokernels (an equivalence preserves all of these), and every module is a cokernel of a map between free modules. This gives $F \cong M \otimes_A -$, and similarly $G \cong M' \otimes_{A'} -$. Then μ comes from $M' \otimes_{A'} M \cong G(M) = GF(A) \cong A$, and τ similarly. For the remaining checks see [Lam99, Chapter 7]. □

Definition 16.9. Two $\Bbbk$-algebras A and A' are *Morita equivalent* if the following equivalent conditions hold:

1) ${}_A\underline{\mathrm{Mod}} \simeq {}_{A'}\underline{\mathrm{Mod}}$;
2) there exists a strict Morita context $(A, A', M, M', \tau, \mu)$;
3) $\underline{\mathrm{Mod}}_A \simeq \underline{\mathrm{Mod}}_{A'}$.

We write $A \overset{\text{Mor.}}{\approx} A'$.

The conditions are equivalent by Morita I and II: 1)⇒2) and 3)⇒2) are Morita II for left and right modules, and 2)⇒1) and 2)⇒3) are Morita I.

Remark 16.10. The equivalence 1)⇔3) is not obvious. There is no evident way to turn left modules into right modules (right A-modules are left A^{op}-modules, and A^{op} need not be isomorphic to A). It is the Morita context, which works on both sides at once, that connects them.

16.5 Exercises

Exercise 16.11. Prove that the multiplication on $\begin{pmatrix} A & M' \\ M & A' \end{pmatrix}$ is associative if and only if (1) and (2) hold. (Hint: by bilinearity it suffices to check triples of "matrices" with a single nonzero entry. Show that all of them follow from the bimodule axioms and the fact that μ and τ are bimodule maps, except the triples of type (M, M', M) and (M', M, M'), which give exactly (1) and (2).)

Exercise 16.12. (a) Show that $M_n(\Bbbk)$ and $M_m(\Bbbk)$ are Morita equivalent for all $n, m \geq 1$.

(b) Show that $\Bbbk \times \Bbbk$ is *not* Morita equivalent to $\Bbbk$, by counting simple modules. (Equivalences preserve simple modules, as in Example 16.7.)

Chapter 17

Morita Equivalence for Skew Group Algebras

We first finish a story begun in Chapter 5: the induced module $A\langle X\rangle \otimes_A M$. Then we turn to Morita theory in practice. A corner eAe of an algebra A is Morita equivalent to A whenever $AeA = A$. For a skew group algebra $R \rtimes G$ and the averaging idempotent e, the corner is the invariant algebra R^G, and the condition $AeA = A$ turns out to mean that $R^G \subset R$ is a Galois extension.

17.1 Induction to $A\langle X\rangle$

Let A be an algebra and $A\langle X\rangle$ the algebra obtained by adjoining a variable X (Chapter 3). The inclusion $A \to A\langle X\rangle$ gives a pair of adjoint functors $\mathrm{Ind}_A^{A\langle X\rangle} \dashv \mathrm{Res}_A^{A\langle X\rangle}$ (Chapter 15), where $\mathrm{Res}_A^{A\langle X\rangle}$ is the restriction functor of Chapter 5 and

$$\mathrm{Ind}_A^{A\langle X\rangle} M = \boxed{A\langle X\rangle \otimes_A M}.$$

This is the left adjoint that Chapter 5 was looking for. What does it look like? Since

$$A\langle X\rangle = \mathrm{span}\{\, a_1 X a_2 X \cdots X a_n \mid n \geq 1,\ a_i \in A \,\},$$

and the last factor a_n can be moved across the tensor sign, we get

$$\begin{aligned} A\langle X\rangle \otimes_A M &= \mathrm{span}\{\, a_1 X a_2 X \cdots X a_n \otimes m \mid a_i \in A,\ m \in M \,\} \\ &= \mathrm{span}\{\, a_1 X a_2 X \cdots a_{n-1} X \otimes m \mid a_i \in A,\ m \in M \,\}, \end{aligned}$$

including the elements $1 \otimes m$ (for $n = 1$). This is exactly the guess "$a_1 X a_2 X \cdots a_n X m$" of Chapter 5.

To say more precisely which of these elements are independent, we need a basis of $A\langle X\rangle$. Choose a basis $\{b_i\}_{i\in I}$ of A with $b_0 = 1$. Then the elements

$$b_{i_0} X b_{i_1} X \cdots X b_{i_n} \qquad (n \geq 0,\ i_0, \ldots, i_n \in I),$$

form a basis of $A\langle X\rangle$, where we do not write the factors $b_0 = 1$ (Exercise 17.7). Equivalently, $a_0 \otimes a_1 \otimes \cdots \otimes a_n \mapsto a_0 X a_1 X \cdots X a_n$ gives an isomorphism of vector spaces $\bigoplus_{n\geq 0} A^{\otimes(n+1)} \cong A\langle X\rangle$. Two consequences:

1) $A\langle X\rangle = A \oplus \big(A\langle X\rangle \cdot XA\big)$, a decomposition of (A, A)-bimodules (the second summand is spanned by the basis words with $n \geq 1$);
2) $A\langle X\rangle X \otimes_{\Bbbk} A \to A\langle X\rangle XA$, $w \otimes a \mapsto wa$, is an isomorphism of right A-modules, since every basis word with $n \geq 1$ is uniquely a basis word ending in X followed by some b_i.

Since tensor products commute with direct sums (both sides have the same universal property),

$$\begin{aligned} A\langle X\rangle \otimes_A M &\cong (A \otimes_A M) \oplus \big((A\langle X\rangle X \otimes_{\Bbbk} A) \otimes_A M\big) \\ &\cong M \oplus \big(A\langle X\rangle X \otimes_{\Bbbk} M\big). \end{aligned}$$

These are isomorphisms of vector spaces (even of left A-modules), but not of $A\langle X\rangle$-modules: X maps the summand M, that is, the elements $1 \otimes m$, into the second summand. Iterating, $A\langle X\rangle \otimes_A M \cong \bigoplus_{n\geq 0} A^{\otimes n} \otimes_{\Bbbk} M$, with $a_1 \otimes \cdots \otimes a_n \otimes m$ corresponding to $a_1 X \cdots a_n X \otimes m$.

17.2 Corners

Let A be an algebra and $e \in A$ an idempotent, $e^2 = e \neq 0$. Then eAe is an algebra with unit $1_{eAe} = e$ (though not a subalgebra of A unless $e = 1$). We try to construct a Morita context (Definition 16.1)

$$\big(A,\ eAe,\ eA,\ Ae,\ \tau,\ \mu\big),$$

where eA is an (eAe, A)-bimodule and Ae an (A, eAe)-bimodule by multiplication, and

$$\begin{aligned} \tau &: eA \otimes_A Ae \longrightarrow eAe, & ea \otimes be &\longmapsto eabe, \\ \mu &: Ae \otimes_{eAe} eA \longrightarrow A, & ae \otimes eb &\longmapsto aeb. \end{aligned}$$

Both maps are well defined bimodule maps, and conditions (1) and (2) of Definition 16.1 say $(xy')y = x(y'y)$ and $x'(xy') = (x'x)y'$, which hold by associativity in A. So this is a Morita context.

The map τ is always surjective: $\tau(ea \otimes e) = eae$. The map μ is surjective if and only if $AeA = A$, that is,

$$1_A = \sum_i a_i e b_i \qquad \text{for some } a_i, b_i \in A,$$

since AeA is a two-sided ideal. By Proposition 16.3, surjective maps in a Morita context are isomorphisms. So τ is always an isomorphism, and we get:

Theorem 17.1. *Let $e \in A$ be a nonzero idempotent with $AeA = A$. Then $(A, eAe, eA, Ae, \tau, \mu)$ is a strict Morita context, hence*

$$A \overset{\text{Mor.}}{\approx} eAe.$$

Example 17.2. Let $A = M_n(\Bbbk)$ and, for some $1 \leq \ell \leq n$,

$$e = \left(\begin{array}{c|c} I_\ell & 0 \\ \hline 0 & 0 \end{array}\right) = \sum_{i=1}^{\ell} E_{ii}.$$

Then $E_{ij} = E_{i1}\, e\, E_{1j}$ for all i, j, so $M_n(\Bbbk) = M_n(\Bbbk)\, e\, M_n(\Bbbk)$, and

$$M_n(\Bbbk) \overset{\text{Mor.}}{\approx} e\, M_n(\Bbbk)\, e = \left(\begin{array}{c|c} M_\ell(\Bbbk) & 0 \\ \hline 0 & 0 \end{array}\right) \cong M_\ell(\Bbbk).$$

For $\ell = 1$ this is Example 16.4 again: eA is the first row and Ae the first column.

17.3 Skew group algebras

Let R be a commutative algebra and G a finite group with $|G| \in \Bbbk^\times$, acting faithfully on R by algebra automorphisms, $G \hookrightarrow \mathrm{Aut}_\Bbbk(R)$. Let

$$A = R \rtimes G = \left\{ \sum_{g\in G} r_g\, g \;\middle|\; r_g \in R \right\}, \qquad g\, r = g(r)\, g \quad (g \in G,\ r \in R),$$

be the skew group algebra (Definition 12.3), and let

$$e = \frac{1}{|G|} \sum_{g\in G} g \in \Bbbk G \subset A.$$

For every $h \in G$, $he = e = eh$, because $g \mapsto hg$ and $g \mapsto gh$ are bijections of G. Hence $e^2 = \frac{1}{|G|}\sum_g ge = e$. Two questions:

1) When is $AeA = A$?
2) What is eAe?

We start with 2). For $r \in R$ and $g \in G$, using $ge = e$ and then $eg = e$,

$$e\, r g\, e = e\, r\, e = \frac{1}{|G|} \sum_{g\in G} g\, r\, e = \frac{1}{|G|} \sum_{g\in G} g(r)\, g\, e = \Big(\frac{1}{|G|} \sum_{g\in G} g(r)\Big) e \in R^G e,$$

where

$$R^G = \{\, r \in R \mid g(r) = r \text{ for all } g \in G \,\}$$

is the algebra of invariants. So $eAe \subseteq R^G e$. In fact

$$eAe = R^G e \cong R^G \qquad \text{as algebras}$$

(Exercise 17.8). By Theorem 17.1,

$$R^G \overset{\text{Mor.}}{\approx} R \rtimes G, \qquad \text{provided } AeA = A.$$

Question 1) has several answers.

Theorem 17.3. *Let $A = R \rtimes G$ and e be as above. The following are equivalent:*

i) $AeA = A$;
ii) there exist $x_1, \dots, x_k, y_1, \dots, y_k \in R$ with $\sum_i x_i\, g(y_i) = \delta_{g,1_G}$ for all $g \in G$;
iii) R is a finitely generated projective R^G-module, and the natural map

$$R \rtimes G \longrightarrow \mathrm{End}_{R^G}(R), \qquad rg \longmapsto \big(s \mapsto r\, g(s)\big),$$

is an isomorphism;
iv) the map

$$R \otimes_{R^G} R \longrightarrow \{\, f :\ G \to R \mid f \text{ any function} \,\}, \qquad r \otimes s \longmapsto \big(g \mapsto r\, g(s)\big),$$

is an isomorphism of algebras, where the functions $G \to R$ have pointwise operations.

The equivalence i)⇔ii) is Exercise 17.9. The equivalence of ii), iii) and iv) is a theorem of Chase, Harrison and Rosenberg [CHR65, Theorem 1.3].

Definition 17.4. $R^G \subset R$ is a *Galois extension with Galois group G* if the conditions of Theorem 17.3 hold.

Corollary 17.5. *If $R^G \subset R$ is a Galois extension (and $|G| \in \mathbb{k}^\times$), then $R^G \overset{\text{Mor.}}{\approx} R \rtimes G$.*

Remark 17.6. In iii) both conditions are needed. Let $R = \mathbb{C}[x, y]$ and let $G = \{\pm 1\}$ act by $x \mapsto -x$, $y \mapsto -y$. By a theorem of Auslander [Auslander62], the natural map $R \rtimes G \to \mathrm{End}_{R^G}(R)$ is an isomorphism here. But ii) fails: evaluating $\sum_i x_i\, g(y_i) = \delta_{g,1}$ at the origin, which is fixed by -1, gives $\sum_i x_i(0)y_i(0) = 1$ for $g = 1$ and $\sum_i x_i(0)y_i(0) = 0$ for $g = -1$. So R is not projective over R^G. The same argument shows that whenever some $g \neq 1$ fixes a point, as in the preview after Example 12.6, the extension is not Galois.

17.4 Exercises

Exercise 17.7. Let $\{b_i\}_{i\in I}$ be a basis of A with $b_0 = 1$, and write $b_i b_j = \sum_k c_{ij}^k b_k$.

(a) Show that $A\langle X\rangle$ is presented by the generators X and b_i ($i \neq 0$) and the relations $b_i b_j = \sum_k c_{ij}^k b_k$ ($i, j \neq 0$), where b_0 is read as the empty word.
(b) Use the Diamond Lemma (Theorem 10.21) to show that the words $b_{i_0} X b_{i_1} X \cdots X b_{i_n}$ (with the factors b_0 omitted) form a basis of $A\langle X\rangle$. (The only ambiguities are $b_i b_j b_k$; compare Exercise 12.10.)

Exercise 17.8. In the setting of the skew group algebra section, show that $eAe = R^G e$ and that $R^G \to eAe$, $s \mapsto se$, is an isomorphism of algebras. (Hints: for $s \in R^G$, $gs = sg$ for all g, so $se = es$; for injectivity, use that $R \rtimes G$ is free over R with basis G.)

Exercise 17.9. Prove i)⇔ii) in Theorem 17.3. (Hint: show that $AeA = ReR$ and that $x\, e\, y = \frac{1}{|G|}\sum_g x\, g(y)\, g$ for $x, y \in R$.)

Exercise 17.10. (a) Let $R = \mathbb{C}[x, x^{-1}]$ and $G = S_2$, with $(12).x = -x$. Show that $R^G \subset R$ is a Galois extension. Conclude that $\mathbb{C}[x^2, x^{-2}] \overset{\text{Mor.}}{\approx} \mathbb{C}[x, x^{-1}] \rtimes S_2$.
(b) Show that $\mathbb{C}[x]^{S_2} \subset \mathbb{C}[x]$, with the same action, is *not* a Galois extension.

Part V

Clifford theory

Chapter 18

Clifford Theory

If B is graded by a group Γ and every component contains a unit, how are B-modules and modules over the identity component $A = B_1$ related? Clifford's Theorem answers this for finite Γ: a simple B-module restricts to a semisimple A-module, whose simple summands form a single orbit under "twisting" by Γ, and the module is induced from a smaller algebra. We work out the example $\mathbb{C}[x] \rtimes S_2$.

18.1 Crossed products, again

We follow Lorenz [Lorenz18, §3.6.4].

Let Γ be a group. Recall from Theorem 12.9 that a Γ-*crossed product* is the same thing as a Γ-graded algebra

$$B = \bigoplus_{x \in \Gamma} B_x, \qquad B_x B_y \subseteq B_{xy}, \qquad 1_B \in B_{1_\Gamma},$$

such that every component B_x contains a unit u_x of B. We put $A = B_1$, the identity component, and choose $u_1 = 1$. By Lemma 12.8, $u_x^{-1} \in B_{x^{-1}}$, so for every $b \in B_x$,

$$b = \underbrace{b\, u_x^{-1}}_{\in B_x B_{x^{-1}} \subseteq A}\, u_x \in A\, u_x, \qquad \text{and similarly} \qquad b = u_x\,(u_x^{-1} b) \in u_x A.$$

So $B_x = A u_x = u_x A$. Conjugation $\sigma_x(a) = u_x a u_x^{-1}$ is an automorphism of A, and $a u_x = u_x\, \sigma_x^{-1}(a)$ for $a \in A$.

Question 18.1. What is the relationship between A-modules and B-modules?

Example 18.2. Let G be a group, $N \trianglelefteq G$ a normal subgroup, and $\Gamma = G/N$. Choose a set $\{\, \bar{x} \mid x \in \Gamma \,\} \subseteq G$ of coset representatives, and write $\Bbbk[\bar{x}N] = \operatorname{span}\{\, g \mid g \in \bar{x}N \,\}$. Then

$$\Bbbk G = \bigoplus_{g \in G} \Bbbk g = \bigoplus_{x \in \Gamma} \Bbbk[\bar{x}N]$$

is a Γ-grading, and $\bar{x} \in \Bbbk[\bar{x}N]$ is a unit. So $B = \Bbbk G$ is a Γ-crossed product, with identity component $A = \Bbbk N$. This is the setting of classical Clifford theory.

18.2 Twisting

Let B be a Γ-crossed product, $A = B_1$, and W an A-module. Each B_x is an (A, A)-bimodule (since $AB_xA \subseteq B_x$), so we can define the *twisted* A-modules

$$ {}^{x}W = B_x \otimes_A W \qquad (x \in \Gamma), $$

with A acting by left multiplication on B_x, and the *stabilizer* of W,

$$ \Gamma_W = \{\, x \in \Gamma \mid {}^{x}W \cong W \,\}. $$

Concretely: since $B_x = u_xA$, every element of ${}^{x}W$ can be written as $u_x \otimes w$, and $w \mapsto u_x \otimes w$ is a bijection $W \to {}^{x}W$. In terms of it,

$$ a \cdot (u_x \otimes w) = au_x \otimes w = u_x\sigma_x^{-1}(a) \otimes w = u_x \otimes \sigma_x^{-1}(a)w. $$

So ${}^{x}W$ is just W with $a \in A$ acting as $\sigma_x^{-1}(a)$. In Example 18.2, ${}^{x}W$ is the familiar conjugate representation of N: n acts as $\bar{x}^{-1}n\bar{x}$.

Lemma 18.3. *For $x, y \in \Gamma$, multiplication $B_x \otimes_A B_y \to B_{xy}$, $b \otimes c \mapsto bc$, is an isomorphism of (A, A)-bimodules. Consequently ${}^{1}W \cong W$ and ${}^{x}({}^{y}W) \cong {}^{xy}W$.*

Proof. The map is a well-defined bimodule map since multiplication is associative. Every element of $B_x \otimes_A B_y$ has the form $u_x \otimes c$ (as $u_xa \otimes c = u_x \otimes ac$), and $u_x \otimes c \mapsto u_xc$ is injective since u_x is a unit. It is surjective: for $d \in B_{xy}$, $d = u_x(u_x^{-1}d)$ with $u_x^{-1}d \in B_{x^{-1}}B_{xy} \subseteq B_y$. Then ${}^{x}({}^{y}W) = B_x \otimes_A (B_y \otimes_A W) \cong (B_x \otimes_A B_y) \otimes_A W \cong {}^{xy}W$ by associativity (Proposition 14.6), and ${}^{1}W = A \otimes_A W \cong W$. □

Lemma 18.4. *Γ_W is a subgroup of Γ, and ${}^{x}W \cong {}^{y}W$ if and only if $x\Gamma_W = y\Gamma_W$.*

Proof. Twisting by x is a functor, so it preserves isomorphisms. If $x, y \in \Gamma_W$, then by Lemma 18.3, ${}^{xy}W \cong {}^{x}({}^{y}W) \cong {}^{x}W \cong W$ and $W \cong {}^{x^{-1}}({}^{x}W) \cong {}^{x^{-1}}W$; also $1 \in \Gamma_W$. Similarly, ${}^{x}W \cong {}^{y}W$ if and only if (twisting by y^{-1}) ${}^{y^{-1}x}W \cong W$, i.e. $y^{-1}x \in \Gamma_W$. □

18.3 Multiplicities and homogeneous components

Definition 18.5. Let V be an A-module of *finite length*, that is, there is a chain of submodules

$$ 0 = V_0 \subset V_1 \subset \cdots \subset V_\ell = V $$

with every V_i/V_{i-1} simple (a *composition series*). For a simple A-module S, the *multiplicity of S in V* is

$$ \ell_S = \#\{\, i \mid V_i/V_{i-1} \cong S \,\}. $$

By the Jordan–Hölder theorem, ℓ_S does not depend on the choice of composition series.

The *S-homogeneous component* of V is

$$ V(S) = \sum \{\, U \subseteq V \mid U \text{ a submodule},\ U \cong S \,\}. $$

If V is semisimple of finite length, then $V(S) \cong S^{\oplus \ell_S}$ and

$$V = \bigoplus_{[S]} V(S),$$

the sum over the isomorphism classes of simple A-modules.

18.4 Clifford's Theorem

Theorem 18.6 (Clifford's Theorem). *Let Γ be a finite group, B a Γ-crossed product, $A = B_1$, and V a simple B-module.*

1) $\mathrm{Res}^B_A V$ *is semisimple of finite length.*
2) *Let S be a simple A-submodule of $\mathrm{Res}^B_A V$, and $\ell = \ell_S$ its multiplicity. Then*

$$\mathrm{Res}^B_A V \cong \bigoplus_{x\Gamma_S \in \Gamma/\Gamma_S} \left({}^xS\right)^{\oplus \ell},$$

   where x runs over representatives of the cosets of Γ_S. Thus the simple summands of $\mathrm{Res}^B_A V$ are exactly the twists ${}^xS$, which are pairwise non-isomorphic for different cosets, and they all occur with the same multiplicity ℓ.
3) *Put $B_S = \bigoplus_{x\in\Gamma_S} B_x$, a subalgebra of B. Then $V(S)$ is a B_S-submodule of V, it is simple as a B_S-module, and*

$$V \cong \mathrm{Ind}^B_{B_S} V(S) = B \otimes_{B_S} V(S).$$

Sketch. 1) $V = Bv$ for any $v \neq 0$, so $\mathrm{Res}^B_A V$ is generated by the finitely many elements $u_x v$. Hence it has a maximal submodule M. Each $u_x M$ is again a maximal A-submodule (as $Au_x = u_x A$), and $\bigcap_x u_x M$ is a B-submodule different from V, hence 0. So V embeds into $\bigoplus_x V/u_x M$, a finite direct sum of simple modules.

2) For each x, $u_x S$ is a simple A-submodule isomorphic to ${}^xS$ (via $u_x \otimes s \mapsto u_x s$). The sum $\sum_x u_x S$ is a B-submodule, hence equal to V, so every simple summand of $\mathrm{Res}^B_A V$ is some ${}^xS$. Moreover u_x maps $V(S)$ bijectively onto $V({}^xS)$, so all the ${}^xS$ have the same multiplicity. Lemma 18.4 says when two of them are isomorphic.

3) For $x \in \Gamma_S$ we get $u_x V(S) = V({}^xS) = V(S)$, so $V(S)$ is a B_S-module. Since B is a free right B_S-module on coset representatives, $B = \bigoplus_{x\Gamma_S} u_x B_S$, and $V = \bigoplus_{x\Gamma_S} u_x V(S)$ by 2), the map $B \otimes_{B_S} V(S) \to V$, $b \otimes v \mapsto bv$, is an isomorphism. If $U \subseteq V(S)$ is a nonzero B_S-submodule, then $BU = \bigoplus u_x U$ is a B-submodule, so $BU = V$ and $U = V(S)$. For the details see [Lorenz18, §3.6.4]. □

18.5 An example: $\mathbb{C}[x] \rtimes S_2$

Let $\Gamma = S_2 = \{1, \sigma\}$, $\sigma = (12)$, act on $\mathbb{C}[x]$ by $\sigma(p(x)) = p(-x)$, and let

$$B = \mathbb{C}[x] \rtimes S_2, \qquad \text{so that} \qquad \sigma x = -x\sigma \ \text{ and } \ \sigma^2 = 1 \ \text{ in } B.$$

(We write $\sigma(p)$ for the action and σp for the product in B; they are related by $\sigma p = \sigma(p)\sigma$.) Then

$$B = \underbrace{\mathbb{C}[x]\cdot 1}_{=A\cong\mathbb{C}[x]} \oplus \mathbb{C}[x]\sigma.$$

The isomorphism classes of simple A-modules are in bijection with $\mathbb{C}$:

$$\lambda \in \mathbb{C} \;\rightsquigarrow\; \mathbb{C}_\lambda = \mathbb{C}\cdot 1_\lambda, \qquad x\cdot 1_\lambda = \lambda 1_\lambda \quad \big(\text{so } p(x)1_\lambda = p(\lambda)1_\lambda\big).$$

Clearly ${}^1\mathbb{C}_\lambda \cong \mathbb{C}_\lambda$. For σ, note that $B_\sigma = \mathbb{C}[x]\sigma = \sigma\mathbb{C}[x]$ is free of rank one as a right $\mathbb{C}[x]$-module, so $\sigma\mathbb{C}[x] \cong \mathbb{C}\sigma \otimes_{\mathbb{C}} \mathbb{C}[x]$ as vector spaces and

$${}^\sigma\mathbb{C}_\lambda = \sigma\mathbb{C}[x] \otimes_{\mathbb{C}[x]} \mathbb{C}_\lambda \cong \mathbb{C}\sigma \otimes_{\mathbb{C}} \mathbb{C}_\lambda = \mathbb{C}\cdot(\sigma\otimes 1_\lambda),$$

with

$$x\cdot(\sigma\otimes 1_\lambda) = x\sigma\otimes 1_\lambda = -\sigma x\otimes 1_\lambda = -\sigma\otimes x1_\lambda = (-\lambda)\cdot(\sigma\otimes 1_\lambda).$$

Hence ${}^\sigma\mathbb{C}_\lambda \cong \mathbb{C}_{-\lambda}$, as the twisting formula (x acts as $\sigma^{-1}(x) = -x$) predicts. Writing $\Gamma_\lambda = \Gamma_{\mathbb{C}_\lambda}$ and $B_\lambda = B_{\mathbb{C}_\lambda}$, we get

$$\Gamma_\lambda = \begin{cases} \{1\}, & \lambda \neq 0, \\ S_2, & \lambda = 0, \end{cases} \qquad B_\lambda = \begin{cases} A = \mathbb{C}[x], & \lambda \neq 0, \\ B, & \lambda = 0. \end{cases}$$

The case $\lambda \neq 0$. Let V be a simple B-module, and suppose its restriction contains the simple $\mathbb{C}[x]$-submodule $S = \mathbb{C}_\lambda$ with $\lambda \neq 0$. By Clifford's Theorem,

$$\operatorname{Res}^B_{\mathbb{C}[x]} V \cong \big({}^1S\big)^{\oplus\ell} \oplus \big({}^\sigma S\big)^{\oplus\ell} = \mathbb{C}_\lambda^{\oplus\ell} \oplus \mathbb{C}_{-\lambda}^{\oplus\ell}, \qquad V \cong \operatorname{Ind}^B_{\mathbb{C}[x]} V(S),$$

where $V(S) \cong \mathbb{C}^\ell$ with x acting as $\lambda\,\mathrm{Id}$. Since $V(S)$ is a simple module over $B_S = \mathbb{C}[x]$, we get $\ell = 1$. So

$$V \cong \operatorname{Ind}^B_{\mathbb{C}[x]} \mathbb{C}_\lambda = B \otimes_{\mathbb{C}[x]} \mathbb{C}_\lambda,$$

which is 2-dimensional with basis $1\otimes 1_\lambda$, $\sigma\otimes 1_\lambda$, on which x acts as $\operatorname{diag}(\lambda, -\lambda)$ and σ swaps the two basis vectors. Conversely, this module is simple for $\lambda \neq 0$: a nonzero submodule contains an eigenvector of x, and σ moves it to the other eigenvector. Note that $\operatorname{Ind}\mathbb{C}_\lambda \cong \operatorname{Ind}\mathbb{C}_{-\lambda}$.

In summary (using Exercise 18.7 for $\lambda = 0$), the simple $\mathbb{C}[x]\rtimes S_2$-modules are, up to isomorphism, the modules $\operatorname{Ind}^B_{\mathbb{C}[x]}\mathbb{C}_\lambda$ for $\lambda \in (\mathbb{C}\setminus\{0\})/\{\pm 1\}$, and two 1-dimensional modules on which x acts by 0 and σ by ± 1.

18.6 Exercises

Exercise 18.7. Let $B = \mathbb{C}[x]\rtimes S_2$ as in the example.

(a) Let V be a simple B-module whose restriction to $\mathbb{C}[x]$ contains $\mathbb{C}_0$. Show that x acts by 0 on V, and conclude that V is one of the two 1-dimensional modules with $\sigma = \pm 1$. (Hint: $\Gamma_0 = S_2$, so Clifford's Theorem gives $\operatorname{Res} V \cong \mathbb{C}_0^{\oplus\ell}$.)

(b) Show that $\operatorname{Ind}^B_{\mathbb{C}[x]}\mathbb{C}_0$ is not simple; in fact it is the direct sum of the two 1-dimensional modules.

Exercise 18.8. Apply Clifford's Theorem to $B = \mathbb{C}S_3$ with $N = A_3 \cong \mathbb{Z}/3\mathbb{Z}$, so that $\Gamma = S_3/A_3 \cong \mathbb{Z}/2\mathbb{Z}$ (Example 18.2).

(a) Show that twisting by a transposition swaps the two nontrivial characters of A_3.
(b) Deduce that the 2-dimensional simple $\mathbb{C}S_3$-module restricts to their direct sum, while the trivial and the sign module both restrict to the trivial module.

Chapter 19

Examples of Clifford Theory Ideas

What survives of Clifford's Theorem when the group is infinite? For $A = R \rtimes \mathbb{Z}$ a simple A-module need not be semisimple over R, so we restrict attention to weight modules*, those that are. For a maximal ideal whose orbit under $\mathbb{Z}$ is infinite, induction produces the unique simple weight module containing that weight.*

19.1 Simple modules and maximal ideals

Let A be a $\Bbbk$-algebra. For v in an A-module M, write $\operatorname{ann}_A(v) = \{\, a \in A \mid a.v = 0 \,\}$, a left ideal, and $\operatorname{Ann}_A(M) = \{\, a \in A \mid aM = 0 \,\} = \bigcap_{v \in M} \operatorname{ann}_A(v)$, a two-sided ideal.

Proposition 19.1 (General fact)**.** *There is a bijection*

$$\left\{ \begin{array}{l} \text{pairs } (M, v) \text{: } M \text{ a simple } A\text{-module,} \\ v \in M \text{ nonzero, up to isomorphism} \end{array} \right\} \longleftrightarrow \{\, I \mid I \subsetneq A \text{ a maximal left ideal} \,\},$$

given by $(M, v) \mapsto \operatorname{ann}_A(v)$ and $I \mapsto (A/I,\ 1 + I)$. Here $(M, v) \cong (M', v')$ means that some isomorphism $M \to M'$ sends v to v'.

Proof. If M is simple and $v \neq 0$, then $a \mapsto av$ is a surjection $A \to M$ with kernel $\operatorname{ann}_A(v)$, so $(M, v) \cong (A/\operatorname{ann}_A(v), 1 + \operatorname{ann}_A(v))$, and $\operatorname{ann}_A(v)$ is a maximal left ideal because M is simple. Conversely, $\operatorname{ann}_A(1 + I) = I$. □

If R is commutative, maximal left ideals are just maximal ideals. Moreover, for a simple R-module $M = Rv$, the annihilator $\operatorname{ann}_R(v) = \operatorname{Ann}_R(M)$ does not depend on v. So the simple R-modules are, up to isomorphism, the modules $R/\mathfrak{m}$, one for each maximal ideal $\mathfrak{m}$. We write $\operatorname{MaxSpec}(R)$ for the set of maximal ideals of R.

Example 19.2. By the Nullstellensatz, if $\Bbbk$ is algebraically closed, every maximal ideal of $\Bbbk[x_1, \dots, x_n]$ has the form

$$\mathfrak{m}_\lambda = (x_1 - \lambda_1,\ x_2 - \lambda_2,\ \dots,\ x_n - \lambda_n), \qquad \lambda = (\lambda_1, \dots, \lambda_n) \in \Bbbk^n.$$

For a $\Bbbk[x_1, \dots, x_n]$-module M and $v \in M$,

$$\mathfrak{m}_\lambda v = 0 \iff (x_i - \lambda_i).v = 0 \text{ for all } i \iff x_i.v = \lambda_i v \text{ for all } i.$$

So v is a common eigenvector of the x_i, with eigenvalues λ_i.

Definition 19.3. Let R be a commutative ring and M an R-module. A nonzero $v \in M$ is a *weight vector* if $\mathfrak{m}v = 0$ for some maximal ideal $\mathfrak{m} \subset R$, its *weight*.

19.2 The algebra $R \rtimes \mathbb{Z}$

From now on, R is a commutative $\Bbbk$-algebra and $\sigma \in \mathrm{Aut}_{\Bbbk}(R)$ an automorphism of infinite order. Let

$$A = R \rtimes \langle \sigma \rangle_{\mathrm{grp}}$$

be the skew group algebra (Definition 12.3) of the infinite cyclic group $\langle \sigma \rangle_{\mathrm{grp}} \cong \mathbb{Z}$. Thus

$$A = \bigoplus_{n \in \mathbb{Z}} R\sigma^n = \bigoplus_{n \in \mathbb{Z}} \sigma^n R, \qquad \sigma r = \sigma(r)\,\sigma,$$

and we have $R \to A$, $r \mapsto r1_A$.

Remark 19.4. A is also a generalized Weyl algebra: $A \cong R(\sigma, 1)$, with $X \mapsto \sigma$ and $Y \mapsto \sigma^{-1}$. Indeed, in $R(\sigma, 1)$ the relations $YX = 1$ and $XY = \sigma(1) = 1$ say that X is invertible with inverse Y. So everything in Section 15.2 applies to A, with $Z^{(n)} = \sigma^n$.

A is a $\mathbb{Z}$-crossed product: $A = \bigoplus_{n \in \mathbb{Z}} A_n$ with $A_n = R\sigma^n$, $A_n A_m \subseteq A_{n+m}$, and $\sigma^n \in A_n$ is a unit. But $\mathbb{Z}$ is an *infinite* group, and Clifford's Theorem fails: a simple A-module need not be semisimple as an R-module.

Example 19.5. Let $R = \mathbb{C}[x]$ and $\sigma(p)(x) = p(x-1)$. Let $M = \mathbb{C}[x]$, with $p \in R$ acting by multiplication and σ acting as σ. This is an A-module, since $\sigma(rp) = \sigma(r)\sigma(p)$. An A-submodule is an ideal (f) of $\mathbb{C}[x]$ with $\sigma(f) \in (f)$. Then $f(x-1) = c\,f(x)$ for some $c \in \mathbb{C}$, so the finite set of roots of f is invariant under $a \mapsto a+1$, hence empty. So the only submodules are 0 and M, and M is simple. But M has no weight vectors at all: $(x-\lambda)p = 0$ forces $p = 0$. In particular M is not semisimple over $\mathbb{C}[x]$.

So instead of all A-modules, we look at a subcategory.

19.3 Weight modules

Definition 19.6. An *(R-)weight A-module* is an A-module M that is semisimple as an R-module, that is,

$$M = \bigoplus_{\mathfrak{m} \in \mathrm{MaxSpec}(R)} M_{\mathfrak{m}}, \qquad \text{where} \quad M_{\mathfrak{m}} = \{\, v \in M \mid \mathfrak{m}v = 0 \,\}.$$

The R-weight A-modules, with all A-module maps between them, form the category of weight modules.

The two descriptions agree. If M is semisimple as an R-module, then $M = \bigoplus_i M_i$ with each M_i simple, so $M_i \cong R/\mathfrak{m}_i$ for some $\mathfrak{m}_i$. Grouping the M_i by their maximal ideal gives the decomposition. Conversely, each $M_{\mathfrak{m}}$ is a vector space over the field $R/\mathfrak{m}$, hence a direct sum of copies of $R/\mathfrak{m}$. (The sum $\sum_{\mathfrak{m}} M_{\mathfrak{m}}$ is always direct, because distinct maximal ideals are comaximal.)

Example 19.7. Let $R = \mathbb{C}[x]$, so $\mathrm{MaxSpec}(R) = \{\, \mathfrak{m}_\lambda = (x-\lambda) \mid \lambda \in \mathbb{C} \,\}$. For an R-weight A-module M,

$$M = \bigoplus_{\lambda \in \mathbb{C}} M_{\mathfrak{m}_\lambda}, \qquad M_{\mathfrak{m}_\lambda} = \{\, v \in M \mid \mathfrak{m}_\lambda v = 0 \,\} = E_\lambda(x|_M),$$

where $E_\lambda(x|_M) = \{ v \in M \mid xv = \lambda v \}$ is the λ-eigenspace of x acting on M. So a $\mathbb{C}[x]$-weight A-module is the same as an A-module on which x is diagonalizable, in the sense that M has a basis of eigenvectors of x (Exercise 19.12).

Example 19.8. A matrix $L \in M_n(\mathbb{C})$ makes $\mathbb{C}^n$ into a $\mathbb{C}[x]$-module. The map $\{x\} \to M_n(\mathbb{C})$, $x \mapsto L$, extends uniquely to an algebra map $\pi : \mathbb{C}[x] \to M_n(\mathbb{C})$, and we let $p(x).v = \pi(p(x))v = p(L)v$. For instance, $Lw = \lambda w$ if and only if $(x - \lambda).w = 0$.

The action on maximal ideals. An automorphism σ of R permutes the maximal ideals: for $\mathfrak{m} \in \mathrm{MaxSpec}(R)$,

$$\sigma(\mathfrak{m}) = \{ \sigma(a) \mid a \in \mathfrak{m} \} \in \mathrm{MaxSpec}(R)$$

(Exercise 19.10).

19.4 Simple weight modules from infinite orbits

Theorem 19.9. *Let $A = R \rtimes \langle \sigma \rangle_{\mathrm{grp}}$, and let $\mathfrak{m} \in \mathrm{MaxSpec}(R)$ be such that*

$$\mathbb{Z} \longrightarrow \mathrm{MaxSpec}(R), \qquad n \longmapsto \sigma^n(\mathfrak{m}),$$

is injective. (We say that $\{\sigma^n(\mathfrak{m})\}_{n\in\mathbb{Z}}$ is a torsion-free orbit.*) Then $M = \mathrm{Ind}_R^A(R/\mathfrak{m})$ is a simple weight A-module with*

$$M = \bigoplus_{n\in\mathbb{Z}} M_{\sigma^n(\mathfrak{m})}, \qquad M_{\sigma^n(\mathfrak{m})} = \sigma^n \otimes R/\mathfrak{m}.$$

Up to isomorphism, M is the unique simple A-module with $M_{\mathfrak{m}} \neq 0$. In particular it is the unique simple weight A-module with $M_{\mathfrak{m}} \neq 0$.

Proof. The decomposition. As a right R-module, $A = \bigoplus_n \sigma^n R$ is free on $\{\sigma^n\}_{n\in\mathbb{Z}}$. Exactly as in Section 15.2, this gives an isomorphism of *vector spaces*

$$M = A \otimes_R R/\mathfrak{m} \cong \bigoplus_{n\in\mathbb{Z}} \sigma^n R \otimes_R R/\mathfrak{m},$$

where each summand $\sigma^n R \otimes_R R/\mathfrak{m} = \sigma^n \otimes R/\mathfrak{m}$ is a copy of $R/\mathfrak{m}$:

$$\sigma^n R \otimes_R R/\mathfrak{m} \cong (\Bbbk\sigma^n \otimes_\Bbbk R) \otimes_R R/\mathfrak{m} \cong \Bbbk\sigma^n \otimes_\Bbbk (R \otimes_R R/\mathfrak{m}) \cong \Bbbk\sigma^n \otimes_\Bbbk R/\mathfrak{m}.$$

These are isomorphisms of vector spaces only. The left R-action on $\sigma^n \otimes R/\mathfrak{m}$ is twisted: from $\sigma r = \sigma(r)\sigma$ we get $\sigma^n r = \sigma^n(r)\sigma^n$, so $r\sigma^n = \sigma^n \sigma^{-n}(r)$ and

$$r \cdot \left(\sigma^n \otimes m\right) = \sigma^n \otimes \sigma^{-n}(r)\, m.$$

In particular, for $a \in \mathfrak{m}$, moving a across the tensor sign,

$$\sigma^n(a) \cdot \left(\sigma^n \otimes (1 + \mathfrak{m})\right) = \sigma^n a \otimes (1 + \mathfrak{m}) = \sigma^n \otimes (a + \mathfrak{m}) = 0.$$

So $\sigma^n(\mathfrak{m})$ kills the summand $\sigma^n \otimes R/\mathfrak{m}$, which is therefore a simple R-module contained in $M_{\sigma^n(\mathfrak{m})}$.

Thus M is a weight module, and $M = \sum_n M_{\sigma^n(\mathfrak{m})}$. Here we use that the orbit is torsion-free: the maximal ideals $\sigma^n(\mathfrak{m})$ are pairwise distinct, and the sum of the weight spaces is direct, so each $M_{\sigma^n(\mathfrak{m})}$ is exactly the n-th summand $\sigma^n \otimes R/\mathfrak{m}$ (and $M_{\mathfrak{n}} = 0$ for $\mathfrak{n}$ outside the orbit). If $\sigma^k(\mathfrak{m}) = \sigma^n(\mathfrak{m})$ for some $k \neq n$, the weight space would contain both summands.

Simplicity is Exercise 19.11.

Uniqueness. Let V be a simple A-module and $v \in V_{\mathfrak{m}}$ nonzero. Then $\mathfrak{m} \subseteq \operatorname{ann}_R(v) \subsetneq R$, so $Rv \cong R/\mathfrak{m}$. By the adjunction $\operatorname{Ind}_R^A \dashv \operatorname{Res}_R^A$, the inclusion $R/\mathfrak{m} \cong Rv \hookrightarrow \operatorname{Res}_R^A V$ corresponds to a nonzero A-module map $M \to V$. Its image is a nonzero submodule of V, so the map is surjective, and its kernel is a proper submodule of the simple module M, so the map is injective. Hence $V \cong M$. (Compare Exercise 15.4.) □

19.5 Exercises

Exercise 19.10. Let σ be an automorphism of a commutative ring R. Show that $\sigma(\mathfrak{m})$ is a maximal ideal for every maximal ideal $\mathfrak{m}$, and that $\mathfrak{m} \mapsto \sigma(\mathfrak{m})$ is a bijection of $\operatorname{MaxSpec}(R)$.

Exercise 19.11. Show that the module M of Theorem 19.9 is simple. (Hint: a nonzero submodule $N \subseteq M$ is again a weight module, so it contains a nonzero vector of some $M_{\sigma^n(\mathfrak{m})} = \sigma^n \otimes R/\mathfrak{m}$. Apply σ^{-n} and use that $1 \otimes (1 + \mathfrak{m})$ generates M.)

Exercise 19.12. Let V be a complex vector space (possibly infinite-dimensional) and $L : V \to V$ a linear map. Make V into a $\mathbb{C}[x]$-module by $p(x).v = p(L)v$. Prove that V is a semisimple $\mathbb{C}[x]$-module if and only if L is diagonalizable, that is, V has a basis of eigenvectors of L.

Part VI

Growth and Gelfand–Kirillov dimension

Chapter 20

Growth of Algebras

How "big" is an infinite-dimensional algebra? Counting how fast the span of products of generators grows gives a useful measure. Polynomial growth of degree k means, roughly, "dimension k". This leads to the Gelfand–Kirillov dimension, which we define here and study in the rest of Part VI.

20.1 Growth

Let A be an algebra over a field $\Bbbk$, and let $V \subseteq A$ be a finite-dimensional generating subspace. Equivalently, V is the $\Bbbk$-span of a finite generating set of A; in particular A is *finitely generated*. Put $V^0 = \Bbbk 1_A$, $V^1 = V$, $V^2 = V \cdot V = \operatorname{span}\{\, xy \mid x, y \in V \,\}$, and in general

$$V^n = \operatorname{span}\{\, x_1 x_2 \cdots x_n \mid x_i \in V \,\}.$$

Let

$$V^{(n)} = \sum_{j=0}^{n} V^j, \qquad d_V(n) = \dim V^{(n)}.$$

Since V generates A, $A = \bigcup_{n \geq 0} V^{(n)}$. (If $1_A \in V$, then simply $V^{(n)} = V^n$.)

The *growth* of A with respect to V is measured by

$$\log_n d_V(n) = \frac{\log d_V(n)}{\log n}.$$

If $d_V(n)$ is roughly n^k, this is roughly k.

Example 20.1. Let $A = \Bbbk[x]$ and $V = \Bbbk x$. Then

$$d_V(n) = \dim(\Bbbk 1 + \Bbbk x + \cdots + \Bbbk x^n) = n + 1,$$

a polynomial in n of degree 1, and $\log_n d_V(n) = \frac{\log(n+1)}{\log n} \to 1$.

Example 20.2. Let $A = \Bbbk\langle x, y \rangle$ be the free algebra on two generators, and $V = \Bbbk x + \Bbbk y$. The words in x, y form a basis, so

$$d_V(n) = \#\{\text{words in } x, y \text{ of length} \leq n\} = 1 + 2 + 2^2 + \cdots + 2^n = 2^{n+1} - 1,$$

an exponential function of n. Here $\log_n d_V(n) \geq \log_n 2^n = \frac{n}{\log n} \log 2 \to \infty$.

Example 20.3. Let $A = \Bbbk[x_1, \dots, x_k]$ and $V = \Bbbk x_1 + \Bbbk x_2 + \cdots + \Bbbk x_k$. Then V^j is spanned by the monomials $x_1^{a_1} \cdots x_k^{a_k}$ of degree $a_1 + \cdots + a_k = j$. Such a monomial corresponds to a row of j stars and $k-1$ bars,

$$\underbrace{* \cdots *}_{a_1} \mid \underbrace{* \cdots *}_{a_2} \mid \cdots \mid \underbrace{* \cdots *}_{a_k},$$

so

$$\dim V^j = \binom{j+k-1}{k-1},$$

a polynomial in j of degree $k-1$. Summing (or counting monomials of degree $\le n$ as monomials of degree exactly n in $k+1$ variables),

$$d_V(n) = \sum_{j=0}^{n} \binom{j+k-1}{k-1} = \binom{n+k}{k} = \frac{(n+k)(n+k-1)\cdots(n+1)}{k!} = \frac{1}{k!}n^k + \cdots,$$

a polynomial in n of degree k.

20.2 Gelfand–Kirillov dimension

Next we let $n \to \infty$. The limit may not exist, so we take the lim sup instead.

Definition 20.4. The *Gelfand–Kirillov dimension* of a finitely generated algebra A is

$$\operatorname{GKdim} A = \limsup_{n\to\infty} \log_n d_V(n),$$

where V is a finite-dimensional generating subspace of A.

Lemma 20.5. $\operatorname{GKdim} A$ *does not depend on the choice of* V.

Proof. Let V and W be finite-dimensional generating subspaces. Since $A = \bigcup_n V^{(n)}$ and W is finite-dimensional, $W \subseteq V^{(m)}$ for some m. Then $W^j \subseteq V^{(mj)}$ for every j, so $W^{(j)} \subseteq V^{(mj)}$ and $d_W(j) \le d_V(mj)$. Hence

$$\log_j d_W(j) \le \log_j d_V(mj) = \frac{\log(mj)}{\log j} \cdot \log_{mj} d_V(mj).$$

The first factor tends to 1 as $j \to \infty$, so

$$\limsup_{j\to\infty} \log_j d_W(j) \le \limsup_{j\to\infty} \log_{mj} d_V(mj) \le \limsup_{n\to\infty} \log_n d_V(n).$$

Exchanging the roles of V and W (using $V \subseteq W^{(m')}$) gives the reverse inequality. □

So the examples give $\operatorname{GKdim} \Bbbk[x] = 1$, $\operatorname{GKdim} \Bbbk\langle x, y\rangle = \infty$ and, from $\frac{n^k}{k!} \le d_V(n) \le (n+k)^k$ in Example 20.3,

$$\operatorname{GKdim} \Bbbk[x_1, \dots, x_k] = k.$$

Definition 20.6. For an algebra A that is not necessarily finitely generated, we define

$$\operatorname{GKdim} A = \sup_{\substack{V \subseteq A \\ \dim V < \infty}} \Big(\limsup_{n \to \infty} \log_n d_V(n) \Big),$$

with $V^{(n)}$ and $d_V(n)$ defined as before.

For finitely generated A this agrees with Definition 20.4. Indeed, every finite-dimensional V is contained in a generating one, $V' = V + W$ with W generating, and $d_V \le d_{V'}$. So the supremum is attained on generating subspaces, where the value is $\operatorname{GKdim} A$ by Lemma 20.5.

20.3 A look ahead

1) If A is commutative and finitely generated, then $\operatorname{GKdim} A$ equals the *Krull dimension* of A,

$$\operatorname{Kdim} A = \sup\{\, n \mid \exists\, P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n \text{ in } \operatorname{Spec}(A) \,\}.$$

   (See Krause and Lenagan [KL00, Chapter 4].) Finite generation is needed. The field $\Bbbk(x)$ has Krull dimension 0, but $\operatorname{GKdim} \Bbbk(x) \ge 1$ (in fact $= 1$): taking $V = \Bbbk 1 + \Bbbk x$ in Definition 20.6 gives $d_V(n) = n + 1$.
2) The possible values are

$$\operatorname{GKdim} A \in \{0\} \cup \{1\} \cup [2, \infty].$$

   Finite-dimensional algebras have $\operatorname{GKdim} A = 0$. Values in $(0, 1)$ cannot occur, because an infinite-dimensional finitely generated algebra grows at least linearly (Borho–Kraft [BK76], Chapter 21). Values in $(1, 2)$ cannot occur by *Bergman's gap theorem* [KL00, Theorem 2.5]. Every value $r \ge 2$ occurs (Warfield, Chapter 22).
3) In Chapter 22 we will see how GKdim behaves under standard constructions: subalgebras, quotients A/I, and skew polynomial rings $A[x; \sigma, \delta]$.

20.4 Exercises

Exercise 20.7. For the Weyl algebra $A_1(\Bbbk)$ and $V = \Bbbk + \Bbbk x + \Bbbk \partial$, show that

$$V^{(n)} = \operatorname{span}\left\{\, x^i \partial^j \;\middle|\; i + j \le n \,\right\}.$$

Conclude directly from Definition 20.4 that $\operatorname{GKdim} A_1(\Bbbk) = 2$.

Exercise 20.8. Let $A = \Bbbk\langle x, y \mid xy = 0 \rangle$. Use the Diamond Lemma (Theorem 10.21) to show that the words $y^a x^b$ $(a, b \ge 0)$ form a basis of A. Compute $d_V(n)$ for $V = \Bbbk x + \Bbbk y$ and show that $\operatorname{GKdim} A = 2$. (So A has the same GK dimension as $\Bbbk[x, y]$, although it is not a domain.)

Chapter 21

Growth of Algebras II

The Gelfand–Kirillov dimension compresses the growth of an algebra into a single number. Here we keep more information: we compare growth functions up to a natural equivalence and attach to every finitely generated algebra its growth $\mathscr{G}(A)$. *Every infinite-dimensional finitely generated algebra grows at least linearly and at most exponentially (Borho–Kraft), and there are algebras whose growth lies strictly between polynomial and exponential.*

21.1 Comparing growth functions

Let A be a $\Bbbk$-algebra and $V \subseteq A$ a finite-dimensional generating subspace. As in Chapter 20, put

$$V^{(n)} = V^0 + V^1 + V^2 + \cdots + V^n \quad (V^0 = \Bbbk), \qquad d_V(n) = \dim_{\Bbbk} V^{(n)}.$$

Then $d_V : \mathbb{N} \to \mathbb{R}$, and we want to compare such functions. Let

$$\Phi = \{\, f : \mathbb{N} \to \mathbb{R} \mid f \text{ is eventually monotone increasing and positive-valued} \,\},$$

where *increasing* means $n \le m \Rightarrow f(n) \le f(m)$. A function in Φ may behave erratically for small n, as in Figure 21.1; only its behaviour for large n matters.

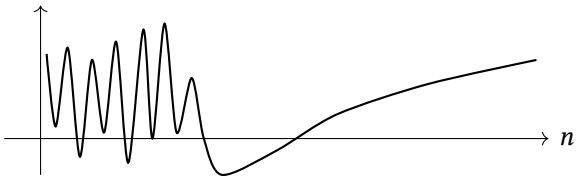


Figure 21.1: A function that is eventually monotone increasing.

Definition 21.1. 1) For $f, g \in \Phi$, write $f \preceq^* g$ if there are $c, m \in \mathbb{N}$ such that $f(n) \le c\, g(mn)$ for all $n \gg 0$.
2) Write $f \sim g$ if $f \preceq^* g$ and $g \preceq^* f$.
3) The equivalence class $\mathscr{G}(f) = [f]$ is called the *growth* of f.
4) $\preceq^*$ induces a *partial* order on $\Phi/\!\sim$, denoted $\preceq$.

Examples 21.2.

(1) Let f, g be monic polynomials, eventually positive,

$$f(n) = n^a + c_{a-1}n^{a-1} + \cdots + c_0, \qquad g(n) = n^b + d_{b-1}n^{b-1} + \cdots + d_0 \qquad (c_i, d_i \in \mathbb{R}),$$

with $a = \deg f$ and $b = \deg g$. Then $f \sim g$ if and only if $a = b$. Indeed, suppose $f \leq^* g$, so

$$n^a + \cdots \leq C\,(m^b n^b + \cdots) \qquad \text{for } n \gg 0.$$

Then

$$\lim_{n\to\infty} \frac{n^a + \cdots}{m^b n^b + \cdots} \leq C, \qquad \text{so } a \leq b.$$

So $f \sim g$ implies $a = b$. Conversely, if $a = b$ then $f(n)/g(n) \to 1$, so $f(n) \leq 2g(n)$ and $g(n) \leq 2f(n)$ for $n \gg 0$.
For $\gamma \in \mathbb{R}_{\geq 0}$, let $\mathscr{P}_\gamma$ be the growth of $n \mapsto n^\gamma$. The same argument shows $\mathscr{P}_\gamma \leq \mathscr{P}_\delta$ if and only if $\gamma \leq \delta$. Note that $\mathscr{P}_0$ is the class of bounded functions.

(2) For $\varepsilon \in \mathbb{R}_{>0}$, let $\mathscr{E}_\varepsilon$ be the growth of $n \mapsto e^{n^\varepsilon}$. Then

$$\varepsilon < \eta \iff \mathscr{E}_\varepsilon < \mathscr{E}_\eta.$$

If $\varepsilon \leq \eta$, then $e^{n^\varepsilon} \leq e^{n^\eta}$ for $n \geq 1$, so $\mathscr{E}_\varepsilon \leq \mathscr{E}_\eta$. If $\mathscr{E}_\eta \leq \mathscr{E}_\varepsilon$, then $e^{n^\eta} \leq c\, e^{(mn)^\varepsilon}$ for $n \gg 0$. Taking logarithms, $n^\eta \leq \log c + m^\varepsilon n^\varepsilon$, which forces $\eta \leq \varepsilon$. Together these give the equivalence. Similarly, $\mathscr{P}_\gamma < \mathscr{E}_\varepsilon$ for all $\gamma \geq 0$ and $\varepsilon > 0$.

(3) Let $f(n) = \log n$ (for $n \geq 2$). Then $\mathscr{G}(f) > \mathscr{P}_0$, but $\mathscr{G}(f) < \mathscr{P}_\varepsilon$ for every $\varepsilon > 0$. Indeed, $1 \leq \log n$ for $n \geq 3$, but $\log n$ is unbounded; and $\log n \leq n^\varepsilon$ for $n \gg 0$, but $n^\varepsilon / \log(mn) \to \infty$ for every m.

21.2 The growth of an algebra

Lemma 21.3. *If V, W are two finite-dimensional generating subspaces of an algebra A, then $\mathscr{G}(d_V) = \mathscr{G}(d_W)$.*

Proof. Since $A = \bigcup_n V^{(n)} = \bigcup_n W^{(n)}$ and V, W are finite-dimensional,

$$W \subseteq V^{(s)} \quad \text{for some } s > 0, \qquad V \subseteq W^{(t)} \quad \text{for some } t > 0.$$

Hence

$$W^{(n)} = \sum_{k=0}^{n} W^k \subseteq \sum_{k=0}^{n} \left(V^{(s)}\right)^k \subseteq V^{(sn)},$$

so $d_W(n) \leq d_V(sn)$ for all $n \geq 0$, and $d_W \leq^* d_V$. By symmetry $d_V \leq^* d_W$, so $d_V \sim d_W$ and $\mathscr{G}(d_V) = \mathscr{G}(d_W)$. □

Definition 21.4. The *growth* of a finitely generated algebra A is

$$\mathscr{G}(A) = \mathscr{G}(d_V),$$

where V is any finite-dimensional generating subspace.

A finitely generated algebra A has

- *polynomially bounded growth* if $\mathscr{G}(A) \le \mathscr{P}_m$ for some $m \in \mathbb{N}$ (for example, *polynomial growth* $\mathscr{G}(A) = \mathscr{P}_m$);
- *exponential growth* if $\mathscr{G}(A) = \mathscr{E}_1$;
- *subexponential growth* if $\mathscr{G}(A) < \mathscr{E}_1$;
- *intermediate growth* if its growth is subexponential but not polynomially bounded.

Example 21.5. $\mathscr{G}(\Bbbk\langle x, y\rangle) = \mathscr{E}_1$. With $V = \Bbbk x + \Bbbk y$ we have $d_V(n) = 2^{n+1} - 1$ (Chapter 20), and $2^{n+1} - 1 \le 2e^n$, while $e^n \le 4^n \le d_V(2n)$.

Example 21.6 (M. Smith)**.** Let

$$A = \Bbbk\langle x, y_1, y_2, y_3, \dots\rangle / I, \qquad I = \langle x y_i - y_i x - y_{i+1},\ y_i y_j - y_j y_i \text{ for all } i, j\rangle_{\text{ideal}}.$$

Since $y_{i+1} = x y_i - y_i x$, the algebra is generated by x and y_1, and we take $V = \Bbbk x + \Bbbk y_1$. In other words, A is the enveloping algebra $U(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$ with basis $x, y_1, y_2, \dots$ and brackets $[x, y_i] = y_{i+1}$, $[y_i, y_j] = 0$. Equivalently, it is the Ore extension $\Bbbk[y_1, y_2, \dots][x; \delta]$, where δ is the derivation with $\delta(y_i) = y_{i+1}$. Smith [Smith76] showed that $d_V(n) \sim e^{\sqrt{n}}$, that is,

$$\mathscr{G}(A) = \mathscr{E}_{1/2}$$

So A has intermediate growth.

Recall that $\operatorname{GKdim} A = \limsup_n \log_n d_V(n)$. Here $\log_n e^{\sqrt{n}} = \sqrt{n}/\log n \to \infty$, so $\operatorname{GKdim} A = \infty$, the same as for the free algebra $\Bbbk\langle x, y\rangle$. But $\mathscr{G}(A) = \mathscr{E}_{1/2} < \mathscr{E}_1 = \mathscr{G}(\Bbbk\langle x, y\rangle)$. So the growth distinguishes A from a free algebra, while the GK dimension does not: $\operatorname{GKdim} A = \infty$ does not force exponential growth.

Proposition 21.7. *Let A be finitely generated. Then $\mathscr{G}(A) = \mathscr{P}_0$ if and only if $\dim_{\Bbbk} A < \infty$.*

Proof. ($\Rightarrow$) If d_V is bounded, then, being increasing and integer-valued, it is eventually constant, so $V^{(n+1)} = V^{(n)}$ for some n. Then $V^{(n+2)} = \Bbbk + V \cdot V^{(n+1)} = \Bbbk + V \cdot V^{(n)} = V^{(n+1)} = V^{(n)}$, and by induction $V^{(n+k)} = V^{(n)}$ for all k. Hence $A = \bigcup_k V^{(k)} = V^{(n)}$ is finite-dimensional.

($\Leftarrow$) Take $V = A$; then d_V is constant. □

21.3 The Borho–Kraft bounds

Proposition 21.8 (Borho–Kraft)**.** *If A is finitely generated but not finite-dimensional, then*

$$\mathscr{P}_1 \le \mathscr{G}(A) \le \mathscr{E}_1.$$

Proof. By Lemma 21.3 we may replace V by $V + \Bbbk 1_A$, so assume $1_A \in V$. Then $V^j \subseteq V^{j+1}$ for all j, so

$$V^{(n)} = V^0 + V^1 + \cdots + V^n = V^n.$$

(For example, if V is spanned by a basis $1, a_1, a_2, \dots$, then $a_1 a_2 \in V^2$ and $a_1 a_2 = a_1 a_2 \cdot 1 \cdots 1 \in V^n$.) The multiplication map

$$V^{\otimes n} \longrightarrow V^n, \qquad a_1 \otimes \cdots \otimes a_n \longmapsto a_1 a_2 \cdots a_n,$$

is surjective, so

$$d_V(n) = \dim V^n \le \dim V^{\otimes n} = (\dim V)^n = e^{n \log \dim V} \le e^{mn}$$

for any integer $m \ge \log \dim V$. Hence $\mathscr{G}(A) = \mathscr{G}(d_V) \le \mathscr{E}_1$.

Also

$$V^n \subsetneq V^{n+1} \subsetneq V^{n+2} \subsetneq \cdots$$

for every n: if $V^n = V^{n+1}$, then, as in the proof of Proposition 21.7, $A = V^n$ would be finite-dimensional. So $d_V(n) = \dim V^n \ge n + 1$, and $\mathscr{G}(A) = \mathscr{G}(d_V) \ge \mathscr{P}_1$. □

See Borho and Kraft [BK76].

Corollary 21.9. *If A is finitely generated and infinite-dimensional, then* $\operatorname{GKdim} A \ge 1$. *Consequently* $\operatorname{GKdim} A \in \{0\} \cup [1, \infty]$ *for every algebra A.*

Proof. $d_V(n) \ge n + 1$ gives $\log_n d_V(n) \ge 1$. For a general algebra, $\operatorname{GKdim} A$ is the supremum of $\operatorname{GKdim} B$ over the finitely generated subalgebras B (each finite-dimensional $V \subseteq A$ generates one, with the same d_V), and each of these is 0 or ≥ 1. □

21.4 Exercises

Exercise 21.10. Let $A = \Bbbk[x]\langle y, z\rangle = \Bbbk\langle x, y, z\rangle / \langle xy - yx,\ xz - zx\rangle_{\text{ideal}}$, the polynomial ring $\Bbbk[x]$ with two new generators y, z adjoined (Chapter 3; this is the same construction $R\langle X, Y\rangle$ from which the generalized Weyl algebras of Definition 5.7 are built). Find the growth $\mathscr{G}(A)$ and $\operatorname{GKdim} A$.

Exercise 21.11. Show that $\le^*$ is reflexive and transitive on Φ. Deduce that $\sim$ is an equivalence relation and that $\le$ is a well-defined partial order on $\Phi/\!\sim$, as claimed in Definition 21.1.

Chapter 22

Gelfand–Kirillov Dimension and Constructions

How does the Gelfand–Kirillov dimension behave under the standard ways of building new algebras from old ones? It cannot go up when we pass to subalgebras or quotients, it is additive in a suitable sense for finite products, it goes up by exactly one for well-behaved skew polynomial rings (but not for all of them), and it drops by at least one when we factor out an ideal containing a regular element.

22.1 GK dimension and growth

Recall (Chapter 20) that for a finitely generated $\Bbbk$-algebra A,

$$\operatorname{GKdim} A = \limsup_{n\to\infty} \log_n d_V(n),$$

where $V \subseteq A$ is a finite-dimensional generating subspace and $d_V(n) = \dim V^{(n)}$; and that for an arbitrary algebra we take the supremum over all finite-dimensional subspaces V (Definition 20.6). We use the growth classes $\mathscr{G}$ and $\mathscr{P}_\rho$ of Chapter 21.

Lemma 22.1. *Let A be finitely generated.*

a)

$$\begin{aligned}\operatorname{GKdim} A &= \inf\{\,\rho \in \mathbb{R}_{\geq 0} \mid d_V(n) \leq n^\rho \text{ for } n \gg 0\,\}\\ &= \inf\left\{\,\rho \in \mathbb{R}_{\geq 0} \,\middle|\, \mathscr{G}(A) \leq \mathscr{P}_\rho\,\right\}.\end{aligned}$$

b) If $\mathscr{G}(A) = \mathscr{G}(B)$, then $\operatorname{GKdim} A = \operatorname{GKdim} B$.

Proof. a) Let $L = \limsup_n \log_n d_V(n)$. If $\rho > L$, then $\log_n d_V(n) < \rho$, i.e. $d_V(n) < n^\rho$, for $n \gg 0$. Conversely, if $d_V(n) \leq n^\rho$ for $n \gg 0$, then $L \leq \rho$. So the first infimum is L. If $d_V(n) \leq n^\rho$ for $n \gg 0$, then $\mathscr{G}(A) \leq \mathscr{P}_\rho$. Conversely, if $d_V(n) \leq c\,(mn)^\rho$ for $n \gg 0$, then $\log_n d_V(n) \leq \log_n c + \rho \log_n(mn) \to \rho$, so $L \leq \rho$. So the second infimum is L as well.

b) follows from a), since the right-hand side of a) only depends on $\mathscr{G}(A)$. □

Theorem 22.2 (Warfield)**.** *For every real number $r \geq 2$ there is a two-generated algebra $A = \Bbbk\langle x, y\rangle/I$ with $\operatorname{GKdim} A = r$.*

See [KL00, Theorem 2.9]. Together with Corollary 21.9 and Bergman's gap theorem, this shows that the possible values of GKdim are exactly $\{0\} \cup \{1\} \cup [2, \infty]$.

22.2 Subalgebras, quotients and products

Proposition 22.3. *If B is a subalgebra or a homomorphic image of A, then*

$$\operatorname{GKdim} B \le \operatorname{GKdim} A.$$

Proof. Say B is a subalgebra of A. Every finite-dimensional subspace $W \subseteq B$ is also a finite-dimensional subspace of A, with the same d_W, so the supremum defining $\operatorname{GKdim} B$ is at most the one defining $\operatorname{GKdim} A$. (Concretely, if A is generated by V, then $W \subseteq V + W$, which also generates A, so $d_W(n) \le d_{V+W}(n)$ and $\limsup \log_n d_W(n) \le \limsup \log_n d_{V+W}(n) = \operatorname{GKdim} A$.)

Now let $\varphi : A \twoheadrightarrow B$ be a surjective algebra map, and let $W \subseteq B$ be finite-dimensional. Choose a finite-dimensional $V \subseteq A$ with $\varphi(V) = W$. Then $\varphi(V^{(n)}) = W^{(n)}$, so $d_W(n) \le d_V(n)$ for all n, and again $\operatorname{GKdim} B \le \operatorname{GKdim} A$. □

Proposition 22.4. *For algebras A_1, A_2,*

$$\operatorname{GKdim}(A_1 \times A_2) = \max\{\operatorname{GKdim} A_1, \operatorname{GKdim} A_2\}.$$

Here $A_1 \times A_2$ is $A_1 \oplus A_2$ as a vector space, with componentwise multiplication.

Proof. The projections $\pi_i : A_1 \times A_2 \to A_i$ are surjective, so ($\ge$) holds by Proposition 22.3.

($\le$): Let γ be the right-hand side. If $\gamma = \infty$ there is nothing to prove. Otherwise, let $W \subseteq A_1 \times A_2$ be finite-dimensional and $U_i = \pi_i(W)$. Then $W \subseteq U_1 \oplus U_2$, and since products are computed componentwise,

$$W^{(n)} \subseteq U_1^{(n)} \oplus U_2^{(n)}.$$

Let $\varepsilon > 0$. Since $\limsup_n \log_n d_{U_i}(n) \le \operatorname{GKdim} A_i \le \gamma$, the argument of Lemma 22.1 gives

$$d_{U_i}(n) \le n^{\gamma+\varepsilon/2} \qquad (i = 1, 2) \quad \text{for } n \gg 0.$$

Hence

$$d_W(n) \le d_{U_1}(n) + d_{U_2}(n) \le 2n^{\gamma+\varepsilon/2} < n^{\varepsilon/2} \cdot n^{\gamma+\varepsilon/2} = n^{\gamma+\varepsilon} \qquad \text{for } n \gg 0.$$

So $\limsup_n \log_n d_W(n) \le \gamma + \varepsilon$ for every $\varepsilon > 0$ and every W, and $\operatorname{GKdim}(A_1 \times A_2) \le \gamma$. □

By induction, $\operatorname{GKdim}(A_1 \times \cdots \times A_k) = \max_i \operatorname{GKdim} A_i$.

Corollary 22.5. *Let $I_1, \dots, I_k \subseteq A$ be ideals. Then*

$$\operatorname{GKdim}\big(A/(I_1 \cap \cdots \cap I_k)\big) = \max_i \operatorname{GKdim}(A/I_i).$$

Proof. The map $a \mapsto (a + I_1, \dots, a + I_k)$ gives an injective algebra map

$$A/(I_1 \cap \cdots \cap I_k) \hookrightarrow A/I_1 \times \cdots \times A/I_k,$$

so ($\le$) holds by Propositions 22.3 and 22.4. On the other hand, $A/(I_1 \cap \cdots \cap I_k) \twoheadrightarrow A/I_i$ for each i, so ($\ge$) holds by Proposition 22.3. □

22.3 Skew polynomial rings

Let A be a $\Bbbk$-algebra, $\sigma \in \mathrm{Aut}_{\Bbbk}(A)$, and $\delta : A \to A$ a σ-derivation, i.e. a $\Bbbk$-linear map with $\delta(ab) = \delta(a)b + \sigma(a)\delta(b)$ for all a, b. The skew polynomial ring (Definition 4.5)

$$B = A[x;\sigma,\delta] = \bigoplus_{n=0}^{\infty} Ax^n \qquad \text{(free as a left } A\text{-module)}$$

has multiplication determined by $xa = \sigma(a)x + \delta(a)$ for $a \in A$.

Example 22.6. The Weyl algebra is

$$A_1(\Bbbk) = \Bbbk\langle x, \partial\rangle / \langle \partial x - x\partial - 1\rangle_{\text{ideal}} \cong \underbrace{\Bbbk[x]}_{A}\Big[\partial; \mathrm{Id}_A,\ \delta = \frac{d}{dx}\Big],$$

where the defining relation for $a = x$ reads $\partial x = x\partial + 1$.

Proposition 22.7. *Let A be finitely generated and $B = A[x;\sigma,\delta]$ as above.*

a) $\mathrm{GKdim}\, B \geq 1 + \mathrm{GKdim}\, A$.

b) If A has a finite-dimensional generating subspace V with $1 \in V$, $\sigma(V) \subseteq V$ and $\delta(V) \subseteq V^2$, then $\mathrm{GKdim}\, B = 1 + \mathrm{GKdim}\, A$.

Proof. a) Let $V \subseteq A$ be a finite-dimensional generating subspace with $1 \in V$, and put $W = V + \Bbbk x$, a generating subspace of B containing 1. Then $V^n x^i \subseteq W^{n+i} \subseteq W^{2n}$ for $i \leq n$, and the sum is direct because B is free over A on the powers of x:

$$V^n \oplus V^n x \oplus \cdots \oplus V^n x^n \subseteq (V + \Bbbk x)^{2n} = W^{2n}.$$

So $(n+1)\, d_V(n) \leq d_W(2n)$. Since $\log_n d_W(2n) = \frac{\log 2n}{\log n} \log_{2n} d_W(2n)$ and $\frac{\log 2n}{\log n} \to 1$,

$$\begin{aligned}\mathrm{GKdim}\, B &\geq \limsup_{n\to\infty} \log_n d_W(2n) \geq \limsup_{n\to\infty} \log_n\big((n+1)d_V(n)\big)\\ &= \lim_{n\to\infty} \log_n(n+1) + \limsup_{n\to\infty} \log_n d_V(n) = 1 + \mathrm{GKdim}\, A.\end{aligned}$$

b) Let V be as in b) and $W = V + \Bbbk x$. For $v_1, \ldots, v_a \in V$,

$$x\, v_1 \cdots v_a = \sigma(v_1 \cdots v_a)\, x + \delta(v_1 \cdots v_a), \qquad \delta(v_1 \cdots v_a) = \sum_{j=1}^{a} \sigma(v_1 \cdots v_{j-1})\, \delta(v_j)\, v_{j+1} \cdots v_a.$$

Since $\sigma(V) \subseteq V$ and $\delta(V) \subseteq V^2$, this shows $xV^a \subseteq V^a x + V^{a+1}$. By induction on n (using $V^n \subseteq V^{n+1}$, as $1 \in V$),

$$W^n \subseteq V^n + V^n x + \cdots + V^n x^n.$$

Hence $d_W(n) = \dim W^n \leq (n+1)\, d_V(n)$, and $\mathrm{GKdim}\, B \leq \limsup_n \log_n\big((n+1)d_V(n)\big) = 1 + \mathrm{GKdim}\, A$. □

Remark 22.8. Some hypothesis like the one in b) is needed. Let $A = \Bbbk[u^{\pm 1}, v^{\pm 1}]$, which has $\mathrm{GKdim}\, A = 2$, and let σ be the automorphism with $\sigma(u) = uv$, $\sigma(v) = u$. The exponents of the monomials $\sigma^n(u)$ grow like Fibonacci numbers, and one can show that $A[x;\sigma]$ has exponential growth, so $\mathrm{GKdim}\, A[x;\sigma] = \infty$. See [KL00, Chapter 3].

Corollary 22.9. *If A is finitely generated, then*

$$\operatorname{GKdim} A[x_1, \dots, x_n] = n + \operatorname{GKdim} A.$$

Proof. Apply Proposition 22.7 b) with $\sigma = \mathrm{Id}$ and $\delta = 0$, where the hypothesis holds for any V, and induct on n. □

Example 22.10. $\operatorname{GKdim} A_1(\Bbbk) = 2$: in $A_1(\Bbbk) = \Bbbk[x][\partial; \mathrm{Id}, \frac{d}{dx}]$, take $V = \Bbbk + \Bbbk x$. Then $\delta(V) \subseteq \Bbbk \subseteq V^2$, so Proposition 22.7 b) gives $\operatorname{GKdim} A_1(\Bbbk) = 1 + \operatorname{GKdim} \Bbbk[x] = 2$.

22.4 Quotients

Proposition 22.11. *Let $I \subseteq A$ be an ideal containing an element c which is* left regular *($ac = 0 \Rightarrow a = 0$) or* right regular *($ca = 0 \Rightarrow a = 0$). Then*

$$\operatorname{GKdim}(A/I) \leq \operatorname{GKdim}(A) - 1.$$

The proof is Exercise 22.12. For example, if $0 \neq f \in \Bbbk[x_1, \dots, x_k]$, then

$$\operatorname{GKdim} \Bbbk[x_1, \dots, x_k]/(f) \leq k - 1.$$

22.5 Generalized Weyl algebras

Let R be an integral domain, $t \in R \setminus \{0\}$ and $\sigma \in \operatorname{Aut}_{\Bbbk}(R)$, and recall the GWA (Definition 5.7)

$$R(\sigma, t) = R\langle X, Y\rangle \Big/ I, \qquad I = \langle YX - t,\ XY - \sigma(t),\ Xr - \sigma(r)X,\ Yr - \sigma^{-1}(r)Y\ (r \in R)\rangle_{\text{ideal}}.$$

By Theorem 6.4,

$$R(\sigma, t) \cong R[Y; \sigma^{-1}][X; \hat{\sigma}, \delta] \Big/ \langle YX - t\rangle_{\text{ideal}},$$

where $\hat{\sigma}(Y) = Y$, $\hat{\sigma}|_R = \sigma$, $\delta(r) = 0$ and $\delta(Y) = \sigma(t) - t$. The value of $\delta(Y)$ comes from rewriting $XY = \sigma(t) = t + (\sigma(t) - t) = YX + \delta(Y)$.

So what is $\operatorname{GKdim} R(\sigma, t)$? By the Remark above, the answer depends on σ. Under a hypothesis like the one in Proposition 22.7 b) it is $\operatorname{GKdim} R + 1$ (Exercise 22.13); for the Weyl algebra this gives 2 again. Without such a hypothesis it can be infinite: for $R = \Bbbk[u^{\pm 1}, v^{\pm 1}]$ and σ as in the Remark, $R(\sigma, 1) \cong R \rtimes \langle\sigma\rangle_{\text{grp}}$ (Chapter 19) contains $R[x; \sigma]$.

22.6 Exercises

Exercise 22.12. Prove Proposition 22.11. (Hint: say c is left regular. Given a finite-dimensional $\bar{V} \subseteq A/I$, choose a finite-dimensional $V \subseteq A$ containing 1 and c whose image contains $\bar{V}$. Choose $W_n \subseteq V^n$ mapping isomorphically onto $\bar{V}^n$. Show that $W_n + W_n c + \cdots + W_n c^n$ is a direct sum contained in V^{2n}, so that $(n+1)\dim \bar{V}^n \leq d_V(2n)$.)

Exercise 22.13. Let R be a finitely generated commutative domain, and suppose R has a finite-dimensional generating subspace V with $1, t \in V$ and $\sigma(V) \subseteq V$, $\sigma^{-1}(V) \subseteq V$. Show that $\operatorname{GKdim} R(\sigma, t) = \operatorname{GKdim} R + 1$. (Hint: for ($\geq$) argue as in Proposition 22.7 a), using Corollary 11.6. For ($\leq$), with $W = V + \Bbbk X + \Bbbk Y$, show that $W^n \subseteq \sum_{|k| \leq n} V^n Z^{(k)}$, where $Z^{(k)}$ is as in Chapter 15.) Check your answer against the Weyl algebra.

Chapter 23

More on Gelfand–Kirillov Dimension

We compute the GK dimension of skew group algebras $R \rtimes \mathbb{Z}$, under a hypothesis that turns out to be necessary. Then we introduce central localization, which does not change the GK dimension, and discuss what goes wrong for generalized Weyl algebras, where the natural localization is not central. Finally, the GK dimension of a filtered algebra equals that of its associated graded algebra. This gives $\operatorname{GKdim} U(\mathfrak{g}) = \dim \mathfrak{g}$.

23.1 Skew group algebras of $\mathbb{Z}$

Let R be an algebra and $\sigma \in \operatorname{Aut}(R)$. A group homomorphism $\alpha : \mathbb{Z} \to \operatorname{Aut}(R)$ is determined by $\sigma := \alpha(1)$, since then $\alpha(n) = \sigma^n$ for all $n \in \mathbb{Z}$. The *skew group algebra with respect to* α (Chapter 19) is

$$R \rtimes \mathbb{Z} = R \rtimes_\alpha \mathbb{Z} = \bigoplus_{n \in \mathbb{Z}} R\, z^n, \qquad (rz^n)(sz^m) = r\,\sigma^n(s)\, z^{n+m} \qquad (r, s \in R,\ n, m \in \mathbb{Z}).$$

In words: to move z^n past s, replace s by $\sigma^n(s)$. (We write z for the generator of $\mathbb{Z}$, since t will be needed for the generalized Weyl algebras below.)

$R \rtimes \mathbb{Z}$ is a quotient of an iterated skew polynomial ring. Let $\hat{\sigma}^{-1}$ be the automorphism of $R[z;\sigma]$ that extends σ^{-1} and fixes z, $\hat{\sigma}^{-1}(z) = z$. (It is well defined because it respects the relation $zr = \sigma(r)z$: $\hat{\sigma}^{-1}(z)\hat{\sigma}^{-1}(r) = z\sigma^{-1}(r) = rz = \hat{\sigma}^{-1}(\sigma(r)z)$.) Put

$$B = R[z;\sigma][z';\hat{\sigma}^{-1}].$$

In B we have $z'r = \sigma^{-1}(r)z'$ and $z'z = \hat{\sigma}^{-1}(z)z' = zz'$, and B is free as a left R-module on $\{z^k(z')^\ell\}_{k,\ell \geq 0}$.

Proposition 23.1. $R \rtimes \mathbb{Z} \cong B/\langle zz' - 1\rangle_{\text{ideal}}$.

Proof. The assignment $r \mapsto r$, $z \mapsto z$, $z' \mapsto z^{-1}$ respects the defining relations of B ($zr = \sigma(r)z$, $z'r = \sigma^{-1}(r)z'$ and $z'z = zz'$ all hold in $R \rtimes \mathbb{Z}$), so it gives a surjective algebra map $B \to R \rtimes \mathbb{Z}$ that kills $zz' - 1$. Let $\bar{B} = B/\langle zz' - 1\rangle_{\text{ideal}}$ and $\varphi : \bar{B} \to R \rtimes \mathbb{Z}$ the induced map. In $\bar{B}$, $zz' = 1 = z'z$, so every $z^k(z')^\ell$ equals $z^{k-\ell}$ or $(z')^{\ell-k}$. Hence $\bar{B}$ is spanned over R by the elements z^n ($n \geq 0$) and $(z')^m$ ($m > 0$). Their images z^n and z^{-m} form a basis of $R \rtimes \mathbb{Z}$ over R, so φ is injective. □

Lemma 23.2. *$zz' - 1$ is left regular in B: if $p\,(zz' - 1) = 0$, then $p = 0$.*

Proof. Write $p = \sum_{k,\ell} r_{k,\ell}\, z^k (z')^\ell$ with $r_{k,\ell} \in R$. Since z and z' commute,

$$p\,(zz' - 1) = \sum_{k,\ell} r_{k,\ell}\, z^{k+1} (z')^{\ell+1} - p.$$

If $p \neq 0$, let (k, ℓ) be lexicographically largest with $r_{k,\ell} \neq 0$. Then the coefficient of $z^{k+1}(z')^{\ell+1}$ in $p\,(zz' - 1)$ is $r_{k,\ell} \neq 0$, since p has no term of that (larger) index. So $p\,(zz' - 1) \neq 0$. □

Theorem 23.3. *Let R be finitely generated, with a finite-dimensional generating subspace V such that $1 \in V$ and $\sigma(V) = V$. Then*

$$\operatorname{GKdim}(R \rtimes \mathbb{Z}) = 1 + \operatorname{GKdim} R.$$

Proof. ($\leq$): By Proposition 22.7 b) with $\delta = 0$, $\operatorname{GKdim} R[z;\sigma] = 1 + \operatorname{GKdim} R$. The subspace $V' = V + \Bbbk z$ generates $R[z;\sigma]$, contains 1, and $\hat{\sigma}^{-1}(V') = \sigma^{-1}(V) + \Bbbk z = V'$. So, applying Proposition 22.7 b) again, $\operatorname{GKdim} B = 2 + \operatorname{GKdim} R$. By Proposition 23.1, Lemma 23.2 and Proposition 22.11,

$$\operatorname{GKdim}(R \rtimes \mathbb{Z}) \leq 2 + \operatorname{GKdim} R - 1 = 1 + \operatorname{GKdim} R.$$

($\geq$): The inclusion

$$R[z;\sigma] = \bigoplus_{n \geq 0} Rz^n \;\hookrightarrow\; \bigoplus_{n \in \mathbb{Z}} Rz^n = R \rtimes \mathbb{Z}, \qquad rz^n \mapsto rz^n,$$

is an injective algebra map, so by Proposition 22.3 and Proposition 22.7 a),

$$1 + \operatorname{GKdim} R \leq \operatorname{GKdim} R[z;\sigma] \leq \operatorname{GKdim}(R \rtimes \mathbb{Z}).$$

□

Remark 23.4. The hypothesis on σ cannot be dropped. For $R = \Bbbk[u^{\pm 1}, v^{\pm 1}]$ and $\sigma(u) = uv$, $\sigma(v) = u$, we saw in Chapter 22 that $R[z;\sigma]$, and hence $R \rtimes \mathbb{Z}$, has infinite GK dimension, although $\operatorname{GKdim} R = 2$. On the other hand, for $R = \Bbbk[u]$ and $\sigma(u) = u - 1$ we can take $V = \Bbbk + \Bbbk u$, and $\operatorname{GKdim}(\Bbbk[u] \rtimes \mathbb{Z}) = 2$.

23.2 Central localization

Localization means inverting some elements; the prototype is $\mathbb{Z} \rightsquigarrow \mathbb{Q}$. Here we focus on *central localizations*, where the "denominators" lie in the center of the algebra.

Let A be a $\Bbbk$-algebra and $S \subseteq A \setminus \{0\}$ a subset such that

1) every $s \in S$ is central: $sa = as$ for all $a \in A$;
2) every $s \in S$ is regular: $sa = 0 \Rightarrow a = 0$ (left and right regularity agree, as s is central);
3) S is multiplicatively closed: $st \in S$ for $s, t \in S$, and $1_A \in S$.

We want fractions $s^{-1}a$, and $s^{-1}a = t^{-1}b$ should mean $ta = sb$. So define an equivalence relation on $S \times A$ by

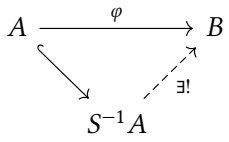
$$(s, a) \sim (t, b) \iff ta = sb,$$

and let $S^{-1}A = (S \times A)/\sim$. We write $s^{-1}a$ for the class of (s, a). Then (Exercise 23.8):

1) $S^{-1}A$ is an algebra with the operations

$$(s^{-1}a)(t^{-1}b) = (st)^{-1}ab, \qquad s^{-1}a + t^{-1}b = (st)^{-1}(ta + sb);$$

2) $A \to S^{-1}A$, $a \mapsto 1^{-1}a$, is an injective algebra map;
3) the image of every $s \in S$ is invertible in $S^{-1}A$;
4) if $\varphi : A \to B$ is an algebra map such that every $\varphi(s)$, $s \in S$, is invertible in B, then there is a unique algebra map $S^{-1}A \to B$ making the diagram



commute.

Theorem 23.5. *In the above setting,*

$$\operatorname{GKdim}(S^{-1}A) = \operatorname{GKdim} A.$$

Proof. Since $A \hookrightarrow S^{-1}A$, $(\geq)$ holds by Proposition 22.3. For $(\leq)$ see [KL00, Theorem 4.2]. □

23.3 Generalized Weyl algebras and localization

Let $R(\sigma, t)$ be a GWA (Definition 5.7). Recall that t is central in R, and suppose t is regular. Let

$$S = \langle \sigma^n(t) \mid n \in \mathbb{Z} \rangle_{\text{mon}} = \{\, \sigma^{n_1}(t)\sigma^{n_2}(t) \cdots \sigma^{n_k}(t) \mid n_i \in \mathbb{Z},\ k \geq 0 \,\} \subseteq R.$$

The elements of S are central and regular in R, but they are *not* central in $R(\sigma, t)$, since $Xs = \sigma(s)X$. So $S^{-1}R(\sigma, t)$ is not a central localization. However, S is an *Ore set* in $R(\sigma, t)$ (since $Xs = \sigma(s)X$, $Ys = \sigma^{-1}(s)Y$ and $\sigma(S) = S$), and the more general Ore localization $S^{-1}R(\sigma, t)$ exists; see [MR01, Chapter 2]. In it t becomes invertible, so X becomes invertible with $X^{-1} = t^{-1}Y$, and one can show

$$S^{-1}R(\sigma, t) \cong (S^{-1}R) \rtimes \mathbb{Z}$$

(Exercise 23.9).

Remark 23.6. This suggests computing

$$\operatorname{GKdim} R(\sigma, t) \overset{?}{=} \operatorname{GKdim} S^{-1}R(\sigma, t) = \operatorname{GKdim}\big((S^{-1}R) \rtimes \mathbb{Z}\big) \overset{?}{=} 1 + \operatorname{GKdim} S^{-1}R = 1 + \operatorname{GKdim} R.$$

Both steps marked "?" need care. Theorem 23.5 is about central localization, and non-central localization can raise the GK dimension: the division ring of fractions of $A_1(\mathbb{C})$ contains a free subalgebra on two generators, so it has infinite GK dimension, while $\operatorname{GKdim} A_1(\mathbb{C}) = 2$ [Makar-Limanov83]. And Theorem 23.3 needs a σ-stable generating subspace of $S^{-1}R$, which is not even finitely generated in general. Under the hypothesis of Exercise 22.13, the formula $\operatorname{GKdim} R(\sigma, t) = 1 + \operatorname{GKdim} R$ does hold, and that exercise proves it directly.

An aside. Let $\varphi : A \to B$ be a surjective algebra map with $\operatorname{GKdim} A = \operatorname{GKdim} B < \infty$, and let $I = \operatorname{Ker} \varphi$. Since $B \cong A/I$, Proposition 22.11 shows that I cannot contain any left or right regular element. (If $\operatorname{GKdim} A = \infty$, the bound $\operatorname{GKdim} A - 1$ gives no information.)

23.4 Filtered algebras

Theorem 23.7. *Let A be a filtered algebra (Definition 8.1) with filtration $(A_{(n)})_{n\geq 0}$ and associated graded algebra $\operatorname{gr} A = \bigoplus_{n\geq 0} A_{(n)}/A_{(n-1)}$. Assume $\dim A_{(n)} < \infty$ for all n and that $\operatorname{gr} A$ is finitely generated. Then*

$$\operatorname{GKdim} A = \operatorname{GKdim} \operatorname{gr} A.$$

Sketch. Note that $\dim A_{(n)} = \dim \bigoplus_{i\leq n} \operatorname{gr}_i A$. Suppose $\operatorname{gr} A$ is generated by homogeneous elements of degree $\leq d$, and let $U = \bigoplus_{i\leq d} \operatorname{gr}_i A$ and $V = A_{(d)}$. Then $U^{(n)} \subseteq \bigoplus_{i\leq dn} \operatorname{gr}_i A$ and $V^{(n)} \subseteq A_{(dn)}$. Conversely, a product of homogeneous generators of total degree $\leq n$ has at most n factors of positive degree, and consecutive factors of degree 0 can be multiplied together inside $\operatorname{gr}_0 A = A_{(0)}$. This gives $\bigoplus_{i\leq n} \operatorname{gr}_i A \subseteq U^{(2n+1)}$, and, lifting symbols and inducting on n, $A_{(n)} \subseteq V^{(2n+1)}$. Hence

$$d_V(n) \leq \dim A_{(dn)} \leq d_U(2dn+1), \qquad d_U(n) \leq \dim A_{(dn)} \leq d_V(2dn+1),$$

so $d_U \sim d_V$, and the GK dimensions agree by Lemma 22.1. For details see [KL00, Chapter 6]. □

Application. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra with basis $x_1, \dots, x_m$, and give $U(\mathfrak{g})$ the filtration of Example 8.4 with all $\deg x_i = 1$. By the Poincaré–Birkhoff–Witt theorem (Section 11.3), the ordered monomials in the x_i form a basis of $U(\mathfrak{g})$. Since $x_i x_j - x_j x_i = [x_i, x_j]$ has degree ≤ 1, the symbols $\bar{x}_i \in \operatorname{gr}_1 U(\mathfrak{g})$ commute, and

$$\operatorname{gr} U(\mathfrak{g}) \cong S(\mathfrak{g}) \cong \Bbbk[\bar{x}_1, \dots, \bar{x}_m],$$

the symmetric algebra on $\mathfrak{g}$. So by Theorem 23.7 and Corollary 22.9,

$$\operatorname{GKdim} U(\mathfrak{g}) = \operatorname{GKdim} \operatorname{gr} U(\mathfrak{g}) = \operatorname{GKdim} S(\mathfrak{g}) = \dim \mathfrak{g}.$$

For example (compare the exercises of Chapter 8),

$$\operatorname{GKdim} U(\mathfrak{sl}_2) = \operatorname{GKdim} \Bbbk[\bar{e}, \bar{f}, \bar{h}] = 3.$$

23.5 Exercises

Exercise 23.8. In the setting of central localization, check that $\sim$ is an equivalence relation and that properties 1)–4) hold. (Where do you use that the elements of S are central, and where that they are regular?)

Exercise 23.9. Let $R(\sigma, t)$ be a GWA with t regular in R, and S as above. Show that $S^{-1}R(\sigma, t) \cong (S^{-1}R) \rtimes \mathbb{Z}$. (You may take for granted that the Ore localization exists and that every element has the form $s^{-1}a$ with $s \in S$, $a \in R(\sigma, t)$. Hint: X becomes invertible, with inverse $t^{-1}Y$.)

Part VII

Quantum algebras and deformations

Chapter 24

Quantum Algebras

Quantum algebras are deformations of familiar commutative or enveloping algebras, depending on a parameter q. We study the quantum plane (its basis, its center, and what happens at roots of unity), its generalizations, the quantum affine spaces and quantum tori, and the quantized enveloping algebra $U_q(\mathfrak{sl}_2)$, which recovers $U(\mathfrak{sl}_2)$ in the limit $q \to 1$.

24.1 The quantum plane

Throughout, $\Bbbk$ is a field and $q \in \Bbbk \setminus \{0\}$. (A typical choice is $\Bbbk = \mathbb{Q}(q)$, with q an indeterminate.)

Example 24.1 (The quantum plane)**.**

$$\Bbbk_q[x,y] = \mathcal{O}_q(\mathbb{A}^2_\Bbbk) = \Bbbk\langle x,y\rangle / \langle yx - qxy\rangle_{\text{ideal}}.$$

We have $\Bbbk_q[x,y] \cong \Bbbk[x][y;\sigma,0]$, where $\sigma : \Bbbk[x] \to \Bbbk[x]$ is the automorphism with $\sigma(x) = qx$. Indeed, in the skew polynomial ring the defining relation for $a = x$ reads $yx = \sigma(x)y = qxy$, which is what we want, and by Exercise 4.11 this one relation suffices. The subspace $V = \Bbbk + \Bbbk x$ generates $\Bbbk[x]$ and satisfies $\sigma(V) \subseteq V$, so Proposition 22.7 b) gives

$$\operatorname{GKdim} \Bbbk_q[x,y] = 1 + \operatorname{GKdim} \Bbbk[x] = 1 + 1 = 2.$$

Claim 24.2. *The monomials $\mathcal{B} = \{x^k y^\ell\}_{k,\ell \geq 0}$ form a basis of $\Bbbk_q[x,y]$.*

Proof. Spanning. Since

$$x \cdot x^k y^\ell = x^{k+1} y^\ell, \qquad y \cdot x^k y^\ell = q^k x^k y^{\ell+1},$$

the span of $\mathcal{B}$ is closed under left multiplication by the generators, so it is a left ideal. It contains 1, so it is all of $\Bbbk_q[x,y]$.

Linear independence. Use the Diamond Lemma (Theorem 10.21) with the single reduction rule

$$yx \longrightarrow qxy$$

and the deg-lex order with $x < y$ (Example 10.16), which is compatible with it. The word yx does not overlap with itself or include itself in a nontrivial way, so there are no ambiguities to resolve. The irreducible words are those without yx, i.e. the $x^k y^\ell$. So $\mathcal{B}$ is linearly independent.

(Alternatively, the basis follows from $\Bbbk_q[x,y] \cong \Bbbk[x][y;\sigma]$ and Theorem 4.4.) □

The center. Let $A = \Bbbk_q[x, y]$. Since A is generated by x and y,

$$Z(A) = \{\, z \in A \mid za = az \text{ for all } a \in A \,\} = \{\, z \in A \mid zx = xz \text{ and } zy = yz \,\}.$$

Write $z = \sum_{k,\ell \geq 0} c_{k\ell}\, x^k y^\ell$ with $c_{k\ell} \in \Bbbk$. Since $y^\ell x = q^\ell x y^\ell$,

$$zx - xz = \sum_{k,\ell \geq 0} c_{k\ell}\,(q^\ell - 1)\, x^{k+1} y^\ell.$$

By the Claim, the monomials are linearly independent, so

$$zx - xz = 0 \iff \big(c_{k\ell} \neq 0 \Rightarrow q^\ell = 1\big).$$

Similarly, $zy - yz = \sum c_{k\ell}(1 - q^k) x^k y^{\ell+1}$, so $zy = yz \iff \big(c_{k\ell} \neq 0 \Rightarrow q^k = 1\big)$. Therefore

$$Z(A) = \begin{cases} \Bbbk & \text{if } q \text{ is not a root of unity,} \\ \Bbbk[x^d, y^d] & \text{if } q \text{ has multiplicative order } d = |\langle q \rangle_{\mathrm{grp}}|. \end{cases}$$

For $q = 1$ (so $d = 1$) this says that $\Bbbk[x, y]$ is commutative, as it should.

Remark 24.3. If q has order d, then A is *free of finite rank over its center*:

$$A = \bigoplus_{k,\ell \geq 0} \Bbbk\, x^k y^\ell = \bigoplus_{0 \leq k,\ell \leq d-1} Z(A)\, x^k y^\ell,$$

of rank d^2. This implies that A is a *PI-algebra* (an algebra satisfying a polynomial identity): there is a nonzero $f \in \Bbbk\langle T_1, T_2, \dots, T_N \rangle$ such that

$$f(a_1, a_2, \dots, a_N) = 0 \qquad \text{for all } a_i \in A$$

(Exercise 24.8). One can show that when q is not a root of unity, A is not PI. So at roots of unity the quantum plane behaves much more like a commutative algebra.

24.2 Quantum affine spaces

What should $\Bbbk_q[x_1, x_2, \dots, x_n]$ be? In general we allow a different parameter for each pair of variables. Let $Q = (q_{ij})_{i,j=1}^n$ be an $n \times n$ matrix with entries in $\Bbbk^\times$ such that

$$q_{ii} = 1 \quad \text{and} \quad q_{ij} q_{ji} = 1 \qquad \text{for all } i, j.$$

We say Q is *multiplicatively skew-symmetric*. (Special case: $q_{ij} = q^{s_{ij}}$, where $S = (s_{ij})$ is a skew-symmetric integer matrix.)

Definition 24.4. The *quantum affine space* (or multiparameter quantum polynomial ring) is

$$A_Q = \Bbbk_Q[x_1, \dots, x_n] = \frac{\Bbbk\langle x_1, \dots, x_n \rangle}{\langle x_i x_j - q_{ij} x_j x_i \text{ for all } i, j \rangle_{\mathrm{ideal}}}.$$

The quantum plane is the case $n = 2$, $x = x_1$, $y = x_2$, $q_{21} = q$.

By Exercise 24.9, A_Q is an iterated skew polynomial ring

$$A_Q \cong \Bbbk[x_1][x_2; \sigma_2, 0] \cdots [x_n; \sigma_n, 0],$$

where σ_i is the automorphism of $\Bbbk_Q[x_1, \dots, x_{i-1}]$ with $\sigma_i(x_j) = q_{ij}x_j$ for $j < i$. (The relation $x_i x_j = \sigma_i(x_j)x_i = q_{ij}x_j x_i$ is exactly the defining relation.)

Proposition 24.5. $\operatorname{GKdim} A_Q = n$.

Proof. Let $A_i = \Bbbk_Q[x_1, \dots, x_i]$ and $V_i = \Bbbk + \Bbbk x_1 + \cdots + \Bbbk x_i$, a generating subspace of A_i containing 1. Since $\sigma_{i+1}(V_i) \subseteq V_i$, Proposition 22.7 b) gives $\operatorname{GKdim} A_{i+1} = 1 + \operatorname{GKdim} A_i$. As $\operatorname{GKdim} A_1 = \operatorname{GKdim} \Bbbk[x_1] = 1$, induction gives $\operatorname{GKdim} A_n = n$. □

24.3 Quantum tori

The usual n-dimensional torus is $(S^1)^n = S^1 \times S^1 \times \cdots \times S^1$, and the complexified torus is $(\mathbb{C}^\times)^n = \mathbb{C}^\times \times \cdots \times \mathbb{C}^\times$. They are related as follows:

$$\underset{\text{real Lie group}}{(S^1)^n} \xrightarrow{\text{Lie alg}} \mathbb{R}^n \xrightarrow[\mathbb{C}\otimes_{\mathbb{R}}\mathfrak{g}]{\text{complexify}} \mathbb{C}^n \xrightarrow{\text{Exp}} \underset{\text{complex Lie group}}{(\mathbb{C}^\times)^n}$$

Here $\mathfrak{g} = \mathbb{R}^n$ is the Lie algebra of $(S^1)^n$, its complexification is $\mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g} = \mathbb{C}^n$, and the exponential map leads to the complex Lie group $(\mathbb{C}^\times)^n$.

Replacing $\mathbb{C}$ by $\Bbbk$, we get the algebraic torus $(\Bbbk^\times)^n$. Its algebra of regular functions is

$$\mathcal{O}\big((\Bbbk^\times)^n\big) \cong \Bbbk[x_1^{\pm 1}, \dots, x_n^{\pm 1}],$$

the Laurent polynomial ring.

Definition 24.6. The *quantum torus* is

$$\Bbbk_Q[x_1^{\pm 1}, \dots, x_n^{\pm 1}] = \frac{\Bbbk\langle x_1, x_1^{-1}, \dots, x_n, x_n^{-1}\rangle}{\langle x_i x_i^{-1} - 1,\ x_i^{-1}x_i - 1,\ x_i x_j - q_{ij}x_j x_i \text{ for all } i, j\rangle_{\text{ideal}}}.$$

It also has GK dimension n (Exercise 24.10).

24.4 Quantum groups

Recall the correspondences

$$\underset{\text{(connected, simply connected)}}{\text{Lie groups}} \longleftrightarrow \text{Lie algebras } \mathfrak{g} \longleftrightarrow \text{enveloping algebras } U(\mathfrak{g}).$$

Quantum groups are deformations of the enveloping algebras. We assume $q^2 \neq 1$, so that $q - q^{-1} \neq 0$.

Definition 24.7. The *quantized enveloping algebra* of $\mathfrak{sl}_2$ is

$$U_q(\mathfrak{sl}_2) = \frac{\mathbb{k}\langle E, F, K, K^{-1}\rangle}{\langle \text{relations}\rangle_{\text{ideal}}},$$

with relations

$$KK^{-1} = 1 = K^{-1}K, \qquad KE = q^2EK, \qquad KF = q^{-2}FK, \qquad EF - FE = \frac{K - K^{-1}}{q - q^{-1}}.$$

The limit $q \to 1$ (heuristic). We cannot simply set $q = 1$ in these relations, because of the denominator $q - q^{-1}$. Instead, assume $\operatorname{char}\mathbb{k} = 0$, let $\hbar$ be a formal variable, and pretend that

$$q = e^{\hbar}, \qquad K = e^{\hbar H} = \sum_{m=0}^{\infty} \frac{1}{m!} H^m \hbar^m,$$

as formal power series in $\hbar$, for a new generator H. Then:

- $KE = q^2EK$ becomes $e^{\hbar H}E = Ee^{\hbar(H+2)}$. This holds if $HE = E(H + 2)$, that is, $[H, E] = 2E$. Comparing coefficients of $\hbar$ shows that this is also necessary.
- Similarly $KF = q^{-2}FK$ becomes $[H, F] = -2F$.
- The last relation becomes

$$EF - FE = \frac{K - K^{-1}}{q - q^{-1}} = \frac{e^{\hbar H} - e^{-\hbar H}}{e^{\hbar} - e^{-\hbar}} = \frac{\sinh(\hbar H)}{\sinh \hbar} = \frac{\hbar H + \frac{\hbar^3 H^3}{6} + \cdots}{\hbar + \frac{\hbar^3}{6} + \cdots} = H + O(\hbar^2).$$

So, as $q \to 1$ (that is, $\hbar \to 0$), the relations become

$$[H, E] = 2E, \qquad [H, F] = -2F, \qquad EF - FE = H,$$

the defining relations of $U(\mathfrak{sl}_2)$.

This can be made precise. Let $U_\hbar(\mathfrak{sl}_2)$ be the $\mathbb{k}[\![\hbar]\!]$-algebra generated by E, F, H, completed in the $\hbar$-adic topology, with relations

$$[H, E] = 2E, \qquad [H, F] = -2F, \qquad [E, F] = \frac{\sinh(\hbar H)}{\sinh \hbar}.$$

It is a *topological algebra.* Setting $\hbar = 0$ gives $U_\hbar(\mathfrak{sl}_2)/\hbar U_\hbar(\mathfrak{sl}_2) \cong U(\mathfrak{sl}_2)$, and the assignment $q \mapsto e^{\hbar}$, $K \mapsto e^{\hbar H}$ relates $U_q(\mathfrak{sl}_2)$ to $U_\hbar(\mathfrak{sl}_2)$. Schematically:

$$\begin{array}{ccc} U_q(\mathfrak{sl}_2) & \longrightarrow & U_\hbar(\mathfrak{sl}_2) \\ & & \downarrow \\ & & U_\hbar(\mathfrak{sl}_2)/\langle\hbar\rangle_{\text{ideal}} \cong U(\mathfrak{sl}_2). \end{array}$$

For the precise statements, see Chari and Pressley [CP94, Chapter 6].

24.5 Exercises

Exercise 24.8. Let A be an algebra that is free of finite rank m over its center Z. Show that A satisfies the *standard identity*

$$s_{m+1}(T_1, \dots, T_{m+1}) = \sum_{\pi \in S_{m+1}} \operatorname{sgn}(\pi)\, T_{\pi(1)} T_{\pi(2)} \cdots T_{\pi(m+1)},$$

so A is a PI-algebra. Apply this to $\Bbbk_q[x, y]$ with q of order d. (Hint: s_{m+1} is Z-multilinear and alternating. A sharper result, the Amitsur–Levitzki theorem [AL50], says that the $n \times n$ matrices over a commutative ring satisfy s_{2n}.)

Exercise 24.9. Show that $A_Q \cong \Bbbk[x_1][x_2; \sigma_2, 0] \cdots [x_n; \sigma_n, 0]$, where σ_i is the automorphism of $\Bbbk_Q[x_1, \dots, x_{i-1}]$ with $\sigma_i(x_j) = q_{ij} x_j$ for $j < i$. (Check first that σ_i is well defined, i.e. that it respects the relations $x_j x_k = q_{jk} x_k x_j$. Then use Exercise 4.11.) Conclude that the ordered monomials $x_1^{a_1} \cdots x_n^{a_n}$ form a basis of A_Q.

Exercise 24.10. (a) Show that the quantum torus is an iterated skew-Laurent ring

$$\Bbbk_Q[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \cong \Bbbk[x_1^{\pm 1}][x_2^{\pm 1}; \sigma_2] \cdots [x_n^{\pm 1}; \sigma_n],$$

where $R[z^{\pm 1}; \sigma]$ denotes the skew group algebra $R \rtimes \mathbb{Z}$ of Chapter 23 and $\sigma_i(x_j^{\pm 1}) = q_{ij}^{\pm 1} x_j^{\pm 1}$ for $j < i$.

(b) Deduce from Theorem 23.3 that $\operatorname{GKdim} \Bbbk_Q[x_1^{\pm 1}, \dots, x_n^{\pm 1}] = n$. (Hint: σ_i maps the span of 1 and the $x_j^{\pm 1}$, $j < i$, onto itself.)

Chapter 25

Limits and the Limit Topology

Many "completed" objects, such as the profinite integers $\widehat{\mathbb{Z}}$, formal power series $\Bbbk[\![t]\!]$, and I-adic completions, are limits of systems of simpler objects. A limit carries a natural topology, in which a map to a discrete set is continuous when it is locally constant. Under suitable hypotheses this means it only depends on finitely much information. This is the setting for the $\hbar$-adic topology used for deformations in Chapter 26.

25.1 Limits

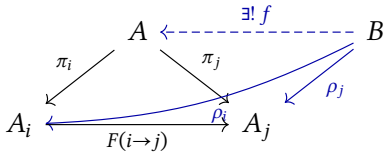
Let $\mathcal{J}$ be a small category and $F : \mathcal{J} \to \mathcal{C}$ a functor, $j \mapsto A_j$. So every arrow $i \to j$ in $\mathcal{J}$ gives a morphism $F(i \to j) : A_i \to A_j$.

Definition 25.1. A *limit* of F is an object A of $\mathcal{C}$ together with morphisms $\pi_j : A \to A_j$ $(j \in \mathcal{J})$ such that

1) $F(i \to j) \circ \pi_i = \pi_j$ for every arrow $i \to j$ in $\mathcal{J}$, and
2) for every object B with morphisms $\rho_j : B \to A_j$ satisfying $F(i \to j) \circ \rho_i = \rho_j$, there is a *unique* morphism $f : B \to A$ with $\pi_j \circ f = \rho_j$ for all j.

We write $A = \varprojlim_j A_j$, or $A = \varprojlim A_j$.

In pictures, for every arrow $i \to j$:



As with tensor products (Chapter 13), uniqueness in 2) implies that a limit is unique up to unique isomorphism, if it exists.

In our examples $\mathcal{J}$ is a partially ordered set, viewed as a category with exactly one arrow $i \to j$ when $i \geq j$ (and none otherwise).

Example 25.2. Let $\mathcal{J} = \{1, 2, 3, \dots\}$, with a (unique) arrow $i \to j$ if and only if $j \mid i$. Let $\mathcal{C} = \underline{\mathrm{Ab}}$ and $F(j) = \mathbb{Z}/j\mathbb{Z}$. For $j \mid i$, the arrow $i \to j$ goes to

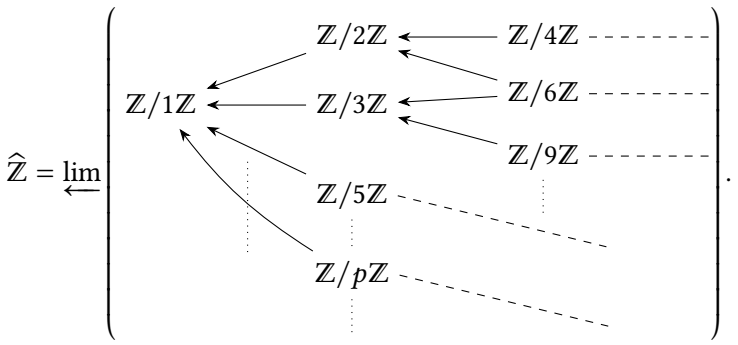
$$\mathbb{Z}/i\mathbb{Z} \twoheadrightarrow \mathbb{Z}/j\mathbb{Z}, \qquad [a]_i \longmapsto [a]_j,$$

which is well defined because $j \mid i \iff i\mathbb{Z} \subseteq j\mathbb{Z}$ (for example $6\mathbb{Z} \subseteq 3\mathbb{Z}$).

Definition 25.3. The *profinite integers* ("projective limit of finite things") are

$$\widehat{\mathbb{Z}} = \varprojlim \mathbb{Z}/n\mathbb{Z},$$

the limit of the system of Example 25.2:



In Ab (and similarly in Set and in rings), the limit is the set of compatible families with componentwise operations. Indeed, given compatible $\rho_j : B \to A_j$, the only possible f is $b \mapsto (\rho_j(b))_j$. So, concretely,

$$\widehat{\mathbb{Z}} = \left\{ (a_n)_{n=1}^{\infty} \in \prod_{n=1}^{\infty} \mathbb{Z}/n\mathbb{Z} \;\middle|\; j \mid i \Rightarrow a_i \equiv a_j \pmod{j} \right\},$$

where $a_i \equiv a_j \pmod{j}$ means that a_i maps to a_j under $\mathbb{Z}/i\mathbb{Z} \to \mathbb{Z}/j\mathbb{Z}$. The projections are $\pi_n : \widehat{\mathbb{Z}} \to \mathbb{Z}/n\mathbb{Z}$, $(a_k)_{k=1}^{\infty} \mapsto a_n$, and there is an injective group homomorphism

$$\mathbb{Z} \hookrightarrow \widehat{\mathbb{Z}}, \qquad a \longmapsto \big([a]_n\big)_{n=1}^{\infty}.$$

Example 25.4. Under this embedding,

$$17 \longmapsto \big(\overset{n=1}{0}, \overset{2}{1}, \overset{3}{2}, \overset{4}{1}, \overset{5}{2}, \overset{6}{5}, 3, \dots, \overset{n=16}{1}, \overset{17}{0}, \overset{18}{17}, \overset{19}{17}, 17, \dots\big).$$

Is every element of $\widehat{\mathbb{Z}}$ of this form? Here is a first attempt at something new. Fix a prime p and consider $\hat{a} = (a_n)_{n=1}^{\infty}$ with

$$a_n = \begin{cases} 1, & p \mid n, \\ 0, & p \nmid n. \end{cases}$$

Is $a_i \equiv a_j \pmod{j}$ whenever $j \mid i$? No: for $p = 2$, $i = 6$ and $j = 3$ we get $a_6 = 1$ but $a_3 = 0$, and $1 \not\equiv 0 \pmod{3}$. So $\hat{a} \notin \widehat{\mathbb{Z}}$. Nevertheless, $\widehat{\mathbb{Z}}$ is strictly bigger than $\mathbb{Z}$ (Exercise 25.15).

Example 25.5. Formal power series are a limit:

$$\Bbbk[\![t]\!] \cong \varprojlim \frac{\Bbbk[t]}{(t)^n} = \varprojlim\Big(\frac{\Bbbk[t]}{(t)} \longleftarrow \frac{\Bbbk[t]}{(t)^2} \longleftarrow \frac{\Bbbk[t]}{(t)^3} \longleftarrow \cdots\Big),$$

with

$$\pi_n : \Bbbk[\![t]\!] \longrightarrow \frac{\Bbbk[t]}{(t)^n}, \qquad \sum_{k=0}^{\infty} c_k t^k \longmapsto \sum_{k=0}^{n-1} c_k t^k.$$

Example 25.6. Let R be a ring and $I \subseteq R$ an ideal. The *I-adic completion* of R is

$$\widehat{R}_I = \varprojlim R/I^n = \varprojlim\big(R/I \longleftarrow R/I^2 \longleftarrow \cdots\big),$$

taken in $\underline{\mathrm{Ab}}$. The componentwise multiplication of compatible families makes $\widehat{R}_I$ a ring. Equivalently, the limit can be taken in $\underline{\mathrm{Rng}}$, the category of rings (with 1) and ring homomorphisms, and it has the same underlying group. For $R = \Bbbk[t]$ and $I = (t)$ we recover Example 25.5; for $R = \mathbb{Z}$ and $I = (p)$ we get the p-adic integers $\mathbb{Z}_p$.

25.2 Topology

Let X be a set.

Definition 25.7. A *topology* on X is a subset $\tau \subseteq \mathscr{P}(X)$ of the power set of X such that

1) $\emptyset, X \in \tau$;
2) if $U_\alpha \in \tau$ for all $\alpha \in I$, then $\bigcup_{\alpha\in I} U_\alpha \in \tau$;
3) if $U_1, \dots, U_n \in \tau$, then $U_1 \cap \cdots \cap U_n \in \tau$.

Elements of τ are called *open* (with respect to τ). A *topological space* is a set X together with a topology τ on X. A map $f : X \to Y$ between topological spaces is *continuous* if $f^{-1}(U) \in \tau_X$ for every $U \in \tau_Y$.

Example 25.8. The *discrete topology* on a set X is $\tau = \mathscr{P}(X)$. Equivalently, it is the topology in which every singleton $\{x\}$ is open.

Definition 25.9. If $\mathscr{A} \subseteq \mathscr{P}(X)$, the topology *generated* by $\mathscr{A}$, written $\langle\mathscr{A}\rangle$, is the smallest topology containing $\mathscr{A}$:

$$\langle\mathscr{A}\rangle = \{\text{arbitrary unions of finite intersections of sets in } \mathscr{A}\},$$

with the conventions that the empty intersection is X and the empty union is $\emptyset$.

Now let $A = \varprojlim A_j$ be a limit in a *concrete* category $\mathcal{C}$: one with a faithful "forgetful" functor $\mathcal{C} \to \underline{\mathrm{Set}}$ that preserves these limits, so that elements of A are compatible families $(a_j)_j$. This holds for $\underline{\mathrm{Ab}}$, $\underline{\mathrm{Rng}}$ and our other examples.

Definition 25.10. Give each A_j the discrete topology. The *limit topology* on A is the topology with the fewest open sets such that all the $\pi_j : A \to A_j$ are continuous. That is, it is the topology generated by

$$\mathscr{A} = \left\{ \pi_j^{-1}(\{x\}) \;\middle|\; j \in \mathcal{J},\ x \in A_j \right\}.$$

For example, in $\mathcal{C} = \underline{\mathrm{Ab}}$, the kernels $\operatorname{Ker}\pi_j = \pi_j^{-1}(\{0\})$ are open, and the sets $\pi_j^{-1}(\{x\})$ are their cosets. For $\Bbbk[\![t]\!]$ these are the sets $f + t^n\Bbbk[\![t]\!]$: this is the *t-adic topology*.

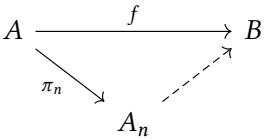
Note 25.11. A map $f : X \to Y$ to a discrete space Y is continuous if and only if it is *locally constant*: every $x \in X$ has an open neighbourhood U such that $f|_U$ is constant (Exercise 25.16).

25.3 Maps to discrete sets

Proposition 25.12. *Let $A = \varprojlim A_j$ with the limit topology, let B be a discrete space, and $f : A \to B$ a map.*

1) If f factors through some π_n, then f is continuous:

$\exists n :$



2) Suppose $\mathcal{J}$ is a directed *poset (any i, j have some k with arrows $k \to i$ and $k \to j$) and every A_j is finite. Then, conversely, every continuous f factors through some π_n.*

Proof. 1) If $f = g \circ \pi_n$, then f is continuous, since π_n is and every map out of the discrete space A_n is.

2) (Sketch.) If $k \to i$, then $\pi_i = F(k \to i) \circ \pi_k$, so $\pi_i^{-1}(x)$ is a union of sets $\pi_k^{-1}(z)$. By directedness, the sets $\pi_k^{-1}(z)$ therefore form a basis of the limit topology. Moreover, A is a closed subset of the product $\prod_j A_j$ of finite discrete spaces, so it is compact by Tychonoff's theorem. Since f is locally constant, A is covered by basic sets on which f is constant, and finitely many of them, say $\pi_{k_1}^{-1}(z_1), \dots, \pi_{k_r}^{-1}(z_r)$, already cover A. Choose n with arrows $n \to k_1, \dots, n \to k_r$. Each nonempty fibre $\pi_n^{-1}(w)$ lies inside one of these basic sets, so f is constant on it. Hence $f = g \circ \pi_n$ for some $g : A_n \to B$. □

Remark 25.13. Some finiteness hypothesis is needed in 2). Let $\Bbbk$ be infinite, choose distinct $\lambda_1, \lambda_2, \dots \in \Bbbk$, and define $f : \Bbbk[\![t]\!] \to \Bbbk$ (discrete) by

$$f\Big(\sum_k c_k t^k\Big) = \begin{cases} c_m & \text{if } c_0 = \lambda_m, \\ 0 & \text{if } c_0 \notin \{\lambda_1, \lambda_2, \dots\}. \end{cases}$$

Then f is locally constant, hence continuous for the t-adic topology. But f does not factor through any single π_n, since for $c_0 = \lambda_n$ it depends on c_n.

Example 25.14. Give $\Bbbk^\times = \mathrm{GL}(1, \Bbbk)$ the discrete topology. The divisibility poset is directed (take the least common multiple) and each $\mathbb{Z}/n\mathbb{Z}$ is finite, so by Proposition 25.12 a continuous group homomorphism

$$\chi : \widehat{\mathbb{Z}} \longrightarrow \Bbbk^\times$$

is the same as a composite $\widehat{\mathbb{Z}} \xrightarrow{\pi_n} \mathbb{Z}/n\mathbb{Z} \xrightarrow{\psi} \Bbbk^\times$ with ψ a group homomorphism, for some n. Such a χ is determined by $\chi(1) = \psi([1])$, where $1 \in \mathbb{Z} \subseteq \widehat{\mathbb{Z}}$, and $\chi(1)^n = 1$. We get a bijection

$$\{\text{roots of unity in } \Bbbk\} \longleftrightarrow \{\text{1-dimensional } \textit{continuous} \text{ representations of } \widehat{\mathbb{Z}}\},$$

sending a root of unity ε of order n to $\widehat{\mathbb{Z}} \to \mathbb{Z}/n\mathbb{Z} \to \Bbbk^\times$, $[1] \mapsto \varepsilon$, with inverse $\chi \mapsto \chi(1)$.

25.4 Exercises

Exercise 25.15. Find an element $\hat{b} \in \widehat{\mathbb{Z}} \setminus \mathbb{Z}$. (Hint: use the Chinese remainder theorem to choose b_n compatibly, so that $\hat{b}$ is congruent to 0 modulo every power of 2 and to 1 modulo every power of 3. Why can no integer do this?)

Exercise 25.16. Show that a map from a topological space to a discrete space is continuous if and only if it is locally constant.

Chapter 26

Deformation Theory

A deformation of an algebra A is a new associative multiplication on formal power series $A[\![\hbar]\!]$ which reduces to the old one at $\hbar = 0$. We have already met an example, $U_\hbar(\mathfrak{sl}_2)$. Asking when such a multiplication is associative to first order leads directly to Hochschild cohomology. Following Hochschild [Hochschild45] and Gerstenhaber [Gerstenhaber64], $H^2(A)$ classifies infinitesimal deformations, and $H^3(A)$ contains the obstructions to extending them.

26.1 Deformations

Let A be an algebra over $\Bbbk$ and $\hbar$ an indeterminate. We have the rings $\Bbbk[\hbar]$ and $\Bbbk[\![\hbar]\!]$, and the inclusions

$$A[\hbar] = \Bbbk[\hbar] \otimes_\Bbbk A \;\subseteq\; \Bbbk[\![\hbar]\!] \otimes_\Bbbk A \;\subseteq\; A[\![\hbar]\!] = \left\{ \sum_{n=0}^{\infty} a_n \hbar^n \;\middle|\; a_n \in A \right\}.$$

The last one is the largest. The middle inclusion is $f \otimes b \mapsto fb$. Its image consists of the series whose coefficients span a finite-dimensional subspace of A. Indeed, if

$$f_1 \otimes b_1 + \cdots + f_m \otimes b_m \longmapsto \sum_{k=0}^{\infty} a_k \hbar^k,$$

then every a_k lies in $\operatorname{span}\{b_1, \ldots, b_m\}$. Conversely, a series whose coefficients lie in the span of $b_1, \ldots, b_m$ is such a sum. So if A is infinite-dimensional, $A[\![\hbar]\!]$ is strictly bigger; for example, $\sum_k x^k \hbar^k \in \Bbbk[x][\![\hbar]\!]$. Note that $A[\![\hbar]\!] = \varprojlim A[\hbar]/(\hbar^n)$, and we give it the corresponding *$\hbar$-adic topology* (Chapter 25).

We look for a new multiplication

$$\mu : A[\![\hbar]\!] \times A[\![\hbar]\!] \longrightarrow A[\![\hbar]\!]$$

such that

- μ is $\Bbbk[\![\hbar]\!]$-bilinear (so it can be viewed as a map on $A[\![\hbar]\!] \otimes_{\Bbbk[\![\hbar]\!]} A[\![\hbar]\!]$);
- μ is continuous with respect to the $\hbar$-adic topology;
- for $a, b \in A$, $\mu(a \otimes b) = \mu_0(a \otimes b) + \mu_1(a \otimes b)\hbar + \mu_2(a \otimes b)\hbar^2 + \cdots$ with linear maps $\mu_k : A \otimes A \to A$, and $\mu_0(a \otimes b) = ab$ is the old multiplication;

- $\mu(1 \otimes b) = b = \mu(b \otimes 1)$ for all $b \in A$, i.e. $\mu_k(1 \otimes b) = 0 = \mu_k(b \otimes 1)$ for $k \geq 1$.

By bilinearity and continuity, μ is determined by the μ_k:

$$\mu\Big(\sum_i a_i \hbar^i \otimes \sum_j b_j \hbar^j\Big) = \sum_{i,j,k} \mu_k(a_i \otimes b_j)\, \hbar^{i+j+k}.$$

Suppose we have such a μ. When is $(A[\![\hbar]\!], \mu, 1)$ an associative algebra?

Definition 26.1. If $(A[\![\hbar]\!], \mu, 1)$ is associative, it is called a (formal) *deformation* of A.

For example, $U_\hbar(\mathfrak{sl}_2)$ of Chapter 24 is a deformation of $U(\mathfrak{sl}_2)$ in this sense.

Definition 26.2. An *infinitesimal deformation* (or *first order deformation*) of A is a triple $(A[\![\hbar]\!], \mu, 1)$ as above such that μ is associative modulo $\hbar^2$.

Associativity modulo $\hbar^2$ only involves μ_1; the higher μ_k play no role. So one could equally work on $A[\hbar]/(\hbar^2)$, with $(a+b\hbar)(c+d\hbar) = ac + \big(ad + bc + \mu_1(a\otimes c)\big)\hbar$. We keep $A[\![\hbar]\!]$ because the eventual goal is to extend a first order deformation to a full deformation. That usually requires infinitely many nonzero μ_k, so the result lives on power series in $\hbar$, not polynomials (see Remark 26.8).

Let us compute the associator to first order. For $a, b, c \in A$,

$$\begin{aligned}
&\mu(a \otimes \mu(b \otimes c)) - \mu(\mu(a \otimes b) \otimes c) \\
&\quad = \mu\big(a \otimes (bc + \mu_1(b \otimes c)\hbar + \cdots)\big) - \mu\big((ab + \mu_1(a \otimes b)\hbar + \cdots) \otimes c\big) \\
&\quad = a\big(bc + \mu_1(b \otimes c)\hbar + \cdots\big) + \mu_1\big(a \otimes (bc + \cdots)\big)\hbar + \cdots \\
&\qquad - \Big[\big(ab + \mu_1(a \otimes b)\hbar + \cdots\big)c + \mu_1\big((ab + \cdots) \otimes c\big)\hbar + \cdots\Big] \\
&\quad = \Big[\boxed{a\mu_1(b \otimes c) + \mu_1(a \otimes bc) - \mu_1(a \otimes b)c - \mu_1(ab \otimes c)}\Big]\hbar + \hbar^2(\cdots).
\end{aligned}$$

The $\hbar^0$ terms $a(bc)$ and $(ab)c$ cancel, by associativity of A. So $(A[\![\hbar]\!], \mu, 1)$ is an infinitesimal deformation of A if and only if

$$(*) \qquad a\mu_1(b \otimes c) - \mu_1(ab \otimes c) + \mu_1(a \otimes bc) - \mu_1(a \otimes b)c = 0 \qquad \text{for all } a, b, c \in A.$$

Definition 26.3. A *Hochschild n-cocycle on A* is a linear map $f : A^{\otimes n} \to A$ such that

$$\begin{aligned}
(**) \qquad a_0 f(a_1 \otimes \cdots \otimes a_n) + \sum_{k=1}^{n} (-1)^k f(a_0 \otimes \cdots \otimes \underbrace{a_{k-1}a_k}_{k\text{-th slot}} \otimes \cdots \otimes a_n) \\
+ (-1)^{n+1} f(a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1})\, a_n = 0
\end{aligned}$$

for all $a_0, a_1, \ldots, a_n \in A$.

So condition $(*)$ says precisely that μ_1 is a Hochschild 2-cocycle.

26.2 Hochschild cohomology

Definition 26.4. An *n-cochain* on A is a linear map $f : A^{\otimes n} \to A$. Let $C^n(A)$ be the space of n-cochains on A. (For $n = 0$, $A^{\otimes 0} = \Bbbk$.)

Definition 26.5. The *coboundary operator* $d = d^n : C^n(A) \to C^{n+1}(A)$ is defined by

$$(df)(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = \text{the left-hand side of } (**),$$

for all $f \in C^n(A)$.

Lemma 26.6. *$d \circ d = 0$, that is, $d^{n+1} \circ d^n = 0$ for all n.*

The proof is Exercise 26.9. We can therefore define:

- $Z^n(A) = \operatorname{Ker} d^n$, the *n-cocycles*;
- $B^n(A) = \operatorname{im} d^{n-1}$, the *n-coboundaries* (with $B^0(A) = 0$);
- $H^n(A) = Z^n(A)/B^n(A)$, the *n-th Hochschild cohomology group.*

Note that $B^n(A) \subseteq Z^n(A)$ because $d^2 = 0$.

Low degrees. For $f \in C^1(A) = \operatorname{End}_\Bbbk(A)$,

$$(df)(a_0 \otimes a_1) = a_0 f(a_1) - f(a_0 a_1) + f(a_0)a_1,$$

so

$$Z^1(A) = \{\, f : A \to A \mid f \text{ is a derivation} \,\} = \operatorname{Der}(A).$$

A 0-cochain is a linear map $\Bbbk \to A$, determined by the image b of 1; so $C^0(A) = \operatorname{Hom}_\Bbbk(A^{\otimes 0}, A) \cong A$, $f \mapsto b = f(1)$. Formula $(**)$ with $n = 0$ gives

$$(db)(a_0) = a_0 b - b a_0 = -[b, a_0].$$

So $H^0(A) = Z^0(A) = \{\, b \in A \mid a_0 b = b a_0 \text{ for all } a_0 \,\} = Z(A)$, the center of A, and db is an *inner derivation*. Writing $\operatorname{Inn}(A) = \{\, [b, -] \mid b \in A \,\}$ for the subspace of inner derivations, we get $B^1(A) = \operatorname{Inn}(A)$ and

$$H^1(A) = \operatorname{Der}(A)/\operatorname{Inn}(A),$$

the space of *outer derivations*.

26.3 What H^2 and H^3 measure

Remark 26.7 (Infinitesimal deformations and H^2). Call two infinitesimal deformations μ, μ' *equivalent* if there is a $\Bbbk[\![\hbar]\!]$-linear map φ with $\varphi(a) \equiv a + \varphi_1(a)\hbar \pmod{\hbar^2}$ for $a \in A$, for some linear $\varphi_1 : A \to A$, such that

$$\varphi\big(\mu(a \otimes b)\big) \equiv \mu'\big(\varphi(a) \otimes \varphi(b)\big) \pmod{\hbar^2}.$$

Comparing the coefficients of $\hbar$ on both sides gives $\varphi_1(ab) + \mu_1(a \otimes b) = \mu_1'(a \otimes b) + a\varphi_1(b) + \varphi_1(a)b$, that is,

$$\mu_1' = \mu_1 - d\varphi_1.$$

So equivalent infinitesimal deformations have cohomologous μ_1's, and conversely. Since an infinitesimal deformation is the same as a 2-cocycle μ_1, we get

$$\{\text{infinitesimal deformations of } A\}/\sim \;\cong\; H^2(A).$$

(Strictly speaking, the unit condition asks for *normalized* cocycles, with $\mu_1(1\otimes b)=0=\mu_1(b\otimes 1)$; one can show that every class in $H^2(A)$ contains one.)

Remark 26.8 (Obstructions and H^3). Suppose $\mu=\mu_0+\mu_1\hbar+\cdots+\mu_n\hbar^n$ is associative modulo $\hbar^{n+1}$, and we look for μ_{n+1} making it associative modulo $\hbar^{n+2}$. The coefficient of $\hbar^{n+1}$ in the associator is

$$\sum_{i+j=n+1}\Big(\mu_i\big(a\otimes\mu_j(b\otimes c)\big)-\mu_i\big(\mu_j(a\otimes b)\otimes c\big)\Big).$$

The terms with $i=0$ or $j=0$ add up to $(d\mu_{n+1})(a\otimes b\otimes c)$. So we need

$$d\mu_{n+1}=-\operatorname{ob}_n,\qquad \operatorname{ob}_n(a\otimes b\otimes c)=\sum_{\substack{i+j=n+1\\ i,j\geq 1}}\Big(\mu_i\big(a\otimes\mu_j(b\otimes c)\big)-\mu_i\big(\mu_j(a\otimes b)\otimes c\big)\Big).$$

Gerstenhaber showed that the *obstruction* ob_n is always a 3-cocycle. So μ_{n+1} exists if and only if the class of ob_n in $H^3(A)$ vanishes. In particular:

If $H^3(A)=0$, then every infinitesimal deformation of A extends, step by step, to a deformation.

The converse does not hold in general. The extension typically needs a nonzero μ_k for every k, which is why deformations live on $A[\![\hbar]\!]$. See [Gerstenhaber64].

26.4 Exercises

Exercise 26.9. Prove Lemma 26.6: $d^{n+1}\circ d^n=0$. (Hint: check $n=0$ and $n=1$ by hand. In general, expand $(d(df))(a_0\otimes\cdots\otimes a_{n+1})$; the terms come in pairs that cancel, using associativity of A when two adjacent products are merged in different orders.)

Exercise 26.10. Let $A=\Bbbk[x,y]$ with $\operatorname{char}\Bbbk=0$.

(a) Show that $\mu_1(f\otimes g)=(\partial_x f)(\partial_y g)$ is a Hochschild 2-cocycle.

(b) Show that μ_1 is not a coboundary. (Hint: for commutative A, every coboundary $d\varphi$ is symmetric, $(d\varphi)(a\otimes b)=(d\varphi)(b\otimes a)$; but $\mu_1(x\otimes y)-\mu_1(y\otimes x)=1$.)

(c) Show that

$$f\star g=\sum_{n\geq 0}\frac{\hbar^n}{n!}\,(\partial_x^n f)(\partial_y^n g)$$

defines an associative deformation of A, and that $x\star y-y\star x=\hbar$: a "Weyl algebra with parameter $\hbar$".

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