THESIS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

Generalized Weyl algebras and elliptic quantum groups

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Abstract

This thesis consists of four papers. In the first paper we present methods and explicit formulas for describing simple weight modules over twisted generalized Weyl algebras. Under certain conditions we obtain a classification of a class of locally finite simple weight modules from simple modules over tensor products of noncommutative tori. As an application we describe simple weight modules over the quantized Weyl algebra of rank two.

In the second paper we derive necessary and sufficient conditions for an ambiskew polynomial ring to have a Hopf algebra structure of a certain type, generalizing many known Hopf algebras, for example $U(\mathfrak{sl}_2)$, $U_q(\mathfrak{sl}_2)$ and the enveloping algebra of the 3-dimensional Heisenberg Lie algebra. In a torsion-free case we describe the finite-dimensional simple modules, and prove a generalized Clebsch-Gordan theorem. We construct a Casimir type operator and prove that any finite-dimensional weight module is semisimple.

In the third paper we define a notion of unitarizability for weight modules over a generalized Weyl algebra (of rank one, with commutative coeffiecient ring R), which is assumed to carry an involution of the form $X^* = Y$, $R^* \subseteq R$. We prove that a weight module V is unitarizable iff it is isomorphic to its finitistic dual $V^\sharp$. Using the classification of weight modules by Drozd, Guzner and Ovsienko, we prove necessary and sufficient conditions for an indecomposable weight module to be isomorphic to its finitistic dual, and thus to be unitarizable. Some examples are given, including $U_q(\mathfrak{sl}_2)$ for q a root of unity.

In the fourth paper, using the language of $\mathfrak{h}$-Hopf algebroids, introduced by Etingof and Varchenko, we construct a dynamical quantum group, $\mathscr{F}_{\mathrm{ell}}(GL(n))$, from Felder's elliptic solution of the quantum dynamical Yang-Baxter equation with spectral parameter associated to the Lie algebra $\mathfrak{sl}_n$. We apply the generalized FRST construction and obtain a bialgebroid $\mathscr{F}_{\mathrm{ell}}(M(n))$ and study analogues of the exterior algebra and elliptic minors. We prove that the elliptic determinant it is grouplike and almost central. Localizing at this determinant and constructing an antipode we obtain the $\mathfrak{h}$-Hopf algebroid $\mathscr{F}_{\mathrm{ell}}(GL(n))$.

Keywords: Generalized Weyl algebra, weight module, quantum Weyl algebra, ambiskew polynomial ring, unitarizable module, Hopf algebra, dynamical quantum group

Papers in this thesis

Paper I

J. T. Hartwig, *Locally finite simple weight modules over twisted generalized Weyl algebras*,
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Paper II

J. T. Hartwig, *Hopf structures on ambiskew polynomial rings*,
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Paper III

J. T. Hartwig, *Unitarizable weight modules over generalized Weyl algebras*,
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Introduction

This work deals with certain algebraic structures such as noncommutative rings, infinite-dimensional algebras, Hopf algebras and their modules. In the following we aim to give a brief overview of some of these areas and their interrelations. Although the subject has a long history, with pioneers such as Noether in the 1920's, we will focus on the more recent activities in the field.

A general principle in mathematics is to strive for defining families of objects with a balance between having good properties and supporting a rich list of examples. For instance, it is often difficult to generalize facts about commutative rings to the setting of arbitrary rings, due to the fact that the class of all rings is so vast. Therefore it is quite natural to consider some smaller family of noncommutative rings which have nice properties such as being "almost" commutative, and/or satisfying various finitistic conditions. One such class is the *Generalized Weyl algebras* to which we now turn.

1 Generalized Weyl algebras of rank one

Definition 1. Let R be a ring, σ be an automorphism of R and t be a central element of R. The associated *generalized Weyl algebra (GWA) of rank (or degree) one*, denoted $R(\sigma, t)$, is defined as the ring extension of R generated by X and Y modulo the following relations:

$$YX = t, \qquad XY = \sigma(t), \tag{1a}$$
$$Xr = \sigma(r)X, \qquad rY = Y\sigma(r) \quad \forall r \in R. \tag{1b}$$

Thus an arbitrary element in $R(\sigma, t)$ can be written

$$\sum_{n=0}^{N} r_n X^n + \sum_{n=1}^{M} s_n Y^n \tag{2}$$

where $r_n, s_n \in R$. When multiplying two expressions of the form (2), one can use relations (1) to write the result on this form again. For example we have

$$(r_1X + r_2Y^2)r_3X = r_1\sigma(r_3)X^2 + r_2\sigma^{-2}(r_3)Yt = r_1\sigma(r_3)X^2 + r_2\sigma^{-2}(r_3)\sigma^{-1}(t)Y$$

for any $r_1, r_2, r_3 \in R$.

Despite their name, generalized Weyl algebras are not algebras in general, but merely rings. They were introduced in [B92], and also studied without a name in [J93] and under the name *hyperbolic ring* in [R]. Many different aspects of these and related rings and their modules have been studied in several papers (see [B91], [B93], [B96], [BJ], [DGO], [J00], [CM], [CL] and references therein). If R is Noetherian, then $R(\sigma,t)$ is also Noetherian. It is very often assumed that R is commutative. If this is so, and if σ is the identity map, then $R(\sigma,t)$ is commutative. An important example of a GWA is the following.

Example 2. The *quantum Weyl algebra* A_1^q where $q\in\mathbb{C}\backslash\{0\}$ is the algebra with generators X,Y and defining relation

$$XY-qYX=1. \tag{3}$$

It is a GWA in the following way. Let $R=\mathbb{C}[x]$, let $\sigma:R\to R$ be the automorphism defined by $\sigma(f(x))=f(qx+1)$ for any $f(x)\in R$, and let t be the polynomial $p(x)=x\in R$. Let $A=R(\sigma,t)$ be the corresponding GWA. Then relations (1a) imply that $YX=x$ and $XY=qx+1$ so that (3) holds. From $Xx=X(YX)=(XY)X=(qx+1)X$ follows that $Xx^k=(qx+1)^kX$ for any $k\in\mathbb{Z}_{\geq 0}$ and thus, by linearity, $Xf(x)=f(qx+1)X$ for any polynomial $f(x)\in R$. Analogously, $f(x)Y=Yf(qx+1)$ for any $f(x)\in R$. This shows that relations (1b) are redundant and that A is generated by X,Y with the single relation (3). Thus A is isomorphic to the quantum Weyl algebra A_1^q. If we take $q=1$ we get the so called *first Weyl algebra*, denoted A_1.

Let us discuss some of the classes of GWAs that have been studied.

Example 3. *Ambiskew polynomial rings* $R(B,\sigma,v,p)$. Let B be a ring, let σ be an automorphism of B, v be a central element of B and p be a central unit in B. Let $R(B,\sigma,v,p)$ be the ring extension of B generated by x and y subject to the relations

$$xy-pyx=v,\quad xb=\sigma(b)x,\quad by=y\sigma(b)\quad \forall b\in B. \tag{4}$$

$R(B,\sigma,v,p)$ is isomorphic to the GWA $B[t](\sigma,t)$ where $t=yx$ and σ is extended to $B[t]$ by $\sigma(t)=pt+v$. Finite-dimensional simple modules were described in [J95] for the case when B is a commutative $\mathbb{K}$-algebra. The relation to down-up algebras (see below) was investigated in [J00]. The quantum Weyl algebra A_1^q is an ambiskew polynomial ring: $A_1^q\simeq R(\mathbb{C},\mathrm{Id},1,q)$.

Example 4. *Generalized down-up (GDU) algebras* $L(f,r,s,\gamma)$ were defined in [CS]. Let $\mathbb{K}$ be an algebraically closed field, $f\in\mathbb{K}[x]$, $r,s,\gamma\in\mathbb{K}$ with $rs\neq 0$. Let $L(f,r,s,\gamma)$ be the $\mathbb{K}$-algebra generated by d,u,h with relations

$$dh-rhd+\gamma d=0,\quad hu-ruh+\gamma d=0,\quad du-sud+f(h)=0. \tag{5}$$

Ordinary *down-up algebras*, introduced in [BR] (see also[CM],[KM]) are obtained as $L(f,r,s,\gamma)$ with $f(x)=x$. In [CS] it was shown that all algebras $L(f,r,s,\gamma)$ (with $rs\neq 0$) are Noetherian domains of Gelfand-Kirillov dimension 3. All simple weight modules (with respect to the subalgebra generated by ud and h) were classified, including all finite-dimensional simple modules. These algebras are examples of ambiskew polynomial rings, hence of GWAs.

Example 5. *Rueda's and Smith's algebras similar to the enveloping algebra of* $\mathfrak{sl}_2$. Let A be the $\mathbb{C}$-algebra with generators X, Y, H and relations

$$HX - XH = X, \quad HY - YH = -Y, \quad XY - \varsigma YX = f(H), \tag{6}$$

where $f \in \mathbb{C}[x]$ and $\varsigma \in \mathbb{C}, \varsigma \neq 0$. These algebras were studied by Rueda [Ru]. If we assume that $\varsigma = 1$ we get the class investigated from many points of view by Smith [S]. If we specialize further and take $f(x) = \frac{1}{2}x$ (or, in fact, any f of degree one) then we get an algebra isomorphic to the enveloping algebra of $\mathfrak{sl}_2$.

Example 6. *Type-A Kleinian singularities.* Let $a \in \mathbb{K}[x]$ and let $A(a)$ be the algebra with generators x, y, h and relations

$$hx - xh = x, \quad hy - yh = -y, \quad yx = a(h), \quad xy = a(h-1). \tag{7}$$

This algebra is isomorphic to the GWA $\mathbb{K}[h](\sigma, a(h))$ where $\sigma(p(h)) = p(h-1)\ \forall p(h) \in \mathbb{K}[h]$. The algebras $A(a)$ have been thoroughly investigated in many papers, see [H], [B92], [B93], [B91], [BJ]. All simple modules were classified in [B92]. In [BJ] the important problem of determining when two algebras $A(a), A(b)$ for $a, b \in \mathbb{K}[x]$ are isomorphic was solved. For $a(x) = x$ the algebra $A(a)$ is isomorphic to the first Weyl algebra A_1.

The list of examples could continue: Witten's seven parameter deformation of $U(\mathfrak{sl}_2)$, Le Bruyn's *conformal* $\mathfrak{sl}_2$ *enveloping algebras*, Woronowicz's deformation (see [CS], [BO] for further information).

Figure 1 shows the relationship between some of these different classes of Generalized Weyl algebras of rank one which have been studied in the literature.

To motivate this picture, some notes are in order. Let $\mathscr{G}$, $\mathscr{W}$, $\mathscr{S}$, and $\mathscr{R}$ be the isomorphism classes of generalized down-up algebras, Witten's deformations, Smith's and Rueda's algebras respectively.

1. $\mathscr{W} \not\subseteq \mathscr{R}$: $\mathscr{W}$ includes the commutative algebra of polynomials in three variables, but all algebras in $\mathscr{R}$ are noncommutative.

2. $\mathscr{S} \subsetneq \mathscr{R}$: If ς is not a root of unity it is proved in [Ru] that Rueda's algebra has trivial center. However in [S] it is shown that all algebras in $\mathscr{S}$ have nontrivial center, a polynomial algebra in one variable Ω (a certain generalized Casimir element).

3. $\mathscr{S} \cap \mathscr{W} \subsetneq \mathscr{R} \cap \mathscr{W}$: An algebra of the form $L(x, 1, s, 1)$ with s not a root of unity is in $\mathscr{R} \cap \mathscr{W}$. Again it has trivial center and thus is not in $\mathscr{S}$.

4. Type-A Kleinian singularities and A_1^q are disjoint from $\mathscr{G}$: The quantum Weyl algebra A_1^q has Gelfand-Kirillov dimension two (see [GZ] for a general result) and any type-A Kleinian singularity $A(a)$ has also Gelfand-Kirillov dimension two (this is mentioned in [BJ], Section 3). But, as shown in [CS], any algebra in $\mathscr{G}$ has Gelfand-Kirillov dimension three (see [CS]).

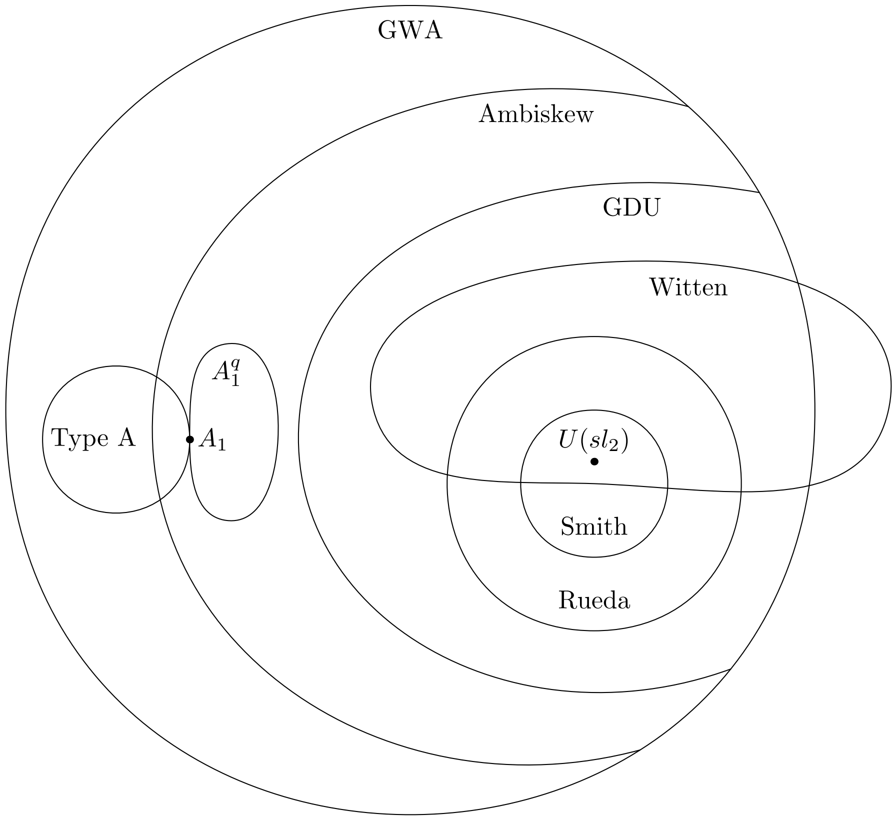


Figure 1: Classes of GWAs of rank one.

5. A quantum Weyl algebra A_1^q is isomorphic to a type-A Kleinian singularity iff $q=1$: Indeed if $A=A_1^q$ and $q\neq 1$ then
$$A/A[A,A]A \simeq \mathbb{K}[X,Y]/\big((1-q)XY-1\big)\simeq \mathbb{K}[X,X^{-1}].$$
However, for any Type A Kleinian singularity $A(a)$ we have
$$B/B[B,B]B \simeq \mathbb{K}[h]/\big(a(h),a(h+1)\big)=\begin{cases}\mathbb{K}[h], & a=0,\\ F, & a\neq 0,\end{cases}$$
where F is a finite-dimensional $\mathbb{K}$-algebra.

6. We believe that not all algebras in Witten's seven-parameter family are generalized Weyl algebras and that not all Type-A Kleinian singularities are ambiskew polynomial rings, but have found no proof of this in the literature.

2 Higher rank and twisted generalized Weyl algebras

Higher rank GWAs were introduced in [B92] and are defined as follows.

Definition 7. Let R be a ring and $\sigma=(\sigma_1,\dots,\sigma_n)$ a set of commuting automorphisms of R and $t=(t_1,\dots,t_n)$ a set of nonzero elements of the center of R such that $\sigma_i(t_j)=t_j\ \forall i\neq j$. The *generalized Weyl algebra of rank (or degree) n*, denoted $R(\sigma,t)$, is the ring extension of R by $X_1,\dots,X_n$, $Y_1,\dots,Y_n$ modulo the relations

$$Y_iX_i=t_i,\qquad X_iY_i=\sigma_i(t_i),\qquad i=1,\dots,n, \tag{8a}$$
$$X_ir=\sigma_i(r)X_i,\qquad rY_i=Y_i\sigma_i(r),\qquad \forall r\in R, i=1,\dots,n, \tag{8b}$$
$$[X_i,Y_j]=0,\qquad \forall i\neq j, \tag{8c}$$
$$[Y_i,Y_j]=[X_i,X_j]=0,\qquad \forall i,j. \tag{8d}$$

Example 8. The *n:th Weyl algebra*, denoted A_n, is the $\mathbb{C}$-algebra with generators $X_1,\dots,X_n$, $Y_1,\dots,Y_n$ and relations
$$[X_i,Y_j]=\delta_{ij},\quad [X_i,X_j]=0,\quad [Y_i,Y_j]=0,\qquad i,j=1,\dots,n. \tag{9}$$

The algebra A_n is isomorphic to the GWA $\mathbb{C}[t_1,\dots,t_n](\sigma,t)$ where $\sigma_i(t_j)=t_j+\delta_{ij}$.

The definition of the n:th Weyl algebra goes back to the pioneers of quantum mechanics in the beginning of the last century. It is also one of the most important examples of an infinite-dimensional simple Noetherian algebra. Despite this fact there are still many unsolved questions about it.

More generally, any tensor product of rank one GWAs is a higher rank GWA.

Twisted GWAs are certain generalizations of these higher rank GWAs and were introduced in [MT99] and further studied in [MT02] and [MPT]. Their definition is more involved but, put simply, it is relation (8d) which is dropped and replaced by taking the quotient by a certain ideal. Also one does not require that $\sigma_i(t_j)=t_j\ \forall i\neq j$. See the first paper in this thesis for a precise definition.

An important example of a twisted GWA is the following.

Example 9. *Quantized Weyl algebras* $A_n^{\bar{q},\Lambda}$. Let $\Lambda = (\lambda_{ij})$ be an $n \times n$ matrix with nonzero complex entries such that $\lambda_{ji} = \lambda_{ij}^{-1}$. Let $\bar{q} = (q_1, \dots, q_n)$ be an n-tuple of elements from $\mathbb{C}\backslash\{0,1\}$. Then n:th quantized Weyl algebra $A_n^{\bar{q},\Lambda}$ is the $\mathbb{C}$-algebra with generators x_i, y_i, $i = 1, \dots, n$ and the following relations for $1 \le i < j \le n$.

$$x_i x_j = q_i \lambda_{ij} x_j x_i, \qquad y_i y_j = \lambda_{ij} y_j y_i, \tag{10a}$$
$$x_i y_j = \lambda_{ji} y_j x_i, \qquad x_j y_i = q_i \lambda_{ij} y_i x_j, \tag{10b}$$

$$x_i y_i - q_i y_i x_i = 1 + \sum_{k=1}^{i-1} (q_k - 1) y_i x_i. \tag{10c}$$

For $\lambda_{ij} = q^{-1/2}$ $\forall i, j$ and $q_1 = \cdots = q_n = q$ this algebra was introduced in [PW]. Then the relations are the canonical commutation relations for annihilation and creation operators corresponding to the (essentially unique) first order differential calculus which is covariant with respect to the quantum group $SL_q(n)$.

Further interesting examples of twisted GWAs were given in [MPT]. These examples are certain algebras related to the Lie algebra $\mathfrak{gl}_n$ (so called *Mickelsson step algebras* and *extended orthogonal Gelfand-Zetlin algebras* respectively).

3 Weight modules

3.1 Generalities

Recall that a module over a ring is called *simple* if it has no nonzero proper submodules, and *semisimple* if it is isomorphic to a direct sum of simple modules.

Let S be a ring containing a commutative subring R with unit. An S-module V is called a *weight module with respect to* R if V is semisimple as an R-module. This is equivalent to that V can be decomposed as

$$V = \bigoplus_{\mathfrak{m} \in \mathrm{Max}(R)} V_{\mathfrak{m}}, \qquad V_{\mathfrak{m}} = \{v \in V \,:\, \mathfrak{m}v = 0\}, \tag{11}$$

where $\mathrm{Max}(R)$ denotes the set of maximal ideals of R. The subgroups $V_{\mathfrak{m}}$ are called *weight spaces* and the set $\mathrm{Supp}(V) = \{\mathfrak{m} \in \mathrm{Max}(R) \,:\, V_{\mathfrak{m}} \neq 0\}$ is called the *support* of V.

Example 10. Let S be the ring of all 2×2 matrices with entries in a field $\mathbb{K}$ and let R be the subring consisting of diagonal matrices. The ring R is isomorphic to $\mathbb{K} \times \mathbb{K}$ and its only maximal ideals are

$$\mathfrak{m}_1 = \left\{ \left[\begin{smallmatrix} \lambda & 0 \\ 0 & 0 \end{smallmatrix}\right] \,:\, \lambda \in \mathbb{K} \right\}, \qquad \mathfrak{m}_2 = \left\{ \left[\begin{smallmatrix} 0 & 0 \\ 0 & \lambda \end{smallmatrix}\right] \,:\, \lambda \in \mathbb{K} \right\}.$$

Consider the natural two-dimensional S-module V=$\mathbb{K}^2$. Its weight spaces are $V_{\mathfrak{m}_1} = \mathbb{K} \cdot \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$ and $V_{\mathfrak{m}_2} = \mathbb{K} \cdot \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$. Thus $V = V_{\mathfrak{m}_1} \oplus V_{\mathfrak{m}_2}$ which shows that V is a weight module with respect to R.

Example 11. Let S be a ring and suppose that $R = \mathbb{C}[x]$ is a subring of S. The maximal ideals of R are all ideals of the form $\mathfrak{m}_\alpha = (x - \alpha)$ where $\alpha \in \mathbb{C}$. Let V be an S-module. The subspaces $V_{\mathfrak{m}_\alpha}$ are precisely the set of all eigenvectors of x on V with eigenvalue α. Thus V is a weight module with respect to R iff x act as a diagonalizable operator on V.

Example 11 shows that weight modules are generalizations of vector spaces on which a linear operator is diagonalizable. As such, they appear in numerous context throughout mathematics and physics. Weight modules have many nice properties, in particular over generalized Weyl algebras.

The following proposition is fundamental in the theory of weight modules.

Proposition 12. *Submodules, quotients, and direct sums of weight modules are weight modules. More precisely, let S be a ring, and R a commutative subring. Then*

(i) if V and W are weight S-modules with respect to R, then the S-module $V \oplus W$ is also a weight module with respect to R and

$$(V \oplus W)_\mathfrak{m} = V_\mathfrak{m} \oplus W_\mathfrak{m} \qquad \forall \mathfrak{m} \in \operatorname{Max}(R),$$

(ii) if V is a weight S-module with respect to R and if W is an S-submodule of V, then W is also a weight module with respect to R and

$$W_\mathfrak{m} = (V_\mathfrak{m}) \cap W \qquad \forall \mathfrak{m} \in \operatorname{Max}(R),$$

(iii) if V is a weight S-module with respect to R and W is an S-submodule of V, then the S-module V/W is also a weight module with respect to R and

$$(V/W)_\mathfrak{m} = \{v + W \,:\, v \in V_\mathfrak{m}\} \qquad \forall \mathfrak{m} \in \operatorname{Max}(R).$$

Proof. Part (i) is straightforward, and (iii) follows from the following general fact: If $\varphi : V \to W$ is a morphism of S-modules, then $\varphi(V_\mathfrak{m}) \subseteq W_\mathfrak{m}$.

Let us prove part (ii). Let $w \in W$ be arbitrary. Since $W \subseteq V$ and V is a weight module, we can decompose w as a sum

$$w = w_1 + w_2 + \cdots + w_n$$

where $w_i \in V_{\mathfrak{m}_i}$ for some maximal ideals $\mathfrak{m}_i$ of R (which we can assume to be pairwise distinct). The problem is to show that, in fact, each term w_i belongs to W. Indeed, if so, then we have proved that $W \subseteq \bigoplus_{\mathfrak{m}\in\operatorname{Max}(R)} W \cap (V_\mathfrak{m})$ and the other inclusion is trivial. If $n = 1$ it is trivial. Assume $n \geq 2$. Since the $\mathfrak{m}_i$ are mutually incomparable with respect to inclusion, there exist $r_i \in \mathfrak{m}_i \backslash \mathfrak{m}_n$ for $i = 1, 2, \ldots, n-1$. Let $r = r_1 r_2 \cdots r_{n-1} \in \mathfrak{m}_1 \cap \cdots \cap \mathfrak{m}_{n-1}$. Since W is an R-module,

$$W \ni rw = rw_1 + \cdots + rw_n = rw_n.$$

Since $\mathfrak{m}_n$ is maximal, hence prime, we have $r \notin \mathfrak{m}_n$. Thus $(r) + \mathfrak{m}_n = R$ so that $sr - 1 \in \mathfrak{m}_n$ for some $s \in R$, which gives $w_n = srw_n \in W$. Subtracting w_n from w and repeating the above argument, we conclude that all the terms w_i indeed belong to W. $\square$

3.2 Weight modules over generalized Weyl algebras

3.2.1 Rank one

Let $A = R(\sigma, t)$ be a GWA of rank one, where R is a commutative ring with unit. By a weight module over A we always mean with respect to the subring R. The following proposition shows that weight modules are of great relevance for GWAs.

Proposition 13. *Let V be an arbitrary A-module. Let V' be the sum of all weight spaces in V:*

$$V' = \sum_{\mathfrak{m}\in \mathrm{Max}(R)} V_{\mathfrak{m}}. \tag{12}$$

Then

a) *the sum in* (12) *is direct and V' is a submodule of V, and*

b) *assuming in addition that R is an algebra over an algebraically closed field $\mathbb{K}$ and that V is finite-dimensional, then V' is nonzero. Thus if V is simple, then it is a weight module.*

Proof. Taking $w = 0$ in the proof of Proposition 12 (ii) we see that the sum in (12) is always direct.

Let $\mathfrak{m}$ be a maximal ideal in R and suppose $v \in V$ is a weight vector of weight $\mathfrak{m}$, i.e. that $\mathfrak{m}v = 0$. Then, due to relation (1b), we have $\sigma(\mathfrak{m})Xv = X\mathfrak{m}v = 0$, proving that $Xv \in V_{\sigma(\mathfrak{m})}$. Similarly, $Yv \in V_{\sigma^{-1}(\mathfrak{m})}$. Since A is generated by X and Y and the elements of R it follows that V' is a submodule of V. This proves part a).

Assuming the conditions in part b), there exists in V a common eigenvector v for all operators from R: $rv = \xi(r)v$ for some $\xi(r) \in \mathbb{K}$. Thus $V_{\mathfrak{m}} \neq 0$, where $\mathfrak{m}$ is the maximal ideal $\ker\xi = \big(r - \xi(r) : r \in R\big)$. Hence $V' \neq 0$. Therefore, if V is simple, $V = V'$. That is, V is a weight module. □

The description of all weight modules over rank one GWAs is rather complete. In fact, given that one understands the orbits in $\mathrm{Max}(R)$ under the action of σ and that one can describe indecomposable elements in certain skew polynomial rings associated to $R(\sigma, t)$, the authors in [DGO] classified all indecomposable weight modules over a generalized Weyl algebra $R(\sigma, t)$, where R is a commutative ring.

3.2.2 Higher rank

Going to the higher rank case, things become considerably more complicated. Even for the n:th Weyl algebra A_n, the problem of describing all indecomposable weight (with respect to $\mathbb{C}[Y_1X_1, \dots, Y_nX_n] \subseteq A_n$) modules is a so called *wild problem* (see [D], [BBF]). This means that the problem contains a classification of all representations of a free algebra with two or more generators as a subproblem.

Therefore, it is essential to restrict oneself to certain subclasses (tame blocks) of weight modules which admit a classification. Such tame blocks were described in [BBF] for A_n, and in [BB] for more general higher rank GWAs (but over an algebraically closed field).

For twisted GWAs, some classes of simple modules were defined and classified in [MT99], [MPT].

There is also the following useful method proved in [MPT]. Any twisted GWA is naturally graded by $\mathbb{Z}^n$. For $\mathfrak{m} \in \mathrm{Max}(R)$, one can consider the maximal graded subalgebra $B_\mathfrak{m}$ which leaves a weight space of weight $\mathfrak{m}$ invariant. Then there is a bijective correspondence between simple weight A-modules V with $V_\mathfrak{m} \neq 0$ and simple modules U over the subalgebra $B_\mathfrak{m}$ such that $\mathfrak{m}U = 0$:

$$\left\{\begin{array}{l}\text{simple weight } A\text{-modules } V \\ \text{with } V_\mathfrak{m} \neq 0\end{array}\right\} \Longleftrightarrow \left\{\begin{array}{l}\text{simple } B_\mathfrak{m}\text{-modules } U \\ \text{annihilated by } \mathfrak{m}.\end{array}\right\}$$

Using this method, the problem of describing simple weight modules over a twisted GWA reduces to describing modules over the subalgebra $B_\mathfrak{m}$ which may be easier in many cases.

4 Hopf algebras

4.1 Definition and examples

Let $\mathbb{K}$ be a field. All tensor products below will be over $\mathbb{K}$. Recall that if A is a $\mathbb{K}$-algebra then $A \otimes A$ is naturally a $\mathbb{K}$-algebra by defining $(a \otimes b)(c \otimes d) = ac \otimes bd$ for $a, b, c, d \in A$ and extending bilinearly.

Definition 14. A *Hopf algebra* $(H, \Delta, \varepsilon, S)$ is a unital $\mathbb{K}$-algebra H together with

$$\begin{array}{rll} \text{a } \mathbb{K}\text{-algebra homomorphism} & \Delta : H \to H \otimes H & (\text{the } \textit{coproduct}), \\ \text{a } \mathbb{K}\text{-algebra homomorphism} & \varepsilon : H \to \mathbb{K} & (\text{the } \textit{counit}), \\ \text{a } \mathbb{K}\text{-algebra antihomomorphism} & S : H \to H & (\text{the } \textit{antipode}), \end{array}$$

such that, for all $x \in H$,

$$(\Delta \otimes \mathrm{Id}_H)\Delta(x) = (\mathrm{Id}_H \otimes \Delta)\Delta(x) \qquad \text{(coassociativity)}, \tag{13a}$$

$$(\varepsilon \otimes \mathrm{Id}_H)\Delta(x) = 1 \otimes x, \quad (\mathrm{Id}_H \otimes \varepsilon)\Delta(x) = x \otimes 1, \qquad \text{(counit axiom)}, \tag{13b}$$

$$m(S \otimes \mathrm{Id}_H)\Delta(x) = \varepsilon(x)1 = m(\mathrm{Id}_H \otimes S)\Delta(x), \qquad \text{(antipode axiom)}, \tag{13c}$$

where $m : H \otimes H \to H$ denotes the multiplication map in the algebra H.

Two of the most important examples of Hopf algebras are the following.

Example 15. Let G be a group and let $A = \mathbb{K}[G]$ be the corresponding group algebra over $\mathbb{K}$. Define for all $g \in G$,

$$\Delta(g) = g \otimes g, \quad \varepsilon(g) = 1, \quad S(g) = g^{-1}, \tag{14}$$

and extend the three maps $\mathbb{K}$-linearly to all of A. Then $(A, \Delta, \varepsilon, S)$ is a Hopf algebra.

Example 16. Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{K}$ and $U = U(\mathfrak{g})$ its universal enveloping algebra. Recall that U can be constructed as $T(\mathfrak{g})/I(\mathfrak{g})$ where $T(\mathfrak{g})$ is the tensor algebra on $\mathfrak{g}$ and $I(\mathfrak{g})$ is the ideal in $T(\mathfrak{g})$ generated by all elements of the form $x \otimes y - y \otimes x - [x, y]$ for $x, y \in \mathfrak{g}$. Define

$$\Delta(x) = x \otimes 1 + 1 \otimes x, \quad \varepsilon(x) = 0, \quad S(x) = -x, \tag{15}$$

for all $x \in \mathfrak{g}$. By the universal property of the tensor algebra, the map Δ extends uniquely to a $\mathbb{K}$-algebra homomorphism $\Delta : T(\mathfrak{g}) \to U \otimes U$. One verifies that its kernel contains the ideal $I(\mathfrak{g})$, thereby inducing a $\mathbb{K}$-algebra homomorphism $\Delta : U \to U \otimes U$. Similarly ε induces a homomorphism $\varepsilon : U \to \mathbb{K}$ and S an antihomomorphism $S : U \to U$. Axioms (13) can be verified. Thus $(U, \Delta, \varepsilon, S)$ is a Hopf algebra.

Example 17. Let $G = GL(n)$ be the group of all invertible $n \times n$ matrices with complex entries. Let $H = \mathscr{F}(G)$ be the commutative algebra (with pointwise operations) of complex-valued functions on G generated by the n^2 coordinate functions $e_{ij} : G \to \mathbb{C}$ given by

$$e_{ij}(g) = g_{ij}$$

for any matrix $g = (g_{kl})_{kl} \in G$. Define $\Delta : H \to H \otimes H$ by

$$\Delta(a)(g, h) = a(gh) \quad \forall a \in H, g, h \in G$$

where we consider $H \otimes H$ as functions on $G \times G$ by $(a \otimes b)(g, h) = a(g)b(h)$ for $a, b \in H$, $g, h \in G$. For the coordinate functions we have

$$\Delta(e_{ij}) = \sum_{x=1}^{n} e_{ix} \otimes e_{xj}.$$

The counit $\varepsilon : H \to \mathbb{C}$ and antipode $S : H \to H$ are defined by

$$\varepsilon(a) = a(I), \qquad S(a)(g) = a(g^{-1}),$$

where I is the identity matrix in G. One can verify that $(H, \Delta, \varepsilon, S)$ is a Hopf algebra.

4.2 Modules over Hopf algebras

Corresponding to each of the maps Δ, ε and S in the definition of Hopf algebra, there is is a module-theoretic construction and to each of the axioms in (13) there is a corresponding morphism of modules.

A module over a Hopf algebra H is just a module over the underlying algebra. If V and W are modules over a Hopf algebra H, then their tensor product $V \otimes_{\mathbb{K}} W$ can be given the structure of an H-module in the following natural way

$$h.(v \otimes w) = \Delta(h)(v \otimes w) = \sum_i (h_i'.v) \otimes (h_i''.w) \tag{16}$$

where $\Delta(h) = \sum_i h'_i \otimes h''_i$. From the coassociativity axiom for Δ follows that, given three H-modules U, V, W, the natural vector space isomorphism $(U \otimes V) \otimes W \simeq U \otimes (V \otimes W)$ is an isomorphism of H-modules.

The counit gives rise to a special one-dimensional module $\mathbf{1}_H = \mathbb{K}$ defined by

$$h.\lambda = \varepsilon(h)\lambda \tag{17}$$

for $h \in H$ and $\lambda \in \mathbf{1}_H$. The counit axiom implies that for any H-module V, the natural isomorphisms $V \otimes \mathbf{1}_H \simeq V \simeq \mathbf{1}_H \otimes V$ of vector spaces are H-module isomorphisms.

Finally, the module-theoretic consequence for the antipode is that the dual vector space $V^* = \mathrm{Hom}(V, \mathbb{K})$ of an H-module V acquires an H-module structure by

$$(h.\varphi)(v) = \varphi\big(S(h).v\big) \tag{18}$$

for any $h \in H, \varphi \in V^*, v \in V$. The antipode axiom implies that the natural evaluation maps $e : V^* \otimes V \to \mathbf{1}_H$, $e' : V \otimes V^* \to \mathbf{1}_H$ are H-module homomorphisms.

4.3 The Clebsch-Gordan formula

Let H be a Hopf algebra. Suppose that any finite-dimensional H-module V is semisimple, that is, can be decomposed as a direct sum of simple modules

$$V \simeq V_1 \oplus V_2 \oplus \cdots \oplus V_n.$$

Given two simple finite-dimensional H-modules V and W, their tensor product $V \otimes W$ is again a finite-dimensional H-module, as described in Section 4.2. It is interesting to ask what the decomposition of $V \otimes W$ into a direct sum of simple H-modules look like.

The answer to this question for the case when $H = U(\mathfrak{sl}_2)$ is given by the classical *Clebsch-Gordan formula*, which is important in quantum mechanics. Let $V(n)$, where $n \in \mathbb{Z}_{\geq 0}$, denote the finite-dimensional simple module over $U(\mathfrak{sl}_2)$ with highest weight n (and of dimension $n+1$). Then

$$V(m) \otimes V(n) \simeq V(m+n) \oplus V(m+n-2) \oplus \cdots \oplus V(m-n) \tag{19}$$

if $m \geq n$. The entries of the matrix of the isomorphism (19) with respect to natural bases on either side are certain special functions known as the *Clebsch-Gordan coefficients*.

5 Quantum groups

5.1 Drinfeld-Jimbo quantum groups

In 1985, Drinfeld and Jimbo independently defined a certain Hopf algebra associated to any semi-simple complex finite-dimensional Lie algebra (more generally, any symmetrizable Kac-Moody algebra). This Hopf algebra is denoted by $U_q(\mathfrak{g})$

where q is a complex parameter. For $q = 1$ one essentially recovers the usual enveloping algebra $U(\mathfrak{g})$. Since the representations of $U(\mathfrak{g})$ are the same as the representations of $\mathfrak{g}$, which in turn (loosely speaking) are the same as representations of the Lie group G corresponding to $\mathfrak{g}$, one calls the algebra $U_q(\mathfrak{g})$ a *quantum group*. The reason for the prefix quantum is that quantum mechanics can be viewed as a deformation of classical mechanics by letting Planck's constant go to zero.

Already from the beginning quantum groups were intimately connected with applications in physics, statistical mechanics and integrable models. Since then, applications has been found in many different areas of mathematics, such as knot theory and special functions. For example, the q-analogues of the Clebsch-Gordan coefficients mentioned in Section 4.3 were known before the definition of quantum groups, as a certain curious family of orthogonal polynomials. Their orthogonality was later explained using quantum groups.

Let us look at the simplest example.

Example 18. $U_q(\mathfrak{sl}_2)$ is by definition the $\mathbb{C}$-algebra with generators $E, F, K^{\pm 1}$ and relations

$$KK^{-1} = 1 = K^{-1}K, \qquad [E,F] = \frac{K - K^{-1}}{q - q^{-1}},$$
$$KEK^{-1} = q^2 E, \qquad KFK^{-1} = q^{-2}F.$$

It has a Hopf structure given by

$$\Delta(E) = 1 \otimes E + E \otimes K, \qquad \Delta(F) = K^{-1} \otimes F + F \otimes 1,$$
$$\Delta(K) = K \otimes K, \qquad \Delta(K^{-1}) = K^{-1} \otimes K^{-1},$$
$$\varepsilon(E) = 0 = \varepsilon(F), \qquad \varepsilon(K) = 1 = \varepsilon(K^{-1}),$$
$$S(E) = -EK^{-1}, \quad S(F) = -KF, \quad S(K) = K^{-1}, \quad S(K^{-1}) = K.$$

The maps Δ, ε should be extended multiplicatively to all of $U_q(\mathfrak{sl}_2)$ while S should be extended to an antihomomorphism and one must verify that the relations above are preserved.

5.2 The FRST construction

There is another type of quantum groups which are in a certain sense dual to those in the previous section. Let us briefly describe them. Let $R = \{R_{ik}^{jl}\}_{1 \le i,j,k,l \le n}$ be a collection of n^4 complex numbers. There is an important construction due to Faddeev-Reshetikhin-Sklyanin-Takhtajan which associates a bialgebra (=Hopf algebra without antipode), $\mathscr{F}_R$, to R in the following way.

Let $\mathscr{F}_R$ be the associative algebra with generators L_{ij}, $1 \le i,j \le n$, and the relation

$$\sum_{x,y=1}^{n} R_{ac}^{xy} L_{xb} L_{yd} = \sum_{x,y=1}^{n} R_{xy}^{bd} L_{cy} L_{ax} \qquad \forall a,b,c,d \in \{1,\dots,n\}. \tag{20}$$

Relation (20) is called the *RLL relation*. Give $\mathscr{F}_R$ the structure of a bialgebra by defining

$$\Delta(L_{ab}) = \sum_{x=1}^{n} L_{ax} \otimes L_{xb}, \qquad \varepsilon(L_{ab}) = \delta_{ab} \tag{21}$$

for $a, b = 1, \ldots, n$.

If we take $R_{ik}^{jl} = \delta_{ij}\delta_{kl}\ \forall i, j, k, l$, then (20) just says that the L_{ij} should commute and thus $\mathscr{F}_R$ is precisely the commutative bialgebra $\mathscr{F}(M(n))$ of all polynomial functions on the semigroup $M(n)$ of all $n \times n$ complex matrices, like in Example 17 but without inverses and antipode. Thus one may view $\mathscr{F}_R$ as a noncommutative deformation of the bialgebra $\mathscr{F}(M(n))$.

From relation (20) one can deduce cubic relations in two nontrivial ways. The relations obtained are

$$\sum_{xyuvrs} R_{ce}^{rs} R_{as}^{uv} R_{ur}^{xy} L_{xb} L_{yd} L_{vf} = \sum_{xyuvrs} R_{xy}^{bd} R_{uv}^{xf} R_{rs}^{yv} L_{es} L_{cr} L_{au}$$

and

$$\sum_{xyuvrs} R_{ac}^{ur} R_{ue}^{xs} R_{rs}^{yv} L_{xb} L_{yd} L_{vf} = \sum_{xyuvrs} R_{yv}^{df} R_{xs}^{bv} R_{ur}^{xy} L_{es} L_{cr} L_{au}.$$

The *Yang-Baxter equation* is a necessary and sufficient condition for these two relations to coincide:

$$\sum_{urs} R_{ce}^{rs} R_{as}^{uv} R_{ur}^{xy} = \sum_{urs} R_{ac}^{ur} R_{ue}^{xs} R_{rs}^{yv} \qquad \forall a, c, e, x, y, v \in \{1, \ldots, n\}. \tag{22}$$

(Another motivation for requiring (22) is that $A(R)$ may then be equipped with a cobraiding, which turns its category of comodules into a braided tensor category.) The solutions to this equation are called *R-matrices*. The Yang-Baxter equation exists in many different versions and generalities. It may involve a so called *spectral parameter* $u \in \mathbb{C}$, and/or a *dynamical parameter* $\lambda \in \mathfrak{h}^*$ (dual of a Cartan subalgebra). There is also a classification theorem due to Belavin and Drinfel'd [BD] which says that R-matrices with spectral parameter fall into one of three categories: rational, trigonometric or elliptic.

6 Summary of papers

6.1 Locally finite simple weight modules over TGWAs

We give a description of a class of simple weight modules over an arbitrary twisted generalized Weyl algebra. We assume that the weight spaces are finite-dimensional. The class of modules, defined in the paper, is the modules without so called *proper inner breaks*. If the ground ring is finitely generated over an algebraically closed field we show that the problem reduces to classifying finite-dimensional simple modules over a tensor product of certain algebras (noncommutative tori) and these have a well-known structure. The description we obtain generalizes previous work in [MT99]. The technique is based on the procedure of induction from the maximal graded weight space preserving subalgebra. We apply these methods to the quantized Weyl algebra $A_n^{\bar{q},\Lambda}$ of rank $n = 2$ and classify all simple weight modules with no proper inner breaks.

6.2 Hopf structures on ambiskew polynomial rings

The algebras $U(\mathfrak{sl}_2)$, $U_q(\mathfrak{sl}_2)$, and the algebra $U_q(f(H,K))$ from [WJY], are all examples of ambiskew polynomial rings which carry Hopf algebra structures. Motivated by this we determine necessary and sufficient conditions for an ambiskew polynomial ring to have a Hopf algebra structure of a certain form. For those algebras which admit such a Hopf algebra structure, we describe the dimensions of finite-dimensional simple modules in terms of their highest weights and prove a generalized Clebsch-Gordan formula. Using this and a Casimir-type operator we prove that all finite-dimensional weight modules are semi-simple. The existence of finite-dimensional non-semisimple modules over $U_q(f(H,K))$ was noted in [WJY].

6.3 Unitarizable weight modules over generalized Weyl algebras

A $*$-*algebra* A is a $\mathbb{C}$-algebra equipped with an *involution* $*$, i.e. a map $* : A \to A$ satisfying

$$(a+b)^* = a^* + b^*, \quad (\lambda a)^* = \bar{\lambda} a^*, \quad (ab)^* = b^* a^*, \quad (a^*)^* = a$$

for all $a, b \in A$, $\lambda \in \mathbb{C}$. A module V over a $*$-algebra A is called *unitarizable* (with respect to $*$) if there is an inner product $(\cdot,\cdot)$ on V such that

$$(av, w) = (v, a^* w) \quad \forall v, w \in V, a \in A.$$

The notion of unitarizability is essential in quantum mechanics as well as representation theory of finite and compact groups. One nice property of unitarizable modules is that they are automatically semisimple.

Unitarizable modules over higher rank and twisted GWAs were studied in [MT01c] and [MT02] respectively.

Unfortunately there may be many modules over a $*$-algebra which are not unitarizable. However, if we relax the condition of the inner product to allow also forms which are not necessarily positive definite (but still non-degenerate), then more modules will be unitarizable in this weaker sense. One looses the semisimplicity feature but some properties remain valid.

In [MT01a], the authors prove general results on existence and uniqueness of indefinite forms on modules over algebras. In [MT01b], and independently in [G], simple weight modules over a semisimple Lie algebra which are unitarizable with an indefinite form were classified.

In this paper we define a notion of unitarizability for weight modules over a generalized Weyl algebra $R(\sigma, t)$ of rank one, where R is assumed to be a commutative ring. The notion does not require the presence of a ground field. The forms are not assumed to be positive definite (for this has no meaning in our setting). We classify all indecomposable weight modules over $R(\sigma, t)$ which are unitarizable with a non-degenerate form. Our method uses the description of weight modules in [DGO].

6.4 The elliptic dynamical $GL(n)$ as an $\mathfrak{h}$-Hopf algebroid

Etingof and Varchenko [EV] have generalized the FRST-construction to the case of R-matrices depending on a dynamical parameter. The result is not a bialgebra, but rather a generalization called an $\mathfrak{h}$-*bialgebroid*. Here $\mathfrak{h}$ is a certain vector space. When $\mathfrak{h} = \{0\}$ one recovers ordinary bialgebras. If an antipode has been assigned they are called $\mathfrak{h}$-*Hopf algebroids*.

In [KNR] the authors extend this FRST procedure to construct an $\mathfrak{h}$-Hopf algebroid from an elliptic R-matrix depending on both a dynamical and a spectral parameter and used it to reprove certain biorthogonality relations for elliptic hypergeometric series previously obtained by Frenkel and Turaev. Their algebra may be considered as a deformation of the algebra of functions $\mathscr{F}(U(2))$ on the group $U(2)$.

In the fourth and final paper we construct an $\mathfrak{h}$-Hopf algebroid associated to $GL(n)$ from an elliptic R-matrix with dynamical and spectral parameter. On the way we are led to consider analogues of the exterior algebra and minors and their Laplace expansions. We prove that the left and right versions of the minors coincide and, using the dynamical version of a *cobraiding*, introduced in [Ro], that the analogue of the determinant is almost central.

7 Outlook and open problems

7.1 Lie algebras and twisted generalized Weyl algebras

According to [MPT], Yu. Drozd posed the question of finding a construction for a family of algebras which would include as examples both higher rank generalized Weyl algebras and universal enveloping algebras of semisimple complex Lie algebras. This question is very interesting, since generalized Weyl algebras of rank one provide a playground for generalizations and deformations of $U(\mathfrak{sl}_2)$.

The twisted generalized Weyl algebras, introduced in [MT99], was an attempt to answer this question and are natural since they allow more noncommutativity than the higher rank GWAs. However, still only indication exists that $U(\mathfrak{g})$ really is a TGWA. In [MT99] some features of the support of a module gave some evidence and in [MPT] the authors proved that a certain extension of $U(\mathfrak{gl}_n)$ is a TGWA.

Recently another candidate for such an algebra has been proposed in [FO], called *Galois orders*.

However, the question for TGWA remains. A first step, which we think may be carried out, would be to associate to each semisimple Lie algebra $\mathfrak{g}$ a twisted generalized Weyl algebra T such that (almost) any finite-dimensional module over $U(\mathfrak{g})$ becomes naturally a module over T.

7.2 Classification of tame blocks in the category of weight modules over a TGWA

The category of weight modules over a twisted or higher rank GWA splits into a sum of *blocks*, which consists of all indecomposable weight modules with support in a given orbit in $\mathrm{Max}(R)$. It is very valuable to have a description of when a given block is tame or wild, in the sense of [D].

In [BB] the authors classified tame blocks in the category of weight modules over a certain class of higher rank GWA. To obtain a similar classification for modules over (some) twisted GWA would be of great value.

7.3 Semisimplicity of the category of weight modules over a TGWA

In [DGO] a necessary and sufficient criterion was obtained as to when the category of weight modules over a rank one GWA is semisimple. Finding such a criterion for twisted GWAs would be very interesting.

7.4 Elliptic Weyl algebra

$SL(n)$ acts naturally as a group of algebra automorphisms on the n:th Weyl algebra A_n by linear change of variables. This carries over to the quantum situation. Namely, in [PW] it is proved that the quantum Weyl algebra, which we consider in Paper I as an example of twisted generalized Weyl algebra (but one should take all q_i's to be equal) is a comodule algebra over the deformed function algebra $\mathscr{F}_q(SL(n))$. This deformed function algebra is a limiting case of the elliptic quantum group considered in Paper IV.

Thus it is natural to ask whether there exists an elliptic dynamical analogue of the Weyl algebra, which carries a comodule algebra structure over the elliptic dynamical $GL(n)$ quantum group, and which perhaps is also an example of a twisted generalized Weyl algebra. One may be able to proceed as in [GZ] where the authors associate a quantum Weyl algebra to any constant R-matrix satisfying the Hecke condition.

One should also try to generalize Woronowicz's theory of differential calculus covariant under quantum groups to the elliptic dynamical setting, and try to view the elliptic Weyl algebra as generated by twisted annihilation and creation operators.

7.5 Selfduality of dynamical quantum groups

In [Ro] a notion of duality for quantum groups associated to dynamical R-matrices was defined. It was also proved that the dynamical trigonometric $SU(2)$ quantum group is essentially self-dual. It means that the algebra is simultaneously a deformation of the enveloping algebra $U(\mathfrak{sl}_2)$ and of the function algebra $\mathscr{F}(SL(2))$. It would be very interesting to prove such a self-duality result for higher rank groups and also for the elliptic case.

7.6 Higher genus quantum groups

So far almost all known quantum groups have a spectral parameter which lives on a Riemann surface of genus 0 or 1. There have been no general results towards a higher genus theory. Perhaps one approach would be to first investigate the what the corresponding special functions should be. Indeed, historically, the special functions corresponding to $U_q(\mathfrak{g})$ as well as elliptic quantum groups predated the definition of the algebras themselves.

References

[B91] V. V. Bavula, *Finite-dimensionality of* Ext^n *and* Tor_n *of simple modules over a class of algebras*, Funktsional. Anal. i Prilozhen. **25** No. 3 (1991) 80–82.

[B92] V. V. Bavula, *Generalized Weyl algebras and their representations*, Algebra i Analiz **4** No. 1 (1992) 75–97; English transl. in St Petersburg Math. J. **4** (1993) 71–92.

[B93] V. V. Bavula, *Generalized Weyl algebras, kernel and tensor-simple algebras, their simple modules*, CMS Conf. Proc. **14** (1993) 83–107.

[B96] V. V. Bavula *Global dimension of Generalized Weyl algebras*, CMS Conf. Proc. **18** (1996) 81–107.

[BB] V. V. Bavula, V. Bekkert, *Indecomposable representations of generalized Weyl algebras*, Comm. Algebra **28** (2000) 5067–5100.

[BBF] V. Bekkert, G. Benkart, V. Futorny, *Weight modules for Weyl algebras*, in "Kac-Moody Lie algebras and related topics (Proc. Ramanujan internat. symp., Chennai, India, 2002)", Contemp. Math. **343**, Amer. Math. Soc., Providence, RI, (2004) 17–42.

[BD] A. A. Belavin, V. G. Drinfel'd, *Solutions of the classical Yang-Baxter equation for simple Lie algebras*, Funct. Anal. Appl. **16** No. 3 (1982) 159–180.

[BJ] V. V. Bavula, D. A. Jordan, *Isomorphism problems and groups of automorphisms for generalized Weyl algebras*, Trans. Amer. Math. Soc. **353** No. 2 (2000) 769–794.

[BO] V. V. Bavula, F. van Oystaeyen, *Simple modules of the Witten-Woronowicz algebra*, J. Algebra **271** (2004) 827–845.

[BR] G. Benkart, T. Roby, *Down-up algebras*, J. Algebra **209** (1998) 305-344; Addendum, J. Algebra **213** No. 1 (1999) 378.

[BW] G. Benkart, S. Witherspoon, *A Hopf structure for down-up algebras*, Math. Z. **238** No. 3 (2001) 523–553.

[CM] P. A. A. B. Carvalho, I. M. Musson, *Down-up algebras and their representation theory*, J. Algebra **228** (2000) 286–310.

[CL] P. A. A. B. Carvalho, S. A. Lopes, *Automorphisms of generalized down-up algebras*, arXiv:0706.3355v1 [math.RA].

[CS] T. Cassidy, B. Shelton, *Basic properties of generalized down-up algebras*, J. Algebra **279** (2004) 402–421.

[D] Yu. A. Drozd, *Tame and wild matrix problems*, in "Representation Theory II (Proc. Second Internat. Conf. Carleton Univ. Ottawa Ontario 1979)" Springer Lect. Notes in Math. **832** (1980) 242–258.

[DGO] Yu. A. Drozd, B. L. Guzner, S. A. Ovsienko, *Weight modules over generalized Weyl algebras*, J. Algebra **184** (1996) 491–504.

[EV] P. Etingof, A. Varchenko, *Solutions of the quantum dynamical Yang-Baxter equation and dynamical quantum groups*, Comm. Math. Phys. **196** No. 3 (1998) 591–640.

[FO] V. Futorny, S. Ovsienko, *Galois orders*, arXiv:math/0610069v2 [math.RT].

[G] J. Germoni, *Contravariant forms for weight modules*, preprint.

[GP] I. M. Gel'fand, V. A. Ponomarev, *Indecomposable representations of the Lorentz group*, Usp. Mat. Nauk **23** No. 2 (1968) 3–60; English transl. in Russ. Math. Surv. **23** No. 2 (1968) 1–58.

[GZ] A. Giaquinto, J. Zhang, *Quantum Weyl algebras*, J. Algebra **176** (1995) 861–881.

[H] T. J. Hodges, *Noncommutative deformation of type-A Kleinian singularities*, J. Algebra **161** No. 2 (1993) 271–290.

[J93] D. A. Jordan, *Primitivity in skew Laurent polynomial rings and related rings*, Math. Z. **213** (1993) 353–371.

[J95] D. A. Jordan, *Finite-dimensional simple modules over certain iterated skew polynomial rings*, J. Pure Appl. Algebra **98** (1995) 45–55.

[J00] D. A. Jordan, *Down-up algebras and ambiskew polynomial rings*, J. Algebra **228** (2000) 311–346.

[KM] E. E. Kirkman, I. M. Musson, *Hopf down-up algebras*, J. Algebra **262** (2003) 42–53.

[KNR] E. Koelink, Y. van Norden, H. Rosengren, *Elliptic* U(2) *quantum group and elliptic hypergeometric series*, Comm. Math. Phys. **245** No. 3 (2004) 519–537.

[KS] E. Kissin, V. Shulman, *Representations on Krein spaces and derivations of C^*-algebras*, Pitman Monographs and Surveys in Pure and Applied Mathematics **89**, Longman, Harlow, (1997).

[MPT] V. Mazorchuk, M. Ponomarenko, L. Turowska, *Some associative algebras related to $U(\mathfrak{g})$ and twisted generalized Weyl algebras*, Math. Scand. **92** (2003) 5–30.

[MT99] V. Mazorchuk, L. Turowska, *Simple weight modules over twisted generalized Weyl algebras*, Comm. Alg. **27** No. 6 (1999) 2613–2625.

[MT01a] V. Mazorchuk, L. Turowska, *Existence and unicity of σ-forms on finite-dimensional modules* Meth. Funct. Anal. Top. **7** (2001) 53–62.

[MT01b] V. Mazorchuk, L. Turowska, *σ-unitarizability of simple weight modules*, Proc. Int. Conf. "Information Processes of formation of Open Systems", Vologda, 26–28 June, (2001) 253–257.

[MT01c] V. Mazorchuk, L. Turowska, *Involutions on generalized Weyl algebras preserving the principal grading*, Rep. Math. Phys. **48** No 3. (2001) 343–351.

[MT02] V. Mazorchuk, L. Turowska, **-Representations of twisted generalized Weyl constructions*, Algebr. Represent. Theory **5** No. 2 (2002) 163–186.

[PW] W. Pusz, S. L. Woronowicz, *Twisted second quantization*, Reports on Math. Phys. **27** No. 2 (1989) 231–257.

[R] A. L. Rosenberg, *Noncommutative algebraic geometry and representations of quantized algebras*, Kluwer, Dordrecht (1995).

[Ro] H. Rosengren, *Duality and self-duality for dynamical quantum groups*, Algebr. Represent. Theory **7** No. 4 (2004) 363–393.

[Ru] S. Rueda, *Some algebras similar to the enveloping algebra of $sl(2)$*, Comm. Algebra **30** No. 3 (2002) 1127–1152.

[S] S. P. Smith, *A class of algebras similar to the enveloping algebra of $sl(2)$*, Trans. Am. Math. Soc. **322** No. 1 (1990) 285–314.

[WJY] D. Wang, Q. Ji, S. Yang, *Finite-dimensional representations of quantum group $U_q(f(H,K))$*, Comm. Alg. **30** No. 5 (2002) 2191–2211.

Paper I

Locally finite simple weight modules over twisted generalized Weyl algebras

Jonas T. Hartwig

Abstract

We present methods and explicit formulas for describing simple weight modules over twisted generalized Weyl algebras. When a certain commutative subalgebra is finitely generated over an algebraically closed field we obtain a classification of a class of locally finite simple weight modules as those induced from simple modules over a subalgebra isomorphic to a tensor product of non-commutative tori. As an application we describe simple weight modules over the quantized Weyl algebra.

1 Introduction

Bavula defined in [1], [2] the notion of a generalized Weyl algebra (GWA) which is a class of algebras which include $U(\mathfrak{sl}(2))$, $U_q(\mathfrak{sl}(2))$, down-up algebras, and the Weyl algebra, as examples. In addition to various ring theoretic properties, the simple modules were also described for some GWAs in [2]. In [6] all simple and indecomposable weight modules of GWAs of rank (or degree) one were classified.

So called higher rank GWAs were defined in [2] and in [3] the authors studied indecomposable weight modules over certain higher rank GWAs.

In [8], with the goal to enrich the representation theory in the higher rank case, the authors defined the twisted generalized Weyl algebras (TGWA). This is a class of algebras which include all higher rank GWAs (if a certain subring R has no zero divisors) and also many algebras which can be viewed as twisted tensor products of rank one GWAs, for example certain Mickelsson step algebras and extended Orthogonal Gelfand-Zetlin algebras [7]. Under a technical assumption on the algebra formulated using a biserial graph, some torsion-free simple weight modules were described in [8]. Simple graded weight modules were studied in [7] using an analogue of the Shapovalov form.

In this paper we describe a more general class of locally finite simple weight modules over TGWAs using the well-known technique of considering the maximal graded subalgebra which preserves the weight spaces. It is known that under quite general assumptions (see Theorem 18 in [5]) any simple weight module over a TGWA is a unique quotient of a module which is induced from a simple module over this subalgebra. Our main results are the description of this subalgebra under various assumptions (Theorem 4.5 and Theorem 4.8) and the explicit formulas (Theorem 5.4) of the associated module of the TGWA. In contrast to [8], we do not assume that the orbits are torsion-free and we allow the modules to have some inner breaks, as long as they do not have any so called *proper* inner breaks (see Definition 3.7). The weight spaces will not in general be one-dimensional in our case, which was the case in [8], [7].

Moreover, as an application we classify the simple weight modules without proper inner breaks over a quantized Weyl algebra of rank two (Theorem 6.14).

The paper is organized as follows. In Section 2 the definitions of twisted generalized Weyl constructions and algebras are given together with some examples. Weight modules and the subalgebra $B(\omega)$ are defined.

In Section 3 we first prove some simple facts and then define the class of simple weight modules with no proper inner breaks. We also show that this class properly contains all the modules studied in [8].

Section 4 is devoted to the description of the subalgebra $B(\omega)$. When the ground field is algebraically closed and a certain subalgebra R is finitely generated, we show that it is isomorphic to a tensor product of noncommutative tori for which the finite-dimensional irreducible representations are easy to describe.

In Section 5 we specify a basis and give explicit formulas for the irreducible quotient of the induced module.

Finally, in Section 6 we consider as an example the quantized Weyl algebra and determine certain important subsets of $\mathbb{Z}^n$ related to $B(\omega)$ and the support of modules as solutions to some systems of equations. In the rank two case we describe all simple weight modules with finite-dimensional weight spaces and no proper inner breaks.

2 Definitions

2.1 The TGWC and TGWA

Fix a positive integer n and set $\underline{n} = \{1,2,\dots,n\}$. Let K be a field, and let R be a commutative unital K-algebra, $\boldsymbol{\sigma} = (\sigma_1,\dots,\sigma_n)$ be an n-tuple of pairwise commuting K-automorphisms of R, $\boldsymbol{\mu} = (\mu_{ij})_{i,j\in\underline{n}}$ be a matrix with entries from $K^* := K\backslash\{0\}$ and $\boldsymbol{t} = (t_1,\dots,t_n)$ be an n-tuple of nonzero elements from R. The *twisted generalized Weyl construction* (TGWC) A' obtained from the data $(R,\boldsymbol{\sigma},\boldsymbol{t},\boldsymbol{\mu})$ is the unital K-algebra generated over R by $X_i, Y_i, (i \in \underline{n})$ with the relations

$$X_i r = \sigma_i(r)X_i, \qquad Y_i r = \sigma_i^{-1}(r)Y_i, \qquad \text{for } r \in R, i \in \underline{n}, \tag{2.1}$$

$$Y_i X_i = t_i, \qquad X_i Y_i = \sigma_i(t_i), \qquad \text{for } i \in \underline{n}, \tag{2.2}$$

$$X_i Y_j = \mu_{ij} Y_j X_i, \qquad \text{for } i,j \in \underline{n}, i \neq j. \tag{2.3}$$

From the relations (2.1)–(2.3) follows that A' carries a $\mathbb{Z}^n$-gradation $\{A'_g\}_{g\in\mathbb{Z}^n}$ which is uniquely defined by requiring

$$\deg X_i = e_i, \quad \deg Y_i = -e_i, \quad \deg r = 0, \quad \text{for } i \in \underline{n}, r \in R,$$

where $e_i = (0,\dots,\overset{i}{1},\dots,0)$. The *twisted generalized Weyl algebra* (TGWA) $A = A(R,\boldsymbol{\sigma},\boldsymbol{t},\boldsymbol{\mu})$ of *rank* n is defined to be A'/I, where I is the sum of all graded two-sided ideals of A' intersecting R trivially. Since I is graded, A inherits a $\mathbb{Z}^n$-gradation $\{A_g\}_{g\in\mathbb{Z}^n}$ from A'.

Note that from relations (2.1)–(2.3) follows the identity

$$X_iX_jt_i = X_jX_i\mu_{ji}\sigma_j^{-1}(t_i) \tag{2.4}$$

which holds for $i,j\in\underline{n}, i\neq j$. Multiplying (2.4) from the left by $\mu_{ij}Y_j$ we obtain

$$X_i\big(t_it_j - \mu_{ij}\mu_{ji}\sigma_i^{-1}(t_j)\sigma_j^{-1}(t_i)\big) = 0 \tag{2.5}$$

for $i,j\in\underline{n}, i\neq j$. One can show that the algebra A', hence A, is nontrivial if one assumes that $t_it_j = \mu_{ij}\mu_{ji}\sigma_i^{-1}(t_j)\sigma_j^{-1}(t_i)$ for $i,j\in\underline{n}, i\neq j$. Analogous identities exist for Y_i.

2.2 Examples

Some of the first motivating examples of a *generalized Weyl algebra* (GWA), i.e. a TGWC of rank 1, are $U(\mathfrak{sl}(2))$, $U_q(\mathfrak{sl}(2))$ and of course the Weyl algebra A_1. We refer to [2] for details.

We give some examples of TGWAs of higher rank.

2.2.1 Quantized Weyl algebras

Let $\Lambda = (\lambda_{ij})$ be an $n\times n$ matrix with nonzero complex entries such that $\lambda_{ij} = \lambda_{ji}^{-1}$. Let $\bar{q} = (q_1,\dots,q_n)$ be an n-tuple of elements of $\mathbb{C}\backslash\{0,1\}$. The n:th quantized Weyl algebra $A_n^{\bar{q},\Lambda}$ is the $\mathbb{C}$-algebra with generators x_i, y_i, $1\le i\le n$, and relations

$$x_ix_j = q_i\lambda_{ij}x_jx_i, \qquad y_iy_j = \lambda_{ij}y_jy_i, \tag{2.6}$$

$$x_iy_j = \lambda_{ji}y_jx_i, \qquad x_jy_i = q_i\lambda_{ij}y_ix_j, \tag{2.7}$$

$$x_iy_i - q_iy_ix_i = 1 + \sum_{k=1}^{i-1}(q_k-1)y_ix_i, \tag{2.8}$$

for $1\le i<j\le n$. Let $R = \mathbb{C}[t_1,\dots,t_n]$ be the polynomial algebra in n variables and σ_i the $\mathbb{C}$-algebra automorphisms defined by

$$\sigma_i(t_j) = \begin{cases} t_j, & j<i,\\ 1+q_it_i+\sum_{k=1}^{i-1}(q_k-1)t_k, & j=i,\\ q_it_j, & j>i.\end{cases} \tag{2.9}$$

One can check that the σ_i commute. Let $\boldsymbol{\mu} = (\mu_{ij})_{i,j\in\underline{n}}$ be defined by $\mu_{ij} = \lambda_{ji}$ and $\mu_{ji} = q_i\lambda_{ij}$ for $i<j$. Let also $\boldsymbol{\sigma} = (\sigma_1,\dots,\sigma_n)$ and $\boldsymbol{t} = (t_1,\dots,t_n)$. One can show that the maximal graded ideal of the TGWC $A'(R,\boldsymbol{\sigma},\boldsymbol{t},\boldsymbol{\mu})$ is generated by the elements

$$X_iX_j - q_i\lambda_{ij}X_jX_i,\ Y_iY_j - \lambda_{ij}Y_jY_i, \quad 1\le i<j\le n.$$

Thus $A_n^{\bar{q},\Lambda}$ is isomorphic to the TGWA $A(R,\boldsymbol{\sigma},\boldsymbol{t},\boldsymbol{\mu})$ via $x_i\mapsto X_i$, $y_i\mapsto Y_i$.

2.2.2 Q_{ij}-CCR

Let $(Q_{ij})_{i,j=1}^d$ be an $d \times d$ matrix with complex entries such that $Q_{ij} = Q_{ji}^{-1}$ if $i \neq j$ and A_d be the algebra generated by elements a_i, a_i^*, $1 \leq i \leq d$ and relations

$$a_i^* a_i - Q_{ii} a_i a_i^* = 1, \qquad a_i^* a_j = Q_{ij} a_j a_i^*,$$
$$a_i a_j = Q_{ji} a_j a_i, \qquad a_i^* a_j^* = Q_{ij} a_j^* a_i^*,$$

where $1 \leq i, j \leq d$ and $i \neq j$. Let $R = \mathbb{C}[t_1, \ldots, t_d]$ and define the automorphisms σ_i of R by $\sigma_i(t_j) = t_j$ if $i \neq j$ and $\sigma_i(t_i) = 1 + Q_{ii} t_i$. Let $\mu_{ij} = Q_{ji}$ for all i, j. Then A_d is isomorphic to the TGWA $A(R, (\sigma_1, \ldots, \sigma_n), (t_1, \ldots, t_n), \boldsymbol{\mu})$.

2.2.3 Mickelsson and OGZ algebras

In both the above examples the generators X_i and X_j commute up to a multiple of the ground field. This need not be the case as shown in [7], where it was shown that Mickelsson step algebras and extended orthogonal Gelfand-Zetlin algebras are TGWAs.

2.3 Weight modules

Let A be a TGWC or a TGWA. Let $\mathrm{Max}(R)$ denote the set of all maximal ideals in R. A module M over A is called a *weight module* if

$$M = \oplus_{\mathfrak{m} \in \mathrm{Max}(R)} M_{\mathfrak{m}},$$

where

$$M_{\mathfrak{m}} = \{v \in M \mid \mathfrak{m} v = 0\}.$$

The *support*, $\mathrm{supp}(M)$, of M is the set of all $\mathfrak{m} \in \mathrm{Max}(R)$ such that $M_{\mathfrak{m}} \neq 0$. A weight module is *locally finite* if all the weight spaces $M_{\mathfrak{m}}$, $\mathfrak{m} \in \mathrm{supp}(M)$, are finite-dimensional over the ground field K.

Since the σ_i are pairwise commuting, the free abelian group $\mathbb{Z}^n$ acts on R as a group of K-algebra automorphisms by

$$g(r) = \sigma_1^{g_1} \sigma_2^{g_2} \ldots \sigma_n^{g_n}(r) \tag{2.10}$$

for $g = (g_1, \ldots, g_n) \in \mathbb{Z}^n$ and $r \in R$. Then $\mathbb{Z}^n$ also acts naturally on $\mathrm{Max}(R)$ by $g(\mathfrak{m}) = \{g(r) \mid r \in \mathfrak{m}\}$. Note that

$$X_i M_{\mathfrak{m}} \subseteq M_{\sigma_i(\mathfrak{m})} \quad \text{and} \quad Y_i M_{\mathfrak{m}} \subseteq M_{\sigma_i^{-1}(\mathfrak{m})} \tag{2.11}$$

for any $\mathfrak{m} \in \mathrm{Max}(R)$. If $a \in A$ is homogeneous of degree $g \in \mathbb{Z}^n$, then by using (2.1) and (2.11) repeatedly one obtains the very useful identities

$$a \cdot r = g(r) \cdot a, \quad r \cdot a = a \cdot (-g)(r), \tag{2.12}$$

for $r \in R$ and

$$a M_{\mathfrak{m}} \subseteq M_{g(\mathfrak{m})} \tag{2.13}$$

for $\mathfrak{m} \in \mathrm{Max}(R)$.

2.4 Subalgebras leaving the weight spaces invariant

Let $\omega \subseteq \mathrm{Max}(R)$ be an orbit under the action of $\mathbb{Z}^n$ on $\mathrm{Max}(R)$ defined in (2.10). Let

$$\mathbb{Z}^n_\omega = \mathbb{Z}^n_\mathfrak{m} = \{g \in \mathbb{Z}^n \mid g(\mathfrak{m}) = \mathfrak{m}\} \tag{2.14}$$

where $\mathfrak{m}$ is some point in ω. Since $\mathbb{Z}^n$ is abelian, $\mathbb{Z}^n_\omega$ does not depend on the choice of $\mathfrak{m}$ from ω. Define

$$B(\omega) = \oplus_{g \in \mathbb{Z}^n_\omega} A_g. \tag{2.15}$$

Since A is $\mathbb{Z}^n$-graded and since $\mathbb{Z}^n_\omega$ is a subgroup of $\mathbb{Z}^n$, $B(\omega)$ is a subalgebra of A and $R = A_0 \subseteq B(\omega)$. Let $\mathfrak{m} \in \omega$ and suppose that M is a simple weight A-module with $\mathfrak{m} \in \mathrm{supp}(M)$. Since M is simple we have $\mathrm{supp}(M) \subseteq \omega$. Using (2.13) it follows that $B(\omega)M_\mathfrak{m} \subseteq M_\mathfrak{m}$ and by definition $M_\mathfrak{m}$ is annihilated by $\mathfrak{m}$ hence also by the two-sided ideal $(\mathfrak{m})$ in $B(\omega)$ generated by $\mathfrak{m}$. Thus $M_\mathfrak{m}$ is naturally a module over the algebra

$$B_\mathfrak{m} := B(\omega)/(\mathfrak{m}). \tag{2.16}$$

By Proposition 7.2 in [7] (see also Theorem 18 in [5] for a general result), $M_\mathfrak{m}$ is a simple $B_\mathfrak{m}$-module, and any simple $B_\mathfrak{m}$-module occurs as a weight space in a simple weight A-module. Moreover, two simple weight A-modules M, N are isomorphic if and only if $M_\mathfrak{m}$ and $N_\mathfrak{m}$ are isomorphic as $B_\mathfrak{m}$-modules. Therefore we are led to study the algebra $B_\mathfrak{m}$ and simple modules over it.

3 Preliminaries

3.1 Reduced words

Let $L = \{X_i\}_{i \in \underline{n}} \cup \{Y_i\}_{i \in \underline{n}}$. By a *word* $(a; Z_1, \ldots, Z_k)$ in A we will mean an element a in A which is a product of elements from the set L, together with a fixed tuple $(Z_1, \ldots, Z_k)$ of elements from L such that $a = Z_1 \cdot \ldots \cdot Z_k$. When referring to a word we will often write $a = Z_1 \ldots Z_k \in A$ to denote the word $(a; Z_1, \ldots, Z_k)$ or just write $a \in A$, suppressing the fixed representation of a as a product of elements from L.

Set $X_i^* = Y_i$ and $Y_i^* = X_i$. For a word $a = Z_1 \ldots Z_k \in A$ we define

$$a^* := Z_k^* \cdot Z_{k-1}^* \cdot \ldots \cdot Z_1^*.$$

In the special case when $\mu_{ij} = \mu_{ji}$ for all i, j then by (2.1)-(2.3) there is an anti-involution $*$ on A' defined by $X_i^* = Y_i$, and $r^* = r$ for $r \in R$. Since $I^* = I$ this anti-involution carries over to A.

Definition 3.1. A word $Z_1 \ldots Z_k$ will be called *reduced* if

$$Z_i \neq Z_j^* \text{ for } i, j \in \underline{k}$$

and

$$Z_i \in \{X_r\}_{r \in \underline{n}} \Longrightarrow Z_j \in \{X_r\}_{r \in \underline{n}} \ \forall j \geq i.$$

For example $Y_1Y_2Y_1X_3$ is reduced whereas $Y_1Y_2X_1$ and $Y_1X_2Y_3$ are not. The following lemma and corollary explains the importance of the reduced words.

Lemma 3.2. *Any word b in A can be written $b = a\cdot r = r'\cdot a$, where a is a reduced word, and $r, r' \in R$.*

Proof. If a and r has been found we can take $r' = (\deg a)(r)$, according to (2.12). Thus we concentrate on finding a and r. Let $b = Z_1 \dots Z_k$ be an arbitrary word in A. We prove the statement by induction on k. If $k = 1$, then b is necessarily reduced so take $a = b, r = 1$. When $k > 1$, use the induction hypothesis to write

$$Z_1 \dots Z_{k-1} = Y_{i_1} \dots Y_{i_l} X_{j_1} \dots X_{j_m} \cdot r',$$

where $1 \le i_u, j_v \le n$ and $i_u \ne j_v$ for any u, v. Consider first the case when $Z_k = Y_j$ for some $j \in \underline{n}$. Then

$$Z_1 \dots Z_k = Y_{i_1} \dots Y_{i_l} X_{j_1} \dots X_{j_m} Y_j \cdot \sigma_j(r').$$

If $j_v \ne j$ for $v = 1, \dots, m$ we are done because using relation (2.3) repeatedly we obtain,

$$Z_1 \dots Z_k = Y_{i_1} \dots Y_{i_l} Y_j X_{j_1} \dots X_{j_m} \cdot \mu\sigma_j(r')$$

for some $\mu \in K^*$. Otherwise, let $v \in \{1, \dots, m\}$ be maximal such that $j_v = j$. Then

$$\begin{aligned} Z_1 \dots Z_k &= Y_{i_1} \dots Y_{i_l} X_{j_1} \dots X_{j_v} Y_j X_{j_{v+1}} \dots X_{j_m} \mu\sigma_j(r') = \\ &= Y_{i_1} \dots Y_{i_l} X_{j_1} \dots X_{j_{v-1}} X_{j_{v+1}} \dots X_{j_m} w(t_j)\mu\sigma_j(r') \end{aligned}$$

for some $\mu \in K^*$ and some $w \in W$. It remains to consider the case $Z_k = X_j$ for some $j \in \underline{n}$. But using that

$$Y_{i_1} \dots Y_{i_l} X_{j_1} \dots X_{j_m} = X_{j_1} \dots X_{j_m} Y_{i_1} \dots Y_{i_l} \mu$$

for some $\mu \in K^*$, it is clear that this case is analogous. □

Corollary 3.3. *Each A_g, $g \in W$, is generated as a right (and also as a left) R-module by the reduced words of degree g.*

Lemma 3.4. *Suppose $*$ defines an anti-involution on A. Let $\mathfrak{p}$ be a prime ideal of R. Let $g \in \mathbb{Z}^n$ and let $a \in A_g$. If $ba \notin \mathfrak{p}$ for some $b \in A_{-g}$ then $a^*a \notin \mathfrak{p}$.*

Proof. Since $\mathfrak{p}$ is prime, and $ba \in R$ we have

$$\mathfrak{p} \not\ni (ba)^2 = (ba)^* ba = a^* b^* ba = a^* a \cdot (-\deg a)(b^* b)$$

so in particular $a^*a \notin \mathfrak{p}$. □

Remark 3.5. If we assume a and b to be words in the formulation of Lemma 3.4, one can easily show that the statement remains true without the restriction on $*$ to be an anti-involution.

3.2 Inner breaks and canonical modules

Let A be a TGWC or a TGWA and let M be a simple weight module over A. In [8] Remark 1 it was noted that the problem of describing simple weight modules over a TGWC is wild in general. This is a motivation for restricting attention to some subclass which has nice properties. In [8] the following definition was made.

Definition 3.6. The support of M has *no inner breaks* if for all $\mathfrak{m} \in \operatorname{supp}(M)$,

$$t_i \in \mathfrak{m} \Longrightarrow \sigma_i(\mathfrak{m}) \notin \operatorname{supp}(M), \text{ and}$$
$$\sigma_i(t_i) \in \mathfrak{m} \Longrightarrow \sigma_i^{-1}(\mathfrak{m}) \notin \operatorname{supp}(M).$$

We introduce the following property.

Definition 3.7. We say that M has *no proper inner breaks* if for any $\mathfrak{m} \in \operatorname{supp}(M)$ and any word a with $aM_\mathfrak{m} \neq 0$ we have $a^*a \notin \mathfrak{m}$.

Observe that whether or not $a^*a \in \mathfrak{m}$ for a word a does not depend on the particular representation of a as a product of generators. Note also that to prove that a simple weight module M has no proper inner breaks, it is sufficient to find for any $\mathfrak{m} \in \operatorname{supp}(M)$ and any word a with $aM_\mathfrak{m} \neq 0$ a word $b \in A$ of degree $-\deg a$ such that $ba \notin \mathfrak{m}$ because then $a^*a \notin \mathfrak{m}$ automatically by Remark 3.5. In fact one can show that a simple weight module M has no proper inner breaks if (and only if) there exists an $\mathfrak{m} \in \operatorname{supp}(M)$ such that for any reduced word $a \in A$ with $aM_\mathfrak{m} \neq 0$ and $aM_\mathfrak{m} \subseteq M_\mathfrak{m}$ there is a word b of degree $-\deg a$ such that $ba \notin \mathfrak{m}$. However we will not use this result.

The choice of terminology in Definition 3.7 is motivated by the following proposition.

Proposition 3.8. *If M has no inner breaks, then M has no proper inner breaks either.*

Proof. Let $\mathfrak{m} \in \operatorname{supp}(M)$ and $a = Z_1 \dots Z_k \in A$ be a word such that $aM_\mathfrak{m} \neq 0$. Thus $Z_i \dots Z_k M_\mathfrak{m} \neq 0$ for $i = 1, \dots, k+1$ so (2.13) implies that

$$(\deg Z_i \dots Z_k)(\mathfrak{m}) \in \operatorname{supp}(M).$$

If M has no inner breaks, it follows that $Z_i^* Z_i \notin (\deg Z_{i+1} \dots Z_k)(\mathfrak{m})$ for $i = 1, \dots, k$. Now using (2.12),

$$\begin{aligned} a^*a = Z_k^* \dots Z_1^* Z_1 \dots Z_k &= Z_k^* \dots Z_2^* Z_2 \dots Z_k (-\deg Z_2 \dots Z_k)(Z_1^* Z_1) = \\ &= \dots = \prod_{i=1}^{k} (-\deg Z_{i+1} \dots Z_k)(Z_i^* Z_i) \notin \mathfrak{m}. \end{aligned} \tag{3.1}$$

Thus M has no proper inner breaks. □

In [8], a simple weight module M was defined to be *canonical* if for any $\mathfrak{m}, \mathfrak{n} \in \operatorname{supp}(M)$ there is an automorphism σ of R of the form

$$\sigma = \sigma_{i_1}^{\varepsilon_1} \cdot \ldots \cdot \sigma_{i_k}^{\varepsilon_k}, \quad \varepsilon_j = \pm 1 \text{ and } 1 \leq i_j \leq n, \text{ for } j = 1, \dots, k,$$

such that $\sigma(\mathfrak{m}) = \mathfrak{n}$ and such that for each $j = 1, \ldots, k$,

$$t_{i_j} \notin \sigma_{i_{j+1}}^{\varepsilon_{j+1}} \ldots \sigma_{i_k}^{\varepsilon_k}(\mathfrak{m}) \quad \text{if } \varepsilon_j = 1, \text{ and} \tag{3.2}$$

$$\sigma_{i_j}(t_{i_j}) \notin \sigma_{i_{j+1}}^{\varepsilon_{j+1}} \ldots \sigma_{i_k}^{\varepsilon_k}(\mathfrak{m}) \quad \text{if } \varepsilon_j = -1. \tag{3.3}$$

This definition can be reformulated as follows.

Proposition 3.9. *M is canonical iff for any $\mathfrak{m}, \mathfrak{n} \in \operatorname{supp}(M)$ there is a word $a \in A$ such that $aM_\mathfrak{m} \subseteq M_\mathfrak{n}$ and $a^*a \notin \mathfrak{m}$.*

Proof. Suppose M is canonical, and let $\mathfrak{m}, \mathfrak{n} \in \operatorname{supp}(M)$. Let σ be as in the definition of canonical module. Define $a = Z_1 \ldots Z_k$ where $Z_j = X_{i_j}$ if $\varepsilon_j = 1$ and $Z_j = Y_{i_j}$ otherwise. Using (2.13) we see that $aM_\mathfrak{m} \subseteq M_\mathfrak{n}$. Also, (3.2) and (3.3) translates into

$$Z_j^* Z_j \notin (\deg Z_{j+1} \ldots Z_k)(\mathfrak{m})$$

for $j = 1, \ldots, k$. Using the calculation (3.1) and that $\mathfrak{m}$ is prime we deduce that $a^*a \notin \mathfrak{m}$.

Conversely, given a word $a = Z_1 \ldots Z_k \in A$ with $aM_\mathfrak{m} \subseteq M_\mathfrak{n}$ and $a^*a \notin \mathfrak{m}$, we define $\varepsilon_i = 1$ if $Z_i = X_i$ and $\varepsilon_i = -1$ otherwise. Then from $a^*a \notin \mathfrak{m}$ follows that $\sigma := \sigma_{i_1}^{\varepsilon_1} \cdot \ldots \cdot \sigma_{i_k}^{\varepsilon_k}$ satisfies (3.2) and (3.3) by the same reasoning as above. □

Corollary 3.10. *If M has no proper inner breaks, then M is canonical.*

Proof. We only need to note that since M is a simple weight module there is for each $\mathfrak{m}, \mathfrak{n} \in \operatorname{supp}(M)$ a word a such that $0 \neq aM_\mathfrak{m} \subseteq M_\mathfrak{n}$. □

Under the assumptions in [8] any canonical module has no inner breaks (see [8], Proposition 1). However we have the following example of a TGWA A and a simple weight module M over A which has no proper inner breaks, and thus is canonical by Corollary 3.10, but nonetheless has an inner break.

Example 3.11. Let $R = \mathbb{C}[t_1, t_2]$ and define the $\mathbb{C}$-algebra automorphisms σ_1 and σ_2 of R by $\sigma_i(t_j) = -t_j$ for $i, j = 1, 2$. Let $\boldsymbol{\mu} = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. Let $A' = A'(R, \boldsymbol{t}, \boldsymbol{\sigma}, \boldsymbol{\mu})$ be the associated TGWC, where $\boldsymbol{t} = (t_1, t_2), \boldsymbol{\sigma} = (\sigma_1, \sigma_2)$. Then one can check that $I = \langle X_1X_2 + X_2X_1, Y_1Y_2 + Y_2Y_1 \rangle$. Let M be a vector space over $\mathbb{C}$ with basis $\{v, w\}$ and define an A'-module structure on M by letting $X_1M = Y_1M = 0$ and

$$\begin{aligned} X_2 v &= w, & X_2 w &= v, \\ Y_2 v &= w, & Y_2 w &= -v. \end{aligned}$$

It is easy to check that the required relations are satisfied and that $IM = 0$, hence M becomes an A-module. Let $\mathfrak{m} = (t_1, t_2 + 1)$ and $\mathfrak{n} = (t_1, t_2 - 1)$. Then

$$M = M_\mathfrak{m} \oplus M_\mathfrak{n}, \quad \text{where } M_\mathfrak{m} = \mathbb{C}v, M_\mathfrak{n} = \mathbb{C}w$$

so M is a weight module. Any proper nonzero submodule of M would also be a weight module by standard results. That no such submodule can exist is easy to check, so M is simple. One checks that M has no proper inner breaks. But $t_1 \in \mathfrak{m}$ and $\sigma_1(\mathfrak{m}) = \mathfrak{n} \in \operatorname{supp}(M)$ so $\mathfrak{m}$ is an inner break.

4 The weight space preserving subalgebra and its irreducible representations

In this section, let A be a TGWC, $\mathfrak{m} \in \mathrm{Max}(R)$ and let ω be the $\mathbb{Z}^n$-orbit of $\mathfrak{m}$. Recall the set $\mathbb{Z}^n_\omega$ defined in (2.14). Define the following subsets of $\mathbb{Z}^n$:

$$\tilde{G}_\mathfrak{m} = \{g \in \mathbb{Z}^n \mid a^*a \notin \mathfrak{m} \text{ for some word } a \in A_g\} \quad \text{and} \quad G_\mathfrak{m} = \tilde{G}_\mathfrak{m} \cap \mathbb{Z}^n_\omega. \tag{4.1}$$

Let also $\varphi_\mathfrak{m} : A \to A/(\mathfrak{m})$ denote the canonical projection, where $(\mathfrak{m})$ is the two-sided ideal in A generated by $\mathfrak{m}$, and let $R_\mathfrak{m} = R/\mathfrak{m}$ be the residue field of R at $\mathfrak{m}$.

Lemma 4.1. *Let $g \in \tilde{G}_\mathfrak{m}$. Then*

$$\varphi_\mathfrak{m}(A_g) = R_\mathfrak{m} \cdot \varphi_\mathfrak{m}(a) = \varphi_\mathfrak{m}(a) \cdot R_\mathfrak{m} \tag{4.2}$$

for any word $a \in A_g$ with $a^*a \notin \mathfrak{m}$.

Proof. Let $b \in A_g$ be any element and $a \in A_g$ a word such that $a^*a \notin \mathfrak{m}$, We must show that there is an $r \in R$ such that $\varphi_\mathfrak{m}(b) = \varphi_\mathfrak{m}(r)\varphi_\mathfrak{m}(a)$. Since $a^*a \notin \mathfrak{m}$ and $\mathfrak{m}$ is maximal, $1 - r_1a^*a \in \mathfrak{m}$ for some $r_1 \in R$. Set $r = br_1a^*$. Then $r \in R$ and

$$b - ra = b(1 - r_1a^*a) \in (\mathfrak{m}).$$

The last equality in (4.2) is immediate using (2.12). □

The following result was proved in [8] Lemma 8 for simple weight modules with so called regular support which in particular means that they have no inner breaks. It is still true in the more general situation when M has no proper inner breaks. Recall the ideal I from the definition of a $TGWA$.

Proposition 4.2. *Suppose A is a TGWC. If M is a simple weight A-module with no proper inner breaks, then $IM = 0$. Hence M is naturally a module over the associated TGWA A/I.*

Proof. Since I is graded and M is a weight modules, it is enough to show that $(I \cap A_g)M_\mathfrak{m} = 0$ for any $g \in \mathbb{Z}^n$ and any $\mathfrak{m} \in \mathrm{supp}(M)$. Assume that $a \in I \cap A_g$ and $av \neq 0$ for some $v \in M_\mathfrak{m}$. Then $a_1v \neq 0$ for some word a_1 in a. Since M has no proper inner breaks, $a_1^*a_1 \notin \mathfrak{m}$ so by Lemma 4.1 there is an $r \in R$ such that $av = a_1rv$. Thus $0 \neq a_1^*a_1rv = a_1^*av$ which implies that $a_1^*a \in R\backslash\{0\}$. This contradicts that $a \in I$. □

We fix now for each $g \in \tilde{G}_\mathfrak{m}$ a word $a_g \in A_g$ such that $a_g^*a_g \notin \mathfrak{m}$. For $g = 0$ we choose $a_g = 1$.

Lemma 4.3. *For any $g \in \tilde{G}_\mathfrak{m}, h \in G_\mathfrak{m}$ we have*
a) $(a_ga_h^*)^*a_ga_h^* \notin \mathfrak{m}$ so in particular $g - h \in \tilde{G}_\mathfrak{m}$ and $G_\mathfrak{m}$ is a subgroup of $\mathbb{Z}^n_\omega$,
b) $\varphi_\mathfrak{m}(A_g)\varphi_\mathfrak{m}(A_h) = \varphi_\mathfrak{m}(A_gA_h) = \varphi_\mathfrak{m}(A_{g+h})$,
c) $A_{g+h}M_\mathfrak{m} = A_gM_\mathfrak{m}$.

Proof. a) We have

$$(a_g a_h^*)^* a_g a_h^* = a_h a_g^* a_g a_h^* = a_h a_h^* h(a_g^* a_g). \tag{4.3}$$

Now $a_g^* a_g \notin \mathfrak{m}$ so $h(a_g^* a_g) \notin h(\mathfrak{m}) = \mathfrak{m}$. And

$$\mathfrak{m} \not\ni (a_h^* a_h)^2 = a_h^*(a_h a_h^*) a_h = a_h^* a_h \cdot (-h)(a_h a_h^*)$$

so $a_h a_h^* \notin h(\mathfrak{m}) = \mathfrak{m}$. Since $\mathfrak{m}$ is maximal the right hand side of (4.3) does not belong to $\mathfrak{m}$. Since $\deg(a_g a_h^*) = g - h$ we obtain $g - h \in \tilde{G}_\mathfrak{m}$. If in addition $g \in G_\mathfrak{m}$ then $g - h \in \mathbb{Z}_\omega^n$ also since $\mathbb{Z}_\omega^n$ is a group. Thus $g - h \in G_\mathfrak{m}$ so $G_\mathfrak{m}$ is a subgroup of $\mathbb{Z}_\omega^n$.

b) Since $\varphi_\mathfrak{m}$ is a homomorphism, the first equality holds. By part a), $-h \in G_\mathfrak{m}$ so by part a) again, $(a_g a_{-h}^*)^* a_g a_{-h}^* \notin \mathfrak{m}$. Hence by Lemma 4.1, we have

$$\varphi_\mathfrak{m}(A_{g+h}) = R_\mathfrak{m} \cdot \varphi_\mathfrak{m}(a_g a_{-h}^*) \subseteq \varphi_\mathfrak{m}(A_g A_h).$$

The reverse inclusion holds since $\{A_g\}_{g \in \mathbb{Z}^n}$ is a gradation of A.

c) By part a), $g + h = g - (-h) \in \tilde{G}_\mathfrak{m}$. Thus by part b),

$$A_{g+h} M_\mathfrak{m} = \varphi_\mathfrak{m}(A_{g+h}) M_\mathfrak{m} = \varphi_\mathfrak{m}(A_g A_h) M_\mathfrak{m} = A_g A_h M_\mathfrak{m} \subseteq A_g M_{h(\mathfrak{m})} = A_g M_\mathfrak{m}.$$

By part a), the same calculation holds if we replace g by $g + h$ and and h by $-h$, which gives the opposite inclusion. □

Lemma 4.4. *Let $g \in \mathbb{Z}^n \backslash \tilde{G}_\mathfrak{m}$. Then $A_g M_\mathfrak{m} = 0$ for any simple weight module M over A with no proper inner breaks.*

Proof. Let $a \in A_g$ be any word. Then $a^* a \in \mathfrak{m}$ and hence if M is a simple weight module over A with no proper inner breaks, $aM_\mathfrak{m} = 0$. Since the words generate A_g as a left R-module, it follows that $A_g M_\mathfrak{m} = 0$. □

4.1 General case

Recall that $(\mathfrak{m})$ denotes the two-sided ideal in A generated by $\mathfrak{m}$. Since $(\mathfrak{m})$ is a graded ideal in A, there is an induced $\mathbb{Z}^n$-gradation of the quotient $A/(\mathfrak{m})$ and $\varphi_\mathfrak{m}(A_g) = (A/(\mathfrak{m}))_g$. Corresponding to the decomposition $\mathbb{Z}_\omega^n$ into the subset $G_\mathfrak{m}$ and its complement are two K-subspaces of the algebra $B_\mathfrak{m} = B(\omega)/\big(B(\omega) \cap (\mathfrak{m})\big)$ which will be denoted by $B_\mathfrak{m}^{(1)}$ and $B_\mathfrak{m}^{(0)}$ respectively. In other words, $B_\mathfrak{m} = B_\mathfrak{m}^{(1)} \oplus B_\mathfrak{m}^{(0)}$, where

$$B_\mathfrak{m}^{(1)} = \bigoplus_{g \in G_\mathfrak{m}} (A/(\mathfrak{m}))_g \quad \text{and} \quad B_\mathfrak{m}^{(0)} = \bigoplus_{g \in \mathbb{Z}_\omega^n \backslash G_\mathfrak{m}} (A/(\mathfrak{m}))_g.$$

By Lemma 4.3a), $G_\mathfrak{m}$ is a subgroup of the free abelian group $\mathbb{Z}^n$, hence is free abelian itself of rank $k \leq n$. Let $s_1, \ldots, s_k$ denote a basis for $G_\mathfrak{m}$ over $\mathbb{Z}$ and let $b_i = \varphi_\mathfrak{m}(a_{s_i})$ for $i = 1, \ldots, k$. Note also that $R_\mathfrak{m}$ is an extension field of K and that $\mathbb{Z}_\omega^n$ acts naturally on $R_\mathfrak{m}$ as a group of K-automorphisms. Let $\{\rho_j\}_{j \in J}$ be a basis for $R_\mathfrak{m}$ over K.

Theorem 4.5. *a) $B_{\mathrm{m}}^{(0)} M_{\mathrm{m}} = 0$ for any simple weight module M over A with no proper inner breaks, and*
b) the b_i are invertible and as a K-linear space, $B_{\mathrm{m}}^{(1)}$ has a basis

$$\{\rho_j b_1^{l_1} \dots b_k^{l_k} \mid j \in J \text{ and } l_i \in \mathbb{Z} \text{ for } 1 \le i \le k\} \tag{4.4}$$

and the following commutation relations hold

$$b_i \lambda = s_i(\lambda) b_i, \quad i = 1, \dots, k, \lambda \in R_{\mathrm{m}}, \tag{4.5}$$

$$b_i b_j = \lambda_{ij} b_j b_i, \quad i, j = 1, \dots, k \tag{4.6}$$

for some nonzero $\lambda_{ij} \in R_{\mathrm{m}}$.

Proof. a) Let $g \in \mathbb{Z}_\omega^n \backslash G_{\mathrm{m}}$. By Lemma 4.4, $A_g M_{\mathrm{m}} = 0$ and thus $\varphi_{\mathrm{m}}(A_g) M_{\mathrm{m}} = 0$.

b) Since $s_i \in G_{\mathrm{m}}$, $\varphi_{\mathrm{m}}(a_{s_i}^*) b_i \in R_{\mathrm{m}} \backslash \{0\}$ and by Lemma 4.3a) with $g = 0$ and $h = s_i$ we have $b_i \varphi_{\mathrm{m}}(a_{s_i}^*) \in R_{\mathrm{m}} \backslash \{0\}$. So the b_i are invertible. The relation (4.5) follows from (2.12). Next we prove (4.6). From Lemma 4.3a) and Lemma 4.1 we have $\varphi(A_{s_i + s_j}) = R_{\mathrm{m}} b_i b_j$. Switching i and j it follows that (4.6) must hold for some nonzero $\lambda_{ij} \in R_{\mathrm{m}}$.

Finally we prove that (4.4) is a basis for $B_{\mathrm{m}}^{(1)}$ over K. Linear independence is clear. Let $g \in G_{\mathrm{m}}$ and write $g = \sum_i l_i s_i$. By repeated use of Lemma 4.3b) we obtain that

$$\varphi_{\mathrm{m}}(A_g) = \varphi_{\mathrm{m}}(A_{\mathrm{sgn}(l_1) s_1})^{|l_1|} \dots \varphi_{\mathrm{m}}(A_{\mathrm{sgn}(l_k) s_k})^{|l_k|}.$$

For $l_i = 0$ the factor should be interpreted as R_{m}. By Lemma 4.1,

$$\varphi_{\mathrm{m}}(A_{\pm s_i})^l = R_{\mathrm{m}} b_i^{\pm l}$$

for $l > 0$ so using (4.5) we get

$$\varphi_{\mathrm{m}}(A_g) = R_{\mathrm{m}} b_1^{l_1} \dots b_k^{l_k}.$$

The proof is finished. □

4.2 Restricted case

In this subsection we will assume that K is algebraically closed. Moreover we will assume that the K-algebra inclusion $K \hookrightarrow R_{\mathrm{m}}$ is onto which is the case when R is finitely generated as a K-algebra by the (weak) Nullstellensatz. Then $\mathbb{Z}_\omega^n$ acts trivially on R_{m}. The structure of $B_{\mathrm{m}}^{(1)}$ given in Theorem 4.5 is then simplified in the following way.

Corollary 4.6. *Let $k = \mathrm{rank}\, G_{\mathrm{m}}$ and let $b_i = \varphi_{\mathrm{m}}(a_{s_i})$ for $i = 1, \dots, k$ where $\{s_1, \dots, s_k\}$ is a $\mathbb{Z}$-basis for G_{m}. Then $B_{\mathrm{m}}^{(1)}$ is the K-algebra with invertible generators $b_1, \dots, b_k$ and the relation*

$$b_i b_j = \lambda_{ij} b_j b_i, \quad 1 \le i, j \le k.$$

Using the normal form of a skew-symmetric integral matrix we will now show that $B_{\mathrm{m}}^{(1)}$ can be expressed as a tensor product of noncommutative tori. Consider the matrix $(\lambda_{ij})_{1\le i,j\le k}$ from (4.6).

Claim 4.7. *If $B_{\mathrm{m}}^{(1)}$ has a nontrivial irreducible finite-dimensional representation, then all the λ_{ij} are roots of unity.*

Proof. Indeed, let N be a finite-dimensional simple module over $B_{\mathrm{m}}^{(1)}$ and let $i \in \{1,\ldots,k\}$. Since K is algebraically closed, b_i has an eigenvector $0 \neq v \in N$ with eigenvalue μ, say. Since b_i is invertible, $\mu \neq 0$. Let $j \neq i$ and consider the vector $b_j v$. It is also nonzero, since b_j is invertible, and it is an eigenvector of b_i with eigenvalue $\lambda_{ij}\mu$. Repeating the process, we obtain a sequence

$$\mu,\ \lambda_{ij}\mu,\ \lambda_{ij}^2\mu,\ \ldots$$

of eigenvalues of b_i. Since N is finite-dimensional, they cannot all be pairwise distinct, and thus λ_{ij} is a root of unity. □

For $\lambda \in K$, let T_λ denote the K-algebra with two invertible generators a and b satisfying $ab = \lambda ba$. T_λ (or its C^*-analogue) is sometimes referred to as a noncommutative torus.

Theorem 4.8. *Let $k = \operatorname{rank} G_{\mathrm{m}}$. If all the λ_{ij} in (4.6) are roots of unity, then there is a root of unity λ, an integer r with $0 \le r \le \lfloor k/2 \rfloor$ and positive integers $p_i, i = 1,\ldots,r$ with $1 = p_1|p_2|\ldots|p_r$ such that*

$$B_{\mathrm{m}}^{(1)} \simeq T_{\lambda^{p_1}} \otimes T_{\lambda^{p_2}} \otimes \cdots \otimes T_{\lambda^{p_r}} \otimes L$$

where L is a Laurent polynomial algebra over K in $k-2r$ variables.

Proof. If $k = 1$, then $B_{\mathrm{m}}^{(1)} \simeq K[b_1, b_1^{-1}]$ and $r = 0$. If $k > 1$, let p be the smallest positive integer such that $\lambda_{ij}^p = 1$ for all i, j. Using that K is algebraically closed, we fix a primitive p:th root of unity $\varepsilon \in K$. Then there are integers θ_{ij} such that

$$\lambda_{ij} = \varepsilon^{\theta_{ij}}$$

and

$$\theta_{ji} = -\theta_{ij}. \tag{4.7}$$

Equation (4.7) means that $\Theta = (\theta_{ij})$ is a $k \times k$ skew-symmetric integer matrix. Next, consider a change of generators of the algebra $B_{\mathrm{m}}^{(1)}$:

$$b_i \mapsto b_i' = b_1^{u_{i1}} \cdots b_k^{u_{ik}} \tag{4.8}$$

Such a change of generators can be done if we are given an invertible $k \times k$ integer matrix $U = (u_{ij})$. The new commutation relations are

$$\begin{aligned} b_i' b_j' &= b_1^{u_{i1}} \cdots b_k^{u_{ik}} b_1^{u_{j1}} \cdots b_k^{u_{jk}} = \\ &= \lambda_{11}^{u_{1i}u_{1j}} \ldots \lambda_{k1}^{u_{ki}u_{1j}} \cdot \ldots \\ &\quad \cdot \lambda_{1k}^{u_{1i}u_{kj}} \ldots \lambda_{kk}^{u_{ki}u_{kj}} \cdot b_j' b_i' = \\ &= \varepsilon^{\sum_{p,q} \theta_{pq} u_{pi} u_{qj}} b_j' b_i' \end{aligned}$$

Hence $\Theta' = U^T \Theta U$. By Theorem IV.1 in [9] there is a U such that Θ' has the skew normal form

$$\begin{pmatrix} 0 & \theta_1 & & & & & & \\ -\theta_1 & 0 & & & & & & \\ & & 0 & \theta_2 & & & & \\ & & -\theta_2 & 0 & & & & \\ & & & & \ddots & & & \\ & & & & & 0 & \theta_r & \\ & & & & & -\theta_r & 0 & \\ & & & & & & & \mathbf{0} \end{pmatrix}$$

where $r \le \lfloor k/2 \rfloor$ is the rank of Θ, the θ_i are nonzero integers, $\theta_i | \theta_{i+1}$ and $\mathbf{0}$ is a $k-2r$ by $k-2r$ zero matrix. Set $\lambda = \varepsilon^{\theta_1}$ and $p_i = \theta_i/\theta_1$ for $i = 1, \dots, r$. The claim follows. $\square$

The following result, describing simple modules over the tensor product of noncommutative tori, is more or less well-known, but we provide a proof for convenience.

Proposition 4.9. *Let M be a finite dimensional simple module over*

$$T := T_{\lambda_1} \otimes \cdots \otimes T_{\lambda_r},$$

where the λ_i are roots of unity in K. Then there are simple modules M_i over T_{λ_i} such that, as T-modules,

$$M \simeq M_1 \otimes \cdots \otimes M_r.$$

Proof. Denote the generators of T_{λ_i} by a_i and b_i. We will view T_{λ_i} as subalgebras of T. Since the elements $a_i, i = 1, \dots, r$ commute and M is finite dimensional and K is algebraically closed, there is a nonzero common eigenvector $w \in M$ of the a_i:

$$a_i w = \mu_i w, \quad i = 1, \dots, r, \tag{4.9}$$

where $\mu_i \in K^*$ because a_i is invertible. Let n_i be the order of λ_i. Then $b_i^{n_i}$ acts as a scalar by Schur's Lemma. By simplicity of M, any element of M has the form (using the commutation relations and (4.9))

$$\sum_{j \in \mathbb{Z}^r,\, 0 \le j_i < n_i} \rho_j b_1^{j_1} \dots b_r^{j_r} \cdot w, \tag{4.10}$$

where $\rho_j \in K$. This shows that

$$\dim_K M \le n_1 \cdot \ldots \cdot n_r.$$

But the terms in (4.10) all belong to different weight spaces with respect to the commutative subalgebra generated by $a_1, \dots, a_r$:

$$a_i \cdot b_1^{j_1} \dots b_r^{j_r} w = \lambda_i^{j_i} \mu_i \cdot b_1^{j_1} \dots b_r^{j_r} w, \quad i = 1, \dots, r,$$

and

$$(\lambda_1^{j_1}\mu_1,\ldots,\lambda_r^{j_r}\mu_r) \neq (\lambda_1^{l_1}\mu_1,\ldots,\lambda_r^{l_r}\mu_r)$$

if $j,l \in \mathbb{Z}^r, 0 \leq j_i, l_i < n_i$ and $j \neq l$. Hence by standard results they must be linearly independent. Thus

$$\dim_K M = n_1 \cdot \ldots \cdot n_r. \tag{4.11}$$

Next, set $M_i = T_{\lambda_i} \cdot w$. Then $M_i = \oplus_{j=0}^{n_i-1} K b_i^j \cdot w$ and

$$\dim_K M_i = n_i. \tag{4.12}$$

Finally, define

$$\psi : M_1 \otimes \ldots \otimes M_r \to M$$

by

$$\psi(w \otimes \ldots \otimes w) = w$$

and by requiring that ψ is a T-module homomorphism. This is possible since $M_1 \otimes \ldots \otimes M_r$ is generated by $w \otimes \ldots \otimes w$ as a T-module. Then ψ is surjective, since M is simple. Also the dimensions on both sides agree, so ψ is an isomorphism of T-modules. □

5 Explicit formulas for the induced modules

In this section we show explicitly how one can obtain simple weight modules with no proper inner breaks over a TGWA (equivalently over a TGWC by Proposition 4.2) from the structure of its weight spaces as $B(\omega)$-modules.

Since the $B(\omega)$-modules were described in the restricted case in Subsection 4.2, we obtain in particular a description of all simple weight modules over A with no proper inner breaks and finite-dimensional weight spaces if R is finitely generated over an algebraically closed field K.

5.1 A basis for M

Let $\{v_i\}_{i\in I}$ be a basis for $M_{\mathfrak{m}}$ over K. By Lemma 4.3a), $\tilde{G}_{\mathfrak{m}}$ is the union of some cosets in $\mathbb{Z}^n/G_{\mathfrak{m}}$. Let $S \subseteq \mathbb{Z}^n$ be a set of representatives of these cosets. For $g \in \tilde{G}_{\mathfrak{m}}$, choose $r_g \in R$ such that $a_g' := r_g a_g^*$ satisfies $\varphi_{\mathfrak{m}}(a_g')\varphi_{\mathfrak{m}}(a_g) = 1$.

Theorem 5.1. *The set $C = \{a_g v_i \mid g \in S, i \in I\}$ is a basis for M over K.*

Proof. First we show that C is linearly independent over K. Assume that

$$\sum_{g,i} \lambda_{gi} a_g v_i = 0.$$

Then $\sum_i \lambda_{gi} a_g v_i = 0$ for each g since the elements belong to different weight spaces. Hence $0 = a_g' \sum_i \lambda_{gi} a_g v_i = \sum_i \lambda_{gi} v_i$ for each g. Since v_i is a basis over K, all the λ_{gi} must be zero.

Next we prove that C spans M over K. Since M is simple and $M_{\mathfrak{m}} \neq 0$,

$$M = AM_{\mathfrak{m}} = \sum_{g\in\mathbb{Z}^n} A_g M_{\mathfrak{m}} = \sum_{g\in\tilde{G}_{\mathfrak{m}}} A_g M_{\mathfrak{m}} = \sum_{h\in S}\sum_{g\in h+G_{\mathfrak{m}}} A_g M_{\mathfrak{m}} = \sum_{h\in S} A_h M_{\mathfrak{m}}$$

by Lemma 4.4 and Lemma 4.3c). □

Corollary 5.2. $\operatorname{supp}(M) = \{g(\mathfrak{m}) \mid g \in S\}$ *and* $g(\mathfrak{m}) \neq h(\mathfrak{m})$ *if* $g,h \in S$, $g \neq h$.

Corollary 5.3. $\dim M = |S| \cdot \dim M_{\mathfrak{m}}$ *with natural interpretation of* ∞.

5.2 The action of A

Our next step is to describe the action of the X_i, Y_i on the basis C for M. Let $\zeta : \tilde{G}_{\mathfrak{m}} \to S$ be the function defined by requiring $g - \zeta(g) \in G_{\mathfrak{m}}$.

Theorem 5.4. *Let* $g \in S$ *and let* $v \in M_{\mathfrak{m}}$. *Then*

$$X_i a_g v = \begin{cases} a_h \cdot b_{g,i} v & \text{if } g + e_i \in \tilde{G}_{\mathfrak{m}}, \\ 0 & \text{otherwise,} \end{cases}$$

where $h = \zeta(g + e_i)$ *and*

$$b_{g,i} = (-h)(X_i a_g a'_{g+e_i-h} a'_h) \cdot a_{g+e_i-h}$$

and

$$Y_i a_g v = \begin{cases} a_k \cdot c_{g,i} v & \text{if } g - e_i \in \tilde{G}_{\mathfrak{m}}, \\ 0 & \text{otherwise,} \end{cases}$$

where $k = \zeta(g - e_i)$ *and*

$$c_{g,i} = (-k)(Y_i a_g a'_{g-e_i-k} a'_k) \cdot a_{g-e_i-k}.$$

Remark 5.5. Note that

$$\deg X_i a_g a'_{g+e_i-h} a'_h = \deg Y_i a_g a'_{g-e_i-k} a'_k = 0$$

so the action of $\mathbb{Z}^n$ on these elements is well defined. Thus we see that $\deg b_{g,i} \in G_{\mathfrak{m}}$ and $\deg c_{g,i} \in G_{\mathfrak{m}}$, i.e. that $b_{g,i}$ and $c_{g,i}$ belong to $B(\omega)$. Therefore the action of these elements on a basis element v_i of $M_{\mathfrak{m}}$ can be determined if we know the structure of $M_{\mathfrak{m}}$ as an $B(\omega)$-module. In the restricted case this was described in Subsection 4.2. Expanding the result in the basis $\{v_i\}$ again and acting by a_h or a_k we obtain a linear combination of basis elements from the set C.

Proof. Assume $g + e_i \in \tilde{G}_{\mathfrak{m}}$. Let $h = \zeta(g + e_i)$. Then

$$\begin{aligned} X_i a_g v &= X_i a_g a'_{g+e_i-h} a_{g+e_i-h} v = \\ &= (X_i a_g a'_{g+e_i-h} a'_h) a_h a_{g+e_i-h} v = \\ &= a_h \cdot (-h)(X_i a_g a'_{g+e_i-h} a'_h) \cdot a_{g+e_i-h} v. \end{aligned}$$

If $g+e_i \notin \tilde{G}_{\mathfrak{m}}$, then $X_i a_g v = 0$ by Lemma 4.4.

Assume $g-e_i \in \tilde{G}_{\mathfrak{m}}$. Let $k=\zeta(g-e_i)$. Then

$$\begin{aligned} Y_i a_g v &= Y_i a_g a'_{g-e_i-k} a_{g-e_i-k} v = \\ &= (Y_i a_g a'_{g-e_i-k} a'_k) a_k a_{g-e_i-k} v = \\ &= a_k \cdot (-k)(Y_i a_g a'_{g-e_i-k} a'_k) \cdot a_{g-e_i-k} v. \end{aligned}$$

If $g-e_i \notin \tilde{G}_{\mathfrak{m}}$, then $Y_i a_g v = 0$ by Lemma 4.4. □

Note that we do not need the technical assumptions in the proof of Theorem 1 in [8] under which the exact formulas for simple weight modules were obtained.

6 Application to quantized Weyl algebras

In this final part we will apply the methods developed in the previous sections to the problem of describing representations of the quantized Weyl algebra, defined in Section 2.2. As mentioned there, it is naturally a TGWA.

First we find the isotropy group and the set $\tilde{G}_{\mathfrak{m}}$ expressed as solution of systems of linear equations (see Proposition 6.3 and Proposition 6.4). These sets are directly related to the structure of the subalgebra $B(\omega)$ (Theorem 4.5) and the support of a module (Corollary 5.2).

Then in Section 6.2 we give a complete classification of all locally finite simple weight modules with no proper inner breaks over a quantized Weyl algebra of rank two. The parameters q_1 and q_2 are allowed to be any numbers from $\mathbb{C}\backslash\{0,1\}$. Example 6.7 shows that the assumption that the modules have no proper inner breaks is not superfluous.

6.1 The isotropy group and $\tilde{G}_{\mathfrak{m}}$

Let $R=\mathbb{C}[t_1,\dots,t_n]$ and fix $\mathfrak{m}=(t_1-\alpha_1,\dots,t_n-\alpha_n)\in \operatorname{Max}(R)$. Let ω be the orbit of $\mathfrak{m}$ under the action (2.10) of $\mathbb{Z}^n$. Set $[k]_q=\sum_{j=0}^{k-1} q^i$ for $k\in\mathbb{Z}$ and $q\in\mathbb{C}$. Recall the definition (2.9) of the automorphisms σ_i of R.

Proposition 6.1. *Let $(g_1,\dots,g_n)\in\mathbb{Z}^n$. Then*

$$\begin{aligned} \sigma_1^{g_1}\dots\sigma_n^{g_n}(\mathfrak{m}) = \\ &\Big([g_1]_{q_1}+q_1^{g_1}t_1-\alpha_1, \quad [g_2]_{q_2}\big(1+(q_1-1)\alpha_1\big)+q_1^{g_1}q_2^{g_2}t_2-\alpha_2,\dots \\ &\dots,[g_j]_{q_j}\big(1+\sum_{r=1}^{j-1}(q_r-1)\alpha_r\big)+q_1^{g_1}\dots q_j^{g_j}t_j-\alpha_j,\dots \\ &\dots,[g_n]_{q_n}\big(1+\sum_{r=1}^{n-1}(q_r-1)\alpha_r\big)+q_1^{g_1}\dots q_n^{g_n}t_n-\alpha_n\Big). \end{aligned}$$

Proof. Induction. □

For notational brevity we set $\beta_i = (q_i - 1)\alpha_i$ and $\gamma_i = 1 + \beta_1 + \beta_2 + \ldots + \beta_i$. We also set $\gamma_0 = 1$. The numbers γ_i will play an important role in the next statements. By a *j-break* we mean an ideal $\mathfrak{n} \in \mathrm{Max}(R)$ such that $t_j \in \mathfrak{n}$.

Corollary 6.2. *For $j = 1, \ldots, n$ we have*

$$t_j \in \sigma_1^{g_1} \ldots \sigma_n^{g_n}(\mathfrak{m}) \Longleftrightarrow \gamma_j = q_j^{g_j} \gamma_{j-1}.$$

Thus ω contains a j-break iff $\gamma_j = q_j^k \gamma_{j-1}$ for some integer k.

Proof. By Proposition 6.1,

$$t_j \in \sigma_1^{g_1} \ldots \sigma_n^{g_n}(\mathfrak{m})$$

iff

$$[g_j]_{q_j}\big(1 + \sum_{r=1}^{j-1}(q_r - 1)\alpha_r\big) = \alpha_j.$$

Multiply both sides with $q_j - 1$ to get

$$(q_j^{g_j} - 1)(1 + \beta_1 + \ldots + \beta_{j-1}) = \beta_j.$$

□

The next Proposition describes the isotropy subgroup $\mathbb{Z}^n_\omega$ defined in (2.14).

Proposition 6.3. *We have*

$$\mathbb{Z}^n_\omega = \{g \in \mathbb{Z}^n \mid (q_1^{g_1} \ldots q_j^{g_j} - 1)\gamma_j = 0 \ \forall j = 1, \ldots, n\}. \tag{6.1}$$

Proof. From Proposition 6.1, $\sigma_1^{g_1} \ldots \sigma_n^{g_n}(\mathfrak{m}) = \mathfrak{m}$ iff

$$\begin{aligned}
\alpha_1 &= [g_1]_{q_1} + q_1^{g_1}\alpha_1 \\
\alpha_2 &= [g_2]_{q_2}\big(1 + (q_1 - 1)\alpha_1\big) + q_1^{g_1} q_2^{g_2}\alpha_2 \\
&\vdots \\
\alpha_n &= [g_n]_{q_n}\big(1 + (q_1 - 1)\alpha_1 + \ldots + (q_{n-1} - 1)\alpha_{n-1}\big) + q_1^{g_1} \ldots q_n^{g_n}\alpha_n
\end{aligned}$$

Multiply the i:th equation by $q_i - 1$. Then the system can be written

$$\begin{aligned}
\beta_1 &= q_1^{g_1} - 1 + q_1^{g_1}\beta_1 \\
\beta_2 &= (q_2^{g_2} - 1)(1 + \beta_1) + q_1^{g_1} q_2^{g_2}\beta_2 \\
&\vdots \\
\beta_n &= (q_n^{g_n} - 1)(1 + \beta_1 + \ldots + \beta_{n-1}) + q_1^{g_1} \ldots q_n^{g_n}\beta_n
\end{aligned}$$

or equivalently

$$\begin{aligned}
1 + \beta_1 &= q_1^{g_1}(1 + \beta_1) \\
1 + \beta_1 + \beta_2 &= q_2^{g_2}(1 + \beta_1) + q_1^{g_1} q_2^{g_2}\beta_2 \\
&\vdots \\
1 + \beta_1 + \ldots + \beta_n &= q_n^{g_n}(1 + \beta_1 + \ldots + \beta_{n-1}) + q_1^{g_1} \ldots q_n^{g_n}\beta_n
\end{aligned}$$

Now for i from 1 to $n-1$, replace the expression $1+\beta_1+\ldots+\beta_i$ in the right hand side of the $i+1$:th equation by the right hand side of the i:th equation. After simplification, the claim follows. $\square$

Note that it follows from (6.1) that the subgroup

$$Q = \{g \in \mathbb{Z}^n \mid q_j^{g_j} = 1 \text{ for } j = 1, \ldots, n\} \tag{6.2}$$

of $\mathbb{Z}^n$ is always contained in $\mathbb{Z}^n_\omega$ for any orbit ω. Moreover $\mathbb{Z}^n_\omega = Q$ if ω (viewed as a subset of $\mathbb{C}^n$) does not intersect the union of the hyperplanes in $\mathbb{C}^n$ defined by the equations $1+(q_1-1)x_1+\ldots+(q_j-1)x_j = 0$ $(1 \le j \le n)$. Of course the group Q can be trivial. This is the case for example when all the q_j are positive real numbers.

Another case of interest is when for any j, $q_1^{g_1}\ldots q_j^{g_j} = 1$ implies $g_1 = \ldots = g_j = 0$. If for instance the q_j are pairwise distinct prime numbers this hold. Then $\mathbb{Z}^n_\omega = \{0\}$ unless $1+\beta_1+\ldots+\beta_j = 0$ for all j, i.e. unless ω contains the point

$$\mathfrak{n}_0 = (t_1 - (1-q_1)^{-1}, t_2, \ldots, t_n).$$

So in this very special case we have $\omega = \{\mathfrak{n}_0\}$ and $\mathbb{Z}^n_\omega = \mathbb{Z}^n$.

We now turn to the set $\tilde{G}_\mathfrak{m}$ defined in (4.1) which can here be described explicitly in terms of $\mathfrak{m}$ in the following way.

Proposition 6.4.

$$\tilde{G}_\mathfrak{m} = \tilde{G}_\mathfrak{m}^{(1)} \times \ldots \times \tilde{G}_\mathfrak{m}^{(n)},$$

where

$$\begin{aligned}\tilde{G}_\mathfrak{m}^{(j)} = \{k \ge 0 \mid \gamma_j \neq q_j^i \gamma_{j-1}\ \forall i = 0,1,\ldots,k-1\} \cup \\ \cup \{k < 0 \mid \gamma_j \neq q_j^i \gamma_{j-1}\ \forall i = -1,-2,\ldots,k\}.\end{aligned}$$

Proof. From the relations of the algebra follows that the subspace spanned by the words in A_g is one-dimensional. Thus $g \in \tilde{G}_\mathfrak{m}$ iff

$$Z_n^{-g_n}\ldots Z_1^{-g_1} Z_1^{g_1}\ldots Z_n^{g_n} \notin \mathfrak{m} \tag{6.3}$$

where $Z_i^k = X_i^k$ if $k \ge 0$ and $Z_i^k = Y_i^{-k}$ if $k<0$. Since $\sigma_i(t_j) = t_j$ for $j<i$, (6.3) is equivalent to

$$Z_n^{-g_n} Z_n^{g_n}\ldots Z_1^{-g_1} Z_1^{g_1} \notin \mathfrak{m}.$$

Since $\mathfrak{m}$ is prime, this holds iff $Z_j^{-g_j} Z_j^{g_j} \notin \mathfrak{m}$ for each j. If $g_j = 0$ this is true. If $g_j > 0$ we have

$$Z_j^{-g_j} Z_j^{g_j} = Y_j^{g_j} X_j^{g_j} = Y_j^{g_j-1} X_j^{g_j-1} \sigma_j^{-g_j+1}(t_j) = \ldots = t_j \sigma_j^{-1}(t_j)\ldots\sigma_j^{-g_j+1}(t_j),$$

while if $g_j < 0$

$$Z_j^{-g_j} Z_j^{g_j} = X_j^{-g_j} Y_j^{-g_j} = X_j^{-g_j-1} Y_j^{-g_j-1} \sigma_j^{-g_j}(t_j) = \ldots = \sigma_j(t_j)\ldots\sigma_j^{-g_j}(t_j).$$

Since $\mathfrak{m}$ is prime, $g \in \tilde{G}_{\mathfrak{m}}$ iff for all $j = 1, \ldots, n$

$$t_j \notin \sigma_j^i(\mathfrak{m}), \quad i = 0, \ldots, g_j - 1 \text{ if } g_j \geq 0,$$

and

$$t_j \notin \sigma_j^i(\mathfrak{m}), \quad i = -1, -2 \ldots, g_j \text{ if } g_j < 0.$$

The claim now follows from Corollary 6.2. □

Corollary 6.5. *If* $\{1, \alpha_1, \alpha_2, \ldots, \alpha_n\}$ *is linearly independent over* $\mathbb{Q}(q_1, \ldots, q_n)$*, then* $\tilde{G}_{\mathfrak{m}} = \mathbb{Z}^n$.

6.2 Description of simple weight modules over rank two algebras

Assume from now on that A is a quantized Weyl algebra of rank two. In this section we will obtain a list of all locally finite simple weight A-modules with no proper inner breaks.

We consider first some families of ideals in $\operatorname{Max}(R)$. Define for $\lambda \in \mathbb{C}$,

$$\mathfrak{n}_\lambda^{(1)} = \big(t_1 - (1-\lambda)(1-q_1)^{-1}, t_2 - \lambda(1-q_2)^{-1}\big),$$
$$\mathfrak{n}_\lambda^{(2)} = \big(t_1 - (1-q_1)^{-1}, t_2 - \lambda\big),$$

and set $\mathfrak{n}_0 = \mathfrak{n}_0^{(1)} = \mathfrak{n}_0^{(2)}$. The following lemma will be useful.

Lemma 6.6. *For* $\lambda \in \mathbb{C}$ *and integers* k, l *we have*

$$\sigma_1^k \sigma_2^l(\mathfrak{n}_\lambda^{(1)}) = \mathfrak{n}^{(1)}_{\lambda q_1^{-k}}, \tag{6.4}$$

$$\sigma_1^k \sigma_2^l(\mathfrak{n}_\lambda^{(2)}) = \mathfrak{n}^{(2)}_{\lambda q_1^{-k} q_2^{-l}}. \tag{6.5}$$

Proof. Follows from Proposition 6.1 or by direct calculation using the definition (2.9) of the σ_i. □

The following example shows the existence of locally finite simple weight modules M over A which have some proper inner breaks.

Example 6.7. Assume that $q_1\lambda_{12}$ is a root of unity of order r. Let M be a vector space of dimension r and let $\{v_0, v_1, \ldots, v_{r-1}\}$ be a basis for M. Define an action of A on M as follows.

$$X_1 v_k = \begin{cases} v_{k+1}, & k < r-1 \\ v_0, & k = r-1 \end{cases} \qquad X_2 v_k = (q_1\lambda_{12})^{-k} v_k$$

$$Y_1 v_k = \begin{cases} (1-q_1)^{-1} v_{k-1}, & k > 0 \\ (1-q_1)^{-1} v_{r-1}, & k = 0 \end{cases} \qquad Y_2 v_k = 0$$

It is easy to check that (2.6)–(2.8) hold so this defines a module over A. It is immediate that $M = M_{\mathfrak{m}}$ where $\mathfrak{m} = \mathfrak{n}_0 = (t_1 - (1-q_1)^{-1}, t_2)$ so M is a weight module and M is simple by standard arguments. However, recalling Definition 3.7, M has some proper inner breaks in the sense that $\mathfrak{m} \in \operatorname{supp}(M)$, $X_2 M_{\mathfrak{m}} \neq 0$ but $Y_2 X_2 M_{\mathfrak{m}} = 0$.

We will describe the isotropy groups of the different ideals in $\mathrm{Max}(R)$. Let K_1 and K_2 denote the kernels of the group homomorphisms from $\mathbb{Z}\times\mathbb{Z}$ to the multiplicative group $\mathbb{C}\backslash\{0\}$ which map (k,l) to q_1^k and $q_1^k q_2^l$ respectively. Then $Q = K_1 \cap K_2$ where Q was defined in (6.2). For $\mathfrak{m}\in\mathrm{Max}(R)$, recall that $\mathbb{Z}^2_{\mathfrak{m}} = \{g \in \mathbb{Z}^2 \mid g(\mathfrak{m}) = \mathfrak{m}\}$. The following corollary describes the isotropy group $\mathbb{Z}^2_{\mathfrak{m}}$ of any $\mathfrak{m}\in\mathrm{Max}(R)$.

Corollary 6.8. *Let $\lambda\in\mathbb{C}\backslash\{0\}$ and $\mathfrak{n}\in\mathrm{Max}(R)\backslash\{\mathfrak{n}^{(i)}_{\mu} \mid \mu\in\mathbb{C}, i=1,2\}$. Then we have the following equalities in the lattice of subgroups of $\mathbb{Z}^2$.*

$$\begin{tikzcd}[column sep=small]
 & \mathbb{Z}^2_{\mathfrak{n}_0} = \mathbb{Z}^2 \arrow[dl, dash] \arrow[dr, dash] & \\
\mathbb{Z}^2_{\mathfrak{n}^{(1)}_{\lambda}} = K_1 \arrow[dr, dash] & & \mathbb{Z}^2_{\mathfrak{n}^{(2)}_{\lambda}} = K_2 \arrow[dl, dash] \\
 & \mathbb{Z}^2_{\mathfrak{n}} = Q &
\end{tikzcd}$$

Proof. The family of ideals $\{\mathfrak{n}^{(1)}_{\lambda} \mid \lambda\in\mathbb{C}\}$ are precisely those for which $\gamma_2 = 0$. And $\{\mathfrak{n}^{(2)}_{\lambda} \mid \lambda\in\mathbb{C}\}$ are exactly those such that $\gamma_1 = 0$. Thus the claim follows from Proposition 6.3. $\square$

Let M be a simple weight A-module with no proper inner breaks and finite dimensional weight spaces, $\mathfrak{m} = (t_1-\alpha_1, t_2-\alpha_2)\in\mathrm{supp}\,M$ and let ω be the orbit of $\mathfrak{m}$. We consider four main cases separately: $\mathfrak{m}=\mathfrak{n}_0$, $\mathfrak{m}=\mathfrak{n}^{(1)}_{\lambda}$ for some $\lambda\neq 0$, $\mathfrak{m}=\mathfrak{n}^{(2)}_{\lambda}$ for some $\lambda\neq 0$ and $\mathfrak{m}\notin\{\mathfrak{n}^{(i)}_{\mu} \mid \mu\in\mathbb{C}, i=1,2\}$. Some of these cases will contain subcases. In each case we will proceed along the following steps, which also illustrate the procedure for a general TGWA.

1. Find the sets $\mathbb{Z}^n_{\mathfrak{m}}$ and $\tilde{G}_{\mathfrak{m}}$ using Corollary 6.8 and Proposition 6.4. Write down $G_{\mathfrak{m}} = \mathbb{Z}^n_{\mathfrak{m}} \cap \tilde{G}_{\mathfrak{m}}$ and choose a basis $\{s_1,\ldots,s_k\}$ for $G_{\mathfrak{m}}$ over $\mathbb{Z}$.
2. For each $g\in\tilde{G}_{\mathfrak{m}}$, choose a word a_g of degree g such that $a_g^* a_g \notin \mathfrak{m}$.
3. Using Corollary 4.6, describe $B^{(1)}_{\mathfrak{m}}$ and the finite-dimensional simple $B^{(1)}_{\mathfrak{m}}$-module $M_{\mathfrak{m}}$.
4. Choose a set of representatives S for $\tilde{G}_{\mathfrak{m}}/G_{\mathfrak{m}}$. By Theorem 5.1 we know then a basis C for M.
5. Calculate the action of X_i, Y_i on the basis using either relations (2.6)–(2.8) or Theorem 5.4.

We will use the following notation: $Z_j^k = X_j^k$ if $k\geq 0$ and $Z_j^k = Y_j^{-k}$ if $k<0$. Note that the k in Z_j^k should only be regarded as an upper index, not as a power. The choice of a_g in step two above is more or less irrelevant for a quantized Weyl algebra because each A_g is one-dimensional. Therefore we will always choose $a_g = Z_1^{g_1} Z_2^{g_2}$ where $g=(g_1,g_2)$.

6.3 The case $\mathfrak{m} = \mathfrak{n}_0$

Here $\alpha_1 = (1-q_1)^{-1}$, $\alpha_2 = 0$ so that $\gamma_1 = \gamma_2 = 0$. By Corollary 6.8 we have $\mathbb{Z}^2_{\mathfrak{m}} = \mathbb{Z}^2$ and from Proposition 6.4 one obtains that $\tilde{G}_{\mathfrak{m}} = \mathbb{Z} \times \{0\}$. Thus $G_{\mathfrak{m}} = \mathbb{Z} \times \{0\} = \mathbb{Z} \cdot s_1$ with $s_1 = (1,0)$. Since $G_{\mathfrak{m}}$ has rank one, Corollary 4.6 implies that $B^{(1)}_{\mathfrak{m}}$ is isomorphic to the Laurent polynomial algebra $\mathbb{C}[T, T^{-1}]$ in one variable. Therefore $M_{\mathfrak{m}}$ is one-dimensional, say $M_{\mathfrak{m}} = \mathbb{C}v_0$ and $b_1 = \varphi_{\mathfrak{m}}(Z_1^1) = \varphi_{\mathfrak{m}}(X_1)$, hence X_1, acts in $M_{\mathfrak{m}}$ as some nonzero scalar ρ. And

$$Y_1 v_0 = \rho^{-1} Y_1 X_1 v_0 = \rho^{-1}(1-q_1)^{-1} v_0.$$

Here $S = \{(0,0)\}$ and $C = \{v_0\}$ is a basis for M with the following action:

$$\begin{aligned} X_1 v_0 &= \rho v_0, & X_2 v_0 &= 0, \\ Y_1 v_0 &= \rho^{-1}(1-q_1)^{-1} v_0, & Y_2 v_0 &= 0. \end{aligned} \tag{6.6}$$

That $Z_2^{\pm 1} v_0 = 0$ follows from Theorem 5.4 since $(0, \pm 1) \notin \tilde{G}_{\mathfrak{m}}$.

6.4 The case $\mathfrak{m} = \mathfrak{n}^{(1)}_\lambda$, $\lambda \neq 0$

Here $\alpha_1 = (1-\lambda)(1-q_1)^{-1}$ and $\alpha_2 = \lambda(1-q_1)^{-1}$ so $\gamma_1 = \lambda$ and $\gamma_2 = 0$. By Proposition 6.4, $\tilde{G}^{(2)}_{\mathfrak{m}} = \mathbb{Z}$ and

$$\tilde{G}^{(1)}_{\mathfrak{m}} = \{k \geq 0 \mid \lambda \neq q_1^i \ \forall i = 0, 1, \ldots, k-1\} \cup \{k < 0 \mid \lambda \neq q_1^i \ \forall i = -1, -2, \ldots, k\}.$$

We consider four subcases according to whether ω contains a 1-break or not and whether q_1 is a root of unity or not.

6.4.1 The case $\mathfrak{m} = \mathfrak{n}^{(1)}_\lambda$, $\lambda \neq 0$, ω contains a 1-break and q_1 is a root of unity

By Corollary 6.2 $\lambda = q_1^k$ for some $k \in \mathbb{Z}$. Let o_1 be the order of q_1. Then $\mathbb{Z}^2_{\mathfrak{m}} = K_1 = (o_1\mathbb{Z}) \times \mathbb{Z}$. We can further assume that $k \in \{0, 1, \ldots, o_1 - 1\}$.

Note that $X_1^k M_{\mathfrak{m}} \neq 0$ because $\deg X_1^k = (k, 0) \in \tilde{G}_{\mathfrak{m}}$ so $Y_1^k X_1^k \notin \mathfrak{m}$. Hence $\sigma_1^k(\mathfrak{m}) \in \operatorname{supp}(M)$. By Lemma 6.6, $\sigma_1^k(\mathfrak{m}) = \mathfrak{n}^{(1)}_{q_1^k q_1^{-k}} = \mathfrak{n}^{(1)}_1$. We can thus change notation and let $\mathfrak{m} = \mathfrak{n}^{(1)}_1$. Then by Proposition 6.4 we have

$$\tilde{G}_{\mathfrak{m}} = \{0, -1, -2, \ldots, -o_1 + 1\} \times \mathbb{Z}.$$

And $G_{\mathfrak{m}} = \tilde{G}_{\mathfrak{m}} \cap \mathbb{Z}^2_{\mathfrak{m}} = \{0\} \times \mathbb{Z}$. By Corollary 4.6, $B^{(1)}_{\mathfrak{m}}$ is a Laurent polynomial algebra in one variable. Thus $M_{\mathfrak{m}}$ is one dimensional with a basis vector, say v_0. X_2 acts by some nonzero scalar ρ on v_0 and $Y_2 X_2 v_0 = (1-q_2)^{-1} v_0$. X_1 and $Y_1^{o_1}$ act as zero on $M_{\mathfrak{m}}$ by Lemma 4.4 because their degrees $(1,0)$ and $(-o_1, 0)$ does not belong to $\tilde{G}_{\mathfrak{m}}$.

As a set of representatives for $\tilde{G}_{\mathfrak{m}}/G_{\mathfrak{m}}$ we choose

$$S = \{(0,0), (-1,0), (-2,0), \ldots, (-o_1+1, 0)\}.$$

By Corollary 5.2 we obtain that

$$\operatorname{supp}(M) = \{\mathfrak{n}_1^{(1)}, \mathfrak{n}_{q_1^{-1}}^{(1)}, \dots, \mathfrak{n}_{q_1^{-o_1+1}}^{(1)}\}.$$

By 5.1, the set

$$C = \{v_j := Y_1^j v_0 \mid j = 0, 1, \dots, o_1 - 1\}$$

is a basis for M. The following picture shows the support of the module and how the X_i act on it. Since the Y_i just act in the opposite direction of the X_i we do not draw their arrows.

$$\begin{tikzcd}
\bullet \arrow[loop above, "X_2"] \arrow[r, "X_1"'] & \bullet \arrow[loop above, "X_2"] \arrow[r, "X_1"'] & \bullet \arrow[loop above, "X_2"] & \cdots\cdots & \bullet \arrow[loop above, "X_2"] \arrow[r, "X_1"'] & \bullet \arrow[loop above, "X_2"]
\end{tikzcd}$$

Using Lemma 6.6,

$$X_1 v_j = X_1 Y_1^j v_0 = Y_1^{j-1} \sigma_1^j(t_1) v_0 = [j]_{q_1} v_{j-1}$$

and from relations (2.6)–(2.8) follow that

$$X_2 v_j = q_1^j \lambda_{12}^j Y_1^j X_2 v_0 = \rho \lambda_{12}^j q_1^j v_j,$$

$$Y_2 v_j = \lambda_{21}^j Y^j Y_2 v_0 = (1 - q_2)^{-1} \rho^{-1} \lambda_{21}^j v_j.$$

Thus the action on the basis $\{v_0, \dots, v_{o_1-1}\}$ is

$$\begin{aligned}
X_1 v_j &= \begin{cases} 0, & j = 0, \\ [j]_{q_1} v_{j-1}, & 0 < j \leq o_1 - 1, \end{cases} \\
Y_1 v_j &= \begin{cases} v_{j+1}, & 0 \leq j < o_1 - 1, \\ 0, & j = o_1 - 1, \end{cases} \\
X_2 v_j &= \rho \lambda_{12}^j q_1^j v_j, \\
Y_2 v_j &= (1 - q_2)^{-1} \rho^{-1} \lambda_{21}^j v_j.
\end{aligned} \tag{6.7}$$

6.4.2 The case $\mathfrak{m} = \mathfrak{n}_\lambda^{(1)}$, $\lambda \neq 0$, ω contains a 1-break and q_1 is not a root of unity

Now there is a unique integer $k \in \mathbb{Z}$ such that $\lambda = q_1^k$. If $k \geq 0$, then $\tilde{G}_{\mathfrak{m}}^{(1)}$ is the set of all integers $\leq k$ while if $k < 0$, then $\tilde{G}_{\mathfrak{m}}^{(1)}$ is all integers $\geq k + 1$.

If $k \geq 0$, $X_1^k M_{\mathfrak{m}} \neq 0$ because $(k, 0) \in \tilde{G}_{\mathfrak{m}}$ so $Y_1^k X_1^k \notin \mathfrak{m}$. Therefore $\sigma_1^k(\mathfrak{m}) = \mathfrak{n}_1^{(1)} \in \operatorname{supp}(M)$. We change notation and let $\mathfrak{m} = \mathfrak{n}_1^{(1)}$. Then $\tilde{G}_{\mathfrak{m}}^{(1)} = \{\dots, -2, -1, 0\}$ and $G_{\mathfrak{m}} = \{0\} \times \mathbb{Z}$. We choose $S = \{(i, 0) \mid i \leq 0\}$. $Y_2 X_2 = (1 - q_2)^{-1}$ on $M_{\mathfrak{m}}$ so $M_{\mathfrak{m}} = \mathbb{C} v_0$, for a basis vector v_0, and $X_2 v_0 = \rho v_0$ for some $\rho \in \mathbb{C}^*$. The set

$C = \{v_j := Y_1^j v_0 \mid j \leq 0\}$ is a basis for M and we have the following picture of $\operatorname{supp}(M)$.

$$\begin{tikzcd} \cdots\cdots \bullet \arrow[loop above, "X_2"] \arrow[r, "X_1"] & \bullet \arrow[loop above, "X_2"] \arrow[r, "X_1"] & \bullet \arrow[loop above, "X_2"] \end{tikzcd}$$

One easily obtains the following action on the basis $\{v_j \mid j \leq 0\}$:

$$\begin{aligned} X_1 v_j &= \begin{cases} 0, & j = 0, \\ [j]_{q_1} v_{j-1}, & j \geq 1, \end{cases} \\ Y_1 v_j &= v_{j+1}, \\ X_2 v_j &= \rho \lambda_{12}^j q_1^j v_j, \\ Y_2 v_j &= (1-q_2)^{-1} \rho^{-1} \lambda_{21}^j v_j. \end{aligned} \tag{6.8}$$

The case $k < 0$ is analogous and yields a lowest weight representation with $\mathfrak{m} = \mathfrak{n}_{q_1^{-1}}^{(1)}$ as its lowest weight. A basis for M is then

$$C = \{v_j := X_1^j v_0 \mid j \geq 0\},$$

where $M_{\mathfrak{m}} = \mathbb{C} v_0$ and the action is given by

$$\begin{aligned} X_1 v_j &= v_{j+1}, \\ Y_1 v_j &= \begin{cases} 0, & j = 0, \\ [-j]_{q_1} v_{j-1}, & j > 0, \end{cases} \\ X_2 v_j &= (q_1 \lambda_{12})^{-j} \rho v_j, \\ Y_2 v_j &= \lambda_{12}^j (1-q_2)^{-1} \rho^{-1} v_j. \end{aligned} \tag{6.9}$$

6.4.3 The case $\mathfrak{m} = \mathfrak{n}_\lambda^{(1)}$, $\lambda \neq 0$, ω contains no 1-break and q_1 is a root of unity

By Corollary 6.2, $\lambda \neq q_1^k$ for all $k \in \mathbb{Z}$. So by Proposition 6.4, $\tilde{G}_{\mathfrak{m}} = \mathbb{Z}^2$. $G_{\mathfrak{m}} = (o_1\mathbb{Z}) \times \mathbb{Z}$ and we can choose $S = \{0, 1, \ldots, o_1 - 1\} \times \{0\}$. From

$$X_1^{o_1} X_2 = (q_1 \lambda_{12})^{o_1} X_2 X_1^{o_1} = \lambda_{12}^{o_1} X_2 X_1^{o_1}$$

and Corollary 4.6 follows that $B_{\mathfrak{m}}^{(1)} \simeq T_{\lambda_{12}^{o_1}}$. It can only have finite-dimensional irreducible representations if $\lambda_{12}^{o_1}$ is a root of unity. Assuming this, any such representation is r-dimensional, where r is the order of $\lambda_{12}^{o_1}$, and is parametrized by $\mathbb{C}^* \times \mathbb{C}^* \ni (\rho, \mu)$ with basis

$$M_{\mathfrak{m}} = \operatorname{Span}\{v_j := X_2^j v_0 \mid j = 0, 1, \ldots, r-1\},$$

where $X_1^{o_1} v_0 = \rho v_0$ and relations

$$\begin{aligned} X_1^{o_1} v_j &= \lambda_{12}^{o_1 j} \rho v_j, \\ X_2 v_j &= \begin{cases} v_{j+1}, & 0 \leq j < r-1, \\ \mu v_0, & j = r-1. \end{cases} \end{aligned}$$

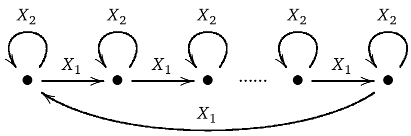
Therefore by Theorem 5.1,

$$M = \mathrm{Span}\{w_{ij} = X_1^i v_j \mid 0 \leq i < o_1, 0 \leq j < r\}.$$

Using the commutation relations and the formulas in Lemma 6.6 we can write down the action as follows.

$$\begin{aligned}
X_1 w_{ij} &= \begin{cases} w_{i+1,j}, & 0 \leq i < o_1 - 1, \\ \lambda_{12}^{o_1 j} \rho w_{0,j}, & i = o_1 - 1, \end{cases} \\
Y_1 w_{ij} &= \begin{cases} (1-\lambda)(1-q_1)^{-1} \lambda_{12}^{-o_1 j} \rho^{-1} w_{o_1-1,j}, & i = 0, \\ (1-\lambda q_1^{-i})(1-q_1)^{-1} w_{i-1,j}, & 0 < i \leq o_1 - 1, \end{cases} \\
X_2 w_{ij} &= \begin{cases} q_1^{-i} \lambda_{21}^i w_{i,j+1}, & 0 \leq j < r - 1, \\ q_1^{-i} \lambda_{21}^i \mu w_{i,0}, & j = r - 1, \end{cases} \\
Y_2 w_{ij} &= \begin{cases} \lambda_{12}^i \mu^{-1} \lambda (1-q_2)^{-1} w_{i,r-1}, & j = 0, \\ \lambda_{12}^i \lambda (1-q_2)^{-1} w_{i,j-1}, & 0 < j \leq r - 1. \end{cases}
\end{aligned} \tag{6.10}$$

The action can be illustrated in the following way.



6.4.4 The case $\mathfrak{m} = \mathfrak{n}_\lambda^{(1)}$, $\lambda \neq 0$, ω contains no 1-break and q_1 is not a root of unity

By Corollary 6.2, $\lambda \neq q_1^k$ for all $k \in \mathbb{Z}$. Now $\mathbb{Z}_\mathfrak{m}^2 = \{0\} \times \mathbb{Z}$ so $G_\mathfrak{m} = \{0\} \times \mathbb{Z}$. $M_\mathfrak{m}$ is one-dimensional with basis v_0, say, and $X_2 = \rho$ on $M_\mathfrak{m}$ while $Y_2 X_2 = \lambda(1-q_2)^{-1} \neq 0$ on $M_\mathfrak{m}$. We choose $S = \mathbb{Z} \times \{0\}$. Then a basis for M is

$$C = \{v_j := X_1^j v_0 \mid j \geq 0\} \cup \{v_j := \zeta_j Y_1^{-j} v_0 \mid j < 0\},$$

where we determine ζ_j by requiring that $X_1 v_j = v_{j+1}$ for all j. Explicitly we have for $j < 0$,

$$\zeta_j = \frac{(1-q_1)^{-j}}{(1-\lambda q_1^{-j})(1-\lambda q_1^{-j-1}) \ldots (1-\lambda q_1)}.$$

Using the commutation relations and the formulas in Lemma 6.6 we get the action on $M = \mathrm{Span}\{v_j \mid j \in \mathbb{Z}\}$.

$$\begin{aligned}
X_1 v_j &= v_{j+1}, & X_2 v_j &= q_1^{-j} \lambda_{12}^{-j} \rho v_j, \\
Y_1 v_j &= \frac{1-\lambda q_1^{-j+1}}{1-q_1} v_{j-1}, & Y_2 v_j &= \lambda_{12}^j \lambda (1-q_2)^{-1} \rho^{-1} v_j,
\end{aligned} \tag{6.11}$$

and a corresponding diagram

$$\begin{tikzcd} \cdots\cdots \bullet \arrow[loop above, "X_2"] \arrow[r, "X_1"] & \bullet \arrow[loop above, "X_2"] \arrow[r, "X_1"] & \bullet \arrow[loop above, "X_2"] \cdots\cdots \end{tikzcd} \quad .$$

6.5 The case $\mathfrak{m} = \mathfrak{n}_\lambda^{(2)}$, $\lambda \neq 0$

Here $\gamma_1 = 0$ while $\gamma_2 = \lambda(q_2 - 1)$. By Corollary 6.2, ω does not contain any breaks. We have $\tilde{G}_\mathfrak{m} = \mathbb{Z}^2$ and $G_\mathfrak{m} = \mathbb{Z}^2_\mathfrak{m} = K_2$.

We will need some lemmas in order to proceed.

Lemma 6.9. *For $k, l \in \mathbb{Z}$ we have*

$$Z_1^k Z_2^l = q_1^{k\bar{l}} \lambda_{12}^{kl} Z_2^l Z_1^k, \tag{6.12}$$

where $\bar{l} = \max\{0, l\}$.

Proof. Relations (2.6)–(2.8) can be rewritten in the more compact form

$$Z_1^k Z_2^l = q_1^{k\delta_{l,1}} \lambda_{12}^{kl} Z_2^l Z_1^k, \quad k, l = \pm 1,$$

where $\delta_{l,1}$ is the Kronecker symbol. After repeated application of this, (6.12) follows. □

By Lemma 6.6 we have for $k, l \in \mathbb{Z}$,

$$\sigma_1^k \sigma_2^l(t_1) = (1 - q_1)^{-1} \mod \mathfrak{m}, \tag{6.13}$$

$$\sigma_1^k \sigma_2^l(t_2) = \lambda q_1^k q_2^l \mod \mathfrak{m}. \tag{6.14}$$

Lemma 6.10. *Let $k, l \in \mathbb{Z}$ and let $m = \min\{|k|, |l|\}$. Then, as operators on $M_\mathfrak{m}$, we have*

$$Z_1^k Z_1^l = \begin{cases} Z_1^{k+l}, & kl \geq 0, \\ (1 - q_1)^{-m} Z_1^{k+l}, & kl < 0, \end{cases} \tag{6.15}$$

$$Z_2^k Z_2^l = \begin{cases} Z_2^{k+l}, & kl \geq 0, \\ \lambda^m q_2^{(1-2l+(\operatorname{sgn} l)m)m/2} Z_2^{k+l}, & kl < 0. \end{cases} \tag{6.16}$$

Proof. Direct calculation using (6.13) and (6.14). For example if $k > 0$ and $l < 0$ we have

$$\begin{aligned} Z_2^k Z_2^l &= X_2^k Y_2^{-l} = X_2^{k-1} \sigma_2(t_2) Y_2^{-l-1} = \\ &= X_2^{k-1} Y_2^{-l-1} \sigma_2^{-l}(t_2) = X_2^{k-1} Y_2^{-l-1} \lambda q_2^{-l} = \ldots = \\ &= \lambda q_2^{-l} \lambda q_2^{-l-1} \ldots \lambda q_2^{-l-(m-1)} Z_2^{k+l} = \\ &= \lambda^m q_2^{-lm-m(m-1)/2} Z_2^{k+l}. \end{aligned}$$

□

Lemma 6.11. *Let $k,l \in \mathbb{Z}$ and let $m = \min\{|k|,|l|\}$. Then, as operators on $M_{\mathfrak{m}}$,*

$$Z_1^k Z_1^l = Z_1^l Z_1^k, \tag{6.17}$$

and

$$Z_2^k Z_2^l = c(k,l) Z_2^l Z_2^k, \tag{6.18}$$

where

$$c(k,l) = \begin{cases} 1, & kl \geq 0, \\ q_2^{(k-l)m-(\operatorname{sgn} k - \operatorname{sgn} l)m^2/2}, & kl < 0. \end{cases} \tag{6.19}$$

Proof. Follows directly from Lemma 6.10. □

Lemma 6.12. *Let $g = (g_1, g_2) \in \mathbb{Z}^2 = \tilde{G}_{\mathfrak{m}}$ and set $r_g = \varphi_{\mathfrak{m}}(a_g^* a_g)^{-1}$ where $\varphi_{\mathfrak{m}}$ is the projection $R \to R/\mathfrak{m} = K$. Then*

$$r_g = (1-q_1)^{|g_1|} (\lambda^{-1} q_2^{(g_2-1)/2})^{|g_2|} \tag{6.20}$$

and $(a_g)^{-1} = r_g a_g^* = r_g Z_2^{-g_2} Z_1^{-g_1}$ as operators on $M_{\mathfrak{m}}$.

Proof. We have

$$a_g^* a_g = (Z_1^{g_1} Z_2^{g_2})^* Z_1^{g_1} Z_2^{g_2} = Z_2^{-g_2} Z_1^{-g_1} Z_1^{g_1} Z_2^{g_2} = Z_1^{-g_1} Z_1^{g_1} Z_2^{-g_2} Z_2^{g_2},$$

by Lemma 6.9. Thus by Lemma 6.10,

$$\varphi_{\mathfrak{m}}(a_g^* a_g) = (1-q_1)^{-|g_1|} \lambda^{|g_2|} q_2^{(1-2g_2+g_2)|g_2|/2}$$

which proves the formula. The last statement is immediate. □

We consider the three subcases corresponding to the rank of the free abelian group K_2.

6.5.1 The case $\mathfrak{m} = \mathfrak{n}_\lambda^{(2)}, \lambda \neq 0, \operatorname{rank} K_2 = 0$

$G_{\mathfrak{m}} = K_2 = \{0\}$ so $B_{\mathfrak{m}}^{(1)} = R$ which is commutative, hence $M_{\mathfrak{m}} = \mathbb{C}v_0$ for some v_0, and $S = \mathbb{Z}^2$. Thus $C = \{a_g v_0 \mid g \in \mathbb{Z}^2\}$ is a basis for M and using Lemma 6.10 and Lemma 6.9 we obtain that the action of X_i is given by

$$\begin{aligned} X_1 a_g v_0 &= \begin{cases} a_{g+e_1} v_0, & g_1 \geq 0, \\ (1-q_1)^{-1} a_{g+e_1} v_0, & g_1 < 0, \end{cases} \\ X_2 a_g v_0 &= \begin{cases} (q_1 \lambda_{12})^{-g_1} a_{g+e_2} v_0, & g_2 \geq 0, \\ (q_1 \lambda_{12})^{-g_1} \lambda q_2^{-g_2} a_{g+e_2} v_0, & g_2 < 0. \end{cases} \end{aligned} \tag{6.21}$$

The action of Y_i on the basis is deduced uniquely from

$$\begin{aligned} Y_1 X_1 a_g v_0 &= (1-q_1)^{-1} a_g v_0, \\ Y_2 X_2 a_g v_0 &= \lambda q_1^{-g_1} q_2^{-g_2} a_g v_0, \end{aligned} \tag{6.22}$$

which hold by (6.13) and (6.14).

6.5.2 The case $\mathfrak{m} = \mathfrak{n}_\lambda^{(2)}, \lambda \neq 0, \operatorname{rank} K_2 = 1$

Let (a,b) be a basis element. Since $G_\mathfrak{m} = K_2$ which is of rank one, $B_\mathfrak{m}^{(1)} \simeq \mathbb{C}[T, T^{-1}]$ by Corollary 4.6 so $M_\mathfrak{m}$ is one-dimensional. As before we let $M_\mathfrak{m} = \mathbb{C}v_0$. Then $Z_1^a Z_2^b v_0 = \rho v_0$ for some $\rho \in \mathbb{C}^*$.

We assume $a \neq 0$. The case $b \neq 0$ can be treated similarly. By changing basis, we can assume that $a > 0$. Choose $S = \{0, 1, \ldots, a-1\} \times \mathbb{Z}$. The corresponding basis for M is

$$C = \{w_{ij} := X_1^i Z_2^j v_0 \mid 0 \leq i \leq a-1, j \in \mathbb{Z}\}.$$

We now aim to apply Theorem 5.4. If $0 \leq i < a-1$ then clearly $X_1 w_{ij} = w_{i+1,j}$. And

$$X_1 w_{a-1,j} = X_1^a Z_2^j v_0 \in \mathbb{C}Z_2^{j-b} v_0 = \mathbb{C}w_{0,j-b}.$$

We want to compute the coefficient of $w_{0,j-b}$. Similarly to the proof of Theorem 5.4 we have, using Lemma 6.12, Lemma 6.9 and (6.16),

$$\begin{aligned} X_1 w_{a-1,j} &= Z_1^a Z_2^j v_0 = (Z_1^a Z_2^j r_{(a,b)} Z_2^{-b} Z_1^{-a}) Z_1^a Z_2^b v_0 = \\ &= r_{(a,b)} (q_1 \lambda_{12})^{ja} q_1^{a \cdot \overline{-b}} \lambda_{12}^{-ab} Z_2^j Z_2^{-b} Z_1^a Z_1^{-a} \rho v_0 = \\ &= (\lambda^{-1} q_2^{(b-1)/2})^{|b|} q_1^{a(j + \overline{-b})} \lambda_{12}^{a(j-b)} \rho C_0 w_{0,j-b}, \end{aligned}$$

where

$$C_0 = \begin{cases} 1, & b \leq 0, \\ \lambda^{\min\{j,b\}} q_2^{(1+2b-\min\{j,b\})\min\{j,b\}/2}, & b > 0. \end{cases}$$

Using Lemma 6.9 one easily get the action of X_2 on the basis. We conclude that

$$\begin{aligned} X_1 w_{ij} &= \begin{cases} w_{i+1,j}, & 0 \leq i < a-1, \\ (\lambda^{-1} q_2^{(b-1)/2})^{|b|} q_1^{a(j+\overline{-b})} \lambda_{12}^{a(j-b)} \rho C_0 w_{0,j-b}, & i = a-1, \end{cases} \\ X_2 w_{ij} &= \begin{cases} q_1^{-i} \lambda_{21}^i w_{i,j+1}, & j \geq 0, \\ q_1^{-i} \lambda_{21}^i \lambda q_2^j w_{i,j+1}, & j < 0. \end{cases} \end{aligned} \tag{6.23}$$

The action of the Y_i is uniquely determined by

$$\begin{aligned} Y_1 X_1 v_{ij} &= (1 - q_1)^{-1} v_{ij}, \\ Y_2 X_2 v_{ij} &= \lambda q_1^{-i} q_2^{-j} v_{ij}, \end{aligned} \tag{6.24}$$

which hold by (6.13)–(6.14). See Figure 1 for a visual representation.

6.5.3 The case $\mathfrak{m} = \mathfrak{n}_\lambda^{(2)}, \lambda \neq 0, \operatorname{rank} K_2 = 2$

Let $s_1 = \mathbf{a} = (a_1, a_2)$, $s_2 = \mathbf{b} = (b_1, b_2)$ be a basis for $G_\mathfrak{m} = K_2$ over $\mathbb{Z}$. We can assume that $a_1, b_1 \geq 0$ and that $d := \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} > 0$.

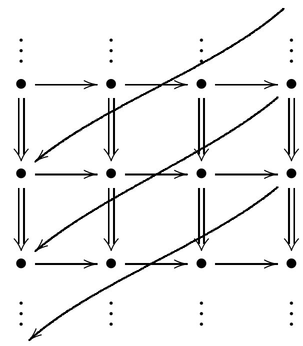


Figure 1: Example of a weight diagram for M when $\mathfrak{m} = \mathfrak{n}_\lambda^{(2)}$, $\lambda \neq 0$ and $\operatorname{rank} K_2 = 1$. Here $a = 4$, $b = -2$. The action of X_1 is indicated by $\rightarrow$ arrows, while $\Rightarrow$ arrows are used for X_2.

By Corollary 4.6, $B_\mathfrak{m}^{(1)} \simeq T_\nu$ for some ν which we will now determine. Using Lemma 6.9 and Lemma 6.11 we have, as operators on $M_\mathfrak{m}$,

$$\begin{aligned} Z_1^{a_1}Z_2^{a_2}Z_1^{b_1}Z_2^{b_2} &= q_1^{-b_1\overline{a_1}}\lambda_{12}^{-b_1a_2}c(a_2,b_2)Z_1^{b_1}Z_1^{a_1}Z_2^{b_2}Z_2^{a_2} = \\ &= q_1^{a_1\overline{b_2}-b_1\overline{a_2}}\lambda_{12}^{a_1b_2-b_1a_2}c(a_2,b_2)Z_1^{b_1}Z_2^{b_2}Z_1^{a_1}Z_2^{a_2}. \end{aligned}$$

We conclude that $B_\mathfrak{m}^{(1)} \simeq T_\nu$ where

$$\nu = \lambda_{12}^d q_1^{a_1\overline{b_2}-b_1\overline{a_2}}c(a_2,b_2). \tag{6.25}$$

The function c was defined in (6.19), $d = a_1b_2 - b_1a_2$ and $\overline{k} := \max\{0,k\}$ for $k \in \mathbb{Z}$. For $M_\mathfrak{m}$ to be finite-dimensional it is thus necessary that this ν is a root of unity. Assume this and let r denote its order. Then $\dim M_\mathfrak{m} = r$. Let

$$\{v_0, v_1, \dots, v_{r-1}\} \tag{6.26}$$

be a basis such that

$$Z_1^{a_1}Z_2^{a_2}v_j = \nu^j\rho v_j, \tag{6.27}$$

$$Z_1^{b_1}Z_2^{b_2}v_j = \begin{cases} v_{j+1}, & 0 \le j < r-1, \\ \mu v_0, & j = r-1, \end{cases} \tag{6.28}$$

where $\rho, \mu \in \mathbb{C}^*$.

The next step is to determine a set $S \subseteq \tilde{G}_\mathfrak{m} = \mathbb{Z}^2$ of representatives for the set of cosets $\tilde{G}_\mathfrak{m}/G_\mathfrak{m} = \mathbb{Z}^2/K_2$ which makes it possible to write down the action of the algebra later. We proceed as follows.

Recall that $K_2 = \mathbb{Z}\cdot(a_1,a_2) \oplus \mathbb{Z}\cdot(b_1,b_2)$. Let d_1 be the smallest positive integer such that $(d_1,0) \in K_2$. We claim that $d_1 = d/\operatorname{GCD}(a_2,b_2)$. Indeed d_1 must be of the form $ka_1 + lb_1$ where $k, l \in \mathbb{Z}$ and $ka_2 + lb_2 = 0$ with $\mathrm{GCD}(k, l) = 1$. For such k, l, $k|b_2$, $l|a_2$ and $b_2/k = -a_2/l =: p > 0$. Then $\mathrm{GCD}(a_2/p, b_2/p) = 1$ which implies that $\mathrm{GCD}(a_2, b_2) = p$. Thus $d_1 = ka_1 + lb_1 = (b_2a_1 - a_2b_1)/p = d/\mathrm{GCD}(a_2, b_2)$ as claimed.

Next, let d_2 denote the smallest positive integer such that some K_2-translation of $(0, d_2)$ lies on the x-axis, i.e. such that

$$\big((0, d_2) + K_2\big) \cap \mathbb{Z} \times \{0\} \neq \emptyset.$$

Such an integer exists because if we write $\mathrm{GCD}(a_2, b_2) = ka_2 + lb_2$, then

$$(0, ka_2 + lb_2) - k(a_1, a_2) - l(b_1, b_2) = (-ka_1 - lb_1, 0).$$

On the other hand if $(0, d_2) + k\mathbf{a} + l\mathbf{b} \in \mathbb{Z} \times \{0\}$, i.e. if $d_2 = ka_2 + lb_2$, then $\mathrm{GCD}(a_2, b_2)|d_2$. Therefore $d_2 = \mathrm{GCD}(a_2, b_2)$.

We also see that for any point in $\mathbb{Z}^2$ of the form (x, d_2) there is a $g \in K_2$ such that $(x, d_2) + g \in \mathbb{Z} \times \{0\}$. Also, $(d_1, 0) \in K_2$ so for any point of the form (d_1, y) there is a $g \in K_2$ (namely $(-d_1, 0)$) such that $(d_1, y) + g \in \{0\} \times \mathbb{Z}$.

Suppose now that for some $k, l \in \mathbb{Z}$,

$$k(a_1, a_2) + l(b_1, b_2) \in K_2 \cap \{0, 1, \dots, d_1 - 1\} \times \{0, 1, \dots, d_2 - 1\}.$$

Then we would have $(0, ka_2 + lb_2) - (k\mathbf{a} + l\mathbf{b}) \in \mathbb{Z} \times \{0\}$ and $ka_2 + lb_2 \in \{0, 1, \dots, d_2 - 1\}$ which contradicts the minimality of d_2 unless $ka_2 + lb_2 = 0$. But in this case $(ka_1 + lb_1, 0) \in K_2$ which contradicts the minimality of d_1 unless $ka_1 + lb_1 = 0$. Hence $K_2 \cap \{0, 1, \dots, d_1 - 1\} \times \{0, 1, \dots, d_2 - 1\} = \{(0, 0)\}$. We have shown that

$$S := \{0, 1, \dots, d_1 - 1\} \times \{0, 1, \dots, d_2 - 1\}$$

is a set of representatives for $\mathbb{Z}^2/K_2$. In particular we get from Corollary 5.3 that $\dim M$ is finite and

$$\dim M/\dim M_{\mathrm{m}} = |S| = d_1d_2 = a_1b_2 - b_1a_2.$$

We fix now integers a_2', b_2' such that

$$d_2 = \mathrm{GCD}(a_2, b_2) = a_2'a_2 + b_2'b_2 \tag{6.29}$$

and such that $-a_2'a_1 - b_2'b_1 \in \{0, 1, \dots, d_1 - 1\}$. This can be done because for any $p \in \mathbb{Z}$, $(a_2'', b_2'') := (a_2' + pb_2/d_2, b_2' - pa_2/d_2)$ also satisfies $a_2''a_2 + b_2''b_2 = d_2$ but now

$$-a_2''a_1 - b_2''b_1 = -(a_2' + pb_2/d_2)a_1 - (b_2' - pa_2/d_2)b_1 = -a_2'a_1 - b_2'b_1 - pd_1.$$

We set

$$s = -a_2'a_1 - b_2'b_1. \tag{6.30}$$

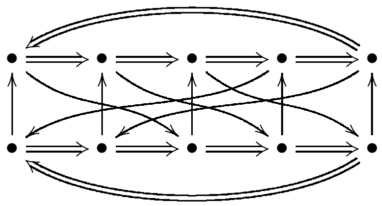


Figure 2: An example of the action on $\operatorname{supp}(M)$ when $\mathfrak{m} = \mathfrak{n}_{\lambda}^{(2)}$, $\lambda \neq 0$ and $\operatorname{rank} K_2 = 2$. Here $\mathbf{a} = (2,-2)$, $\mathbf{b} = (3,2)$, $d_1 = 5, d_2 = 2$ and $s = 2$. The $\Rightarrow$ arrows indicate the action of X_1 and the $\rightarrow$ arrows show the action of X_2.

Let $(i,j) \in S$. We have the following reductions in $\mathbb{Z}^2$ modulo K_2.

$$(1,0)+(i,j) = \begin{cases} (i+1,j), & 0 \leq i < d_1 - 1, \\ (0,j), & i = d_1 - 1, \end{cases}$$

$$(0,1)+(i,j) = \begin{cases} (i,j+1), & 0 \leq j < d_2 - 1, \\ (i+s,0), & j = d_2 - 1, i+s \leq d_1 - 1, \\ (i+s-d_1,0), & j = d_2 - 1, j+s > d_1 - 1. \end{cases}$$

From this we can understand how the X_i act on the support of M, see Figure 2 for an example. By Theorem 5.1 the set

$$C = \{w_{ijk} := X_1^i X_2^j v_k \mid 0 \leq i < d_1, 0 \leq j < d_2, 0 \leq k < r\}$$

is a basis for M where v_k is the basis (6.26) for $M_{\mathfrak{m}}$.

If $0 \leq i < d_1 - 1$ we clearly have $X_1 w_{ijk} = w_{i+1,j,k}$. Suppose $i = d_1 - 1$. Then by Lemma 6.9,

$$X_1 w_{ijk} = X_1^{d_1} X_2^j v_k = q_1^{d_1 j} \lambda_{12}^{d_1 j} X_2^j X_1^{d_1} v_k.$$

Thus we must express $X_1^{d_1}$ in terms of $Z_1^{a_1} Z_2^{a_2}$ and $Z_1^{b_1} Z_2^{b_2}$. Since $(d_1, 0) = b_2/d_2 \mathbf{a} - a_2/d_2 \mathbf{b}$ we have

$$(Z_1^{a_1} Z_2^{a_2})^{b_2/d_2} (Z_1^{b_1} Z_2^{b_2})^{-a_2/d_2} = C_1^{-1} X_1^{d_1} \tag{6.31}$$

as operators on $M_{\mathfrak{m}}$ for some constant C_1^{-1} which we must calculate.

Lemma 6.13. *The constant C_1 defined in (6.31) is given by*

$$C_1^{-1} = r_{\mathbf{a}}^{\overline{-b_2/d_2}} (q_1^{-a_1 \overline{a_2}} \lambda_{12}^{-a_1 a_2})^{\frac{b_2}{d_2}(\frac{b_2}{d_2}-1)/2} \cdot r_{\mathbf{b}}^{\overline{a_2/d_2}} (q_1^{-b_1 \overline{b_2}} \lambda_{12}^{-b_1 b_2})^{\frac{a_2}{d_2}(\frac{a_2}{d_2}+1)/2} \cdot \\ \cdot q_1^{b_1 a_2 \overline{a_2 b_2}/d_2^2} \lambda_{12}^{b_1 a_2^2 b_2/d_2^2} r_{(0,-b_2 a_2/d_2)}^{-1} C_1', \tag{6.32}$$

where the r_g, $g \in \mathbb{Z}^2$ are given by (6.20),

$$C_1' = \begin{cases} (1-q_1)^{-\min\{|a_1 b_2/d_2|, |b_1 a_2/d_2|\}}, & a_2 b_2 > 0, \\ 1, & a_2 b_2 \leq 0, \end{cases}$$

$\overline{k} = \max\{0,k\}$ *for $k \in \mathbb{Z}$ and $d_2 = \operatorname{GCD}(a_2, b_2)$.*

Proof. If $b_2 \geq 0$ for example, we have by Lemma 6.9

$$\begin{aligned}(Z_1^{a_1}Z_2^{a_2})^{b_2/d_2} = q_1^{-a_1\overline{a_2}}\lambda_{12}^{-a_1a_2}\cdot(q_1^{-a_1\overline{a_2}}\lambda_{12}^{-a_1a_2})^2\cdot\ldots\\ \ldots\cdot(q_1^{-a_1\overline{a_2}}\lambda_{12}^{-a_1a_2})^{b_2/d_2-1}Z_1^{a_1b_2/d_2}Z_2^{a_2b_2/d_2} =\\ = (q_1^{-a_1\overline{a_2}}\lambda_{12}^{-a_1a_2})^{\frac{b_2}{d_2}(\frac{b_2}{d_2}-1)/2}Z_1^{a_1b_2/d_2}Z_2^{a_2b_2/d_2}\end{aligned}$$

When $b_2 < 0$ we get a similar calculation where $r_{\mathbf{a}}^{-b_2/d_2}$ appears by Lemma 6.12. $(Z_1^{b_1}Z_2^{b_2})^{-a_2/d_2}$ can analogously be expressed as a multiple of $Z_1^{-b_1a_2/d_2}Z_2^{-b_2a_2/d_2}$. We then commute $Z_2^{a_2b_2/d_2}$ and $Z_1^{-b_1a_2/d_2}$ using Lemma 6.9. As a last step we use Lemma 6.10 and obtain two more factors. □

We conclude that

$$X_1w_{ijk} = \begin{cases} w_{i+1,j,k}, & i < d_1 - 1,\\ q_1^{jd_1}\lambda_{12}^{jd_2}C_1\nu^{b_2/d_2k_1''}\rho^{b_2/d_2}\mu^{k_1'}w_{0,j,k_1''}, & i = d_1 - 1.\end{cases}$$

Here

$$k - a_2/d_2 = rk_1' + k_1'' \quad \text{with } 0 \leq k_1'' < r. \tag{6.33}$$

Next we turn to the description of how X_2 acts on the basis C. If $0 \leq j < d_2 - 1$ we have $X_2w_{ijk} = q_1^{-i}\lambda_{12}^{-i}w_{i,j+1,k}$ by Lemma 6.9. Suppose $j = d_2 - 1$. Then, as in the first step of the proof of Theorem 5.4,

$$X_2w_{ijk} = q_1^{-i}\lambda_{12}^{-i}X_1^iX_2^{d_2}v_k = q_1^{-i}\lambda_{12}^{-i}X_1^i(X_2^{d_2}r_{(-s,d_2)}Z_2^{-d_2}Z_1^s)(Z_1^{-s}Z_2^{d_2})v_k. \tag{6.34}$$

By (6.16) and (6.20),

$$\begin{aligned}X_2^{d_2}r_{(-s,d_2)}Z_2^{-d_2}Z_1^s &= r_{(-s,d_2)}r_{(0,-d_2)}^{-1}Z_1^s =\\ &= (1-q_1)^s(\lambda^{-1}q_2^{(d_2-1)/2})^{d_2}(\lambda^{-1}q_2^{(-d_2-1)/2})^{d_2}Z_1^s =\\ &= (1-q_1)^s(\lambda^2q_2)^{-d_2}Z_1^s.\end{aligned} \tag{6.35}$$

We must express $Z_1^{-s}Z_2^{d_2}$ in the generators of the algebra $B_{\mathrm{m}}^{(1)}$ in order to calculate its action on v_k.

$$(Z_1^{a_1}Z_2^{a_2})^{a_2'}(Z_1^{b_1}Z_2^{b_2})^{b_2'} = C_2^{-1}Z_1^{-s}Z_2^{d_2}, \tag{6.36}$$

for some $C_2 \in \mathbb{C}^*$ since the degree on both sides are equal by (6.29) and (6.30). Similarly to the proof of Lemma 6.13,

$$\begin{aligned}C_2^{-1} = r_{\mathbf{a}}^{\overline{-a_2'}}(q_1^{-a_1\overline{a_2}}\lambda_{12}^{-a_1a_2})^{a_2'(a_2'-1)/2}\cdot r_{\mathbf{b}}^{\overline{-b_2'}}(q_1^{-b_1\overline{b_2}}\lambda_{12}^{-b_1b_2})^{b_2'(b_2'-1)/2}\cdot\\ \cdot q_1^{-b_1b_2'\overline{a_2a_2'}}\lambda^{-b_1b_2'a_2a_2'}C_2'C_2'',\end{aligned} \tag{6.37}$$

and

$$C_2' = \begin{cases} 1, & a_2'b_2' \geq 0,\\ (1-q_1)^{-\min\{|a_1a_2'|,|b_1b_2'|\}}, & a_2'b_2' < 0,\end{cases}$$

$$C_2'' = \begin{cases} 1, & a_2a_2'b_2b_2' \geq 0,\\ \lambda^{m'}q_2^{(1-2b_2b_2'+(\operatorname{sgn} b_2b_2')m')m'/2}, & a_2a_2'b_2b_2' < 0,\end{cases}$$

where $m' = \min\{|a_2 a_2'|, |b_2 b_2'|\}$. Furthermore, letting

$$b_2' + k = r k_2' + k_2'', \quad \text{where } 0 \le k_2'' < r \tag{6.38}$$

we have by (6.27)–(6.28),

$$(Z_1^{a_1} Z_2^{a_2})^{a_2'} (Z_1^{b_1} Z_2^{b_2})^{b_2'} v_k = \nu^{a_2' k_2''} \rho^{a_2'} \mu^{k_2'} v_{k_2''}. \tag{6.39}$$

If $i + s \le d_1 - 1$ we can now write down the action of X_2 on w_{ijk} by combining (6.34)–(6.37), (6.39) to get a multiple of $w_{i+s,0,k_2''}$. However if $i + s > d_1 - 1$, we must reduce further because then $(i + s, 0) \notin S$. Let

$$k_2'' - a_2/d_2 = r k_3' + k_3'', \quad \text{where } 0 \le k_3'' < r. \tag{6.40}$$

Then by the calculations for the action of $X_1^{d_1}$ on M_{m},

$$X_1^{d_1} v_{k_2''} = X_1^{i+s-d_1} X_1^{d_1} v_{k_2''} = C_1 \mu^{k_3'} \nu^{k_3'' b_2/d_2} \rho^{b_2/d_2} w_{i+s-d_1,0,k_3''}.$$

Summing up, M has a basis

$$\{w_{ijk} \mid 0 \le i < d_1, 0 \le j < d_2, 0 \le k < r\}$$

and X_1, X_2 act on this basis as follows.

$$X_1 w_{ijk} = \begin{cases} w_{i+1,j,k}, & i < d_1 - 1, \\ q_1^{jd_1} \lambda_{12}^{jd_2} C_1 \nu^{b_2/d_2 k_1''} \rho^{b_2/d_2} \mu^{k_1'} w_{0,j,k_1''}, & i = d_1 - 1. \end{cases}$$

$$X_2 w_{ijk} = (q_1 \lambda_{12})^{-i} \cdot$$

$$\cdot \begin{cases} w_{i,j+1,k}, \\ \qquad \text{if } 0 \le j < d_2 - 1, \\ (1 - q_1)^s (\lambda^2 q_2)^{-d_2} C_2 \nu^{a_2' k_2''} \rho^{a_2'} \mu^{k_2'} w_{i+s,0,k_2''}, \\ \qquad \text{if } j = d_2 - 1 \text{ and } i + s \le d_1 - 1, \\ (1 - q_1)^s (\lambda^2 q_2)^{-d_2} C_2 \nu^{a_2' k_2'' + k_3'' b_2/d_2} \rho^{a_2' + b_2/d_2} \mu^{k_2' + k_3'} C_1 w_{i+s-d_1,0,k_3''}, \\ \qquad \text{if } j = d_2 - 1 \text{ and } i + s > d_1 - 1, \end{cases} \tag{6.41}$$

where C_1 is given by (6.32), C_2 by (6.37) and ν by (6.25). The parameters ρ and μ comes from the action (6.27), (6.28) of $B_{\mathrm{m}}^{(1)}$ on M_{m} and k_i', k_i'' are defined in (6.33), (6.38) and (6.40).

The action of the Y_i is uniquely determined by

$$\begin{aligned} Y_1 X_1 w_{ijk} &= (1 - q_1)^{-1} w_{ijk}, \\ Y_2 X_2 w_{ijk} &= \lambda q_1^{-i} q_2^{-j} w_{ijk}. \end{aligned} \tag{6.42}$$

We remark that the case $q_1 = q_2$ corresponds to $\mathbf{a} = (a_1, a_2) = (1, -1)$. Then $d_2 = 1$, $d_1 = d = |b_1 + b_2|$ and $s = 1$. X_1 and X_2 will act on the support in the same direction, cyclically as in Figure 3. The explicit action can be deduced from the above more general case noting that here $k_2'' = k$, $k_2' = 0$ and

$$k_1' = k_3' = \begin{cases} 0, & k < r - 1, \\ 1, & k = r - 1, \end{cases} \qquad k_1'' = k_3'' = \begin{cases} k, & k < r - 1, \\ 0, & k = r - 1. \end{cases}$$

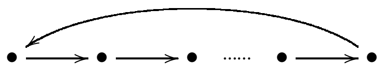


Figure 3: Weight diagram when $\mathfrak{m} = \mathfrak{n}^{(2)}_\lambda$, $\lambda \neq 0$, $\operatorname{rank} K_2 = 2$ and $q_1 = q_2$.

6.6 The case $\mathfrak{m} \notin \{\mathfrak{n}^{(i)}_\mu \mid \mu \in \mathbb{C}, i = 1, 2\}$

This is the generic case. We have $\mathbb{Z}^2_{\mathfrak{m}} = Q$ by Corollary 6.8. Our statements here generalize without any problem to the case of arbitrary rank.

Assume first that the q_i are roots of unity of orders o_i $(i = 1, 2)$ and that ω does not contain any 1-breaks or 2-breaks. Then by Corollary 6.2 and Proposition 6.4 we have $\tilde{G}_{\mathfrak{m}} = \mathbb{Z}^2$. Thus $G_{\mathfrak{m}} = (o_1\mathbb{Z}) \times (o_2\mathbb{Z})$. Moreover,

$$X_1^{o_1} X_2^{o_2} = \lambda_{12}^{o_1 o_2} X_2^{o_2} X_1^{o_1}$$

so $B^{(1)}_{\mathfrak{m}} \simeq T_{\lambda_{12}^{o_1 o_2}}$ by Corollary 4.6. This algebra has only finite dimensional representations if $\lambda_{12}^{o_1 o_2}$ is a root of unity. Assuming this, let r be the order of $\lambda_{12}^{o_1 o_2}$. Then there are $\rho, \mu \in \mathbb{C}^*$ and $M_{\mathfrak{m}}$ has a basis $v_0, v_1, \dots, v_{r-1}$ such that

$$\begin{aligned} X_1^{o_1} v_i &= \lambda_{12}^{i o_1 o_2} \rho v_i \\ X_2^{o_1} v_i &= \begin{cases} v_{i+1} & 0 \le i < p - 1 \\ \mu v_0 & i = p - 1 \end{cases} \end{aligned}$$

Choose $S = \{0, 1, \dots, o_1 - 1\} \times \{0, 1, \dots, o_2 - 1\}$. The corresponding basis for M is $C = \{w_{ijk} := X_1^i X_2^j v_k \mid 0 \le i < o_1, 0 \le j < o_2, 0 \le k < r\}$. The following formulas are easily deduced using (2.6)–(2.8).

$$\begin{aligned} X_1 w_{ijk} &= \begin{cases} w_{i+1,j,k}, & k < o_1 - 1, \\ \lambda_{12}^{o_1(o_2 k + j)} \rho w_{0jk}, & k = o_1 - 1, \end{cases} \\ X_2 w_{ijk} &= (q_1 \lambda_{12})^{-i} \cdot \begin{cases} w_{i,j+1,l}, & l < o_2 - 1, \\ w_{i,0,l+1}, & l = o_2 - 1, i < r - 1, \\ \mu w_{i00}, & l = o_2 - 1, i = r - 1. \end{cases} \end{aligned} \tag{6.43}$$

The action of Y_1, Y_2 is determined by

$$\begin{aligned} Y_1 X_1 w_{ijk} &= q_1^{-i}(\alpha_1 - [i]_{q_1}) w_{ijk}, \\ Y_2 X_2 w_{ijk} &= q_1^{-i} q_2^{-j}(\alpha_2 - [j]_{q_2}(1 + (q_1 - 1)\alpha_1)) w_{ijk}. \end{aligned} \tag{6.44}$$

In all other cases one can show using the same argument that $\dim M_{\mathfrak{n}} = 1$ for all $\mathfrak{n} \in \operatorname{supp}(M)$ and that M can be realized in a vector space with basis $\{w_{ij}\}_{(i,j) \in I}$, where $I = I_1 \times I_2$ is one of the following sets

$$\begin{aligned} &\mathbb{N}_{d_1} \times \mathbb{N}_{d_2}, \quad \mathbb{N}_{d_1} \times \mathbb{Z}^{\pm}, \quad \mathbb{Z}^{\pm} \times \mathbb{N}_{d_2}, \quad \mathbb{Z} \times \mathbb{Z}, \\ &\mathbb{Z}^{\pm} \times \mathbb{Z}, \quad \mathbb{Z} \times \mathbb{Z}^{\pm}, \quad \mathbb{Z}^{\pm} \times \mathbb{Z}^{\pm}, \quad \mathbb{Z}^{\pm} \times \mathbb{Z}^{\mp}, \end{aligned}$$

where $\mathbb{N}_d = \{0, 1, \dots, d-1\}$, $\mathbb{Z}^{\pm} = \{k \in \mathbb{Z} \mid \pm k \geq 0\}$ and d_i is the order of q_i if finite. The action of the generators is given by the following formulas.

$$
\begin{aligned}
X_1 w_{ij} &= \begin{cases} w_{i+1,j}, & (i+1,j) \in I, \\ \rho \lambda_{12}^{d_1 j} w_{0,j}, & (i+1,j) \notin I, I_1 = \mathbb{N}_{d_1} \text{ and } \alpha_1 \neq [i]_{q_1}, \\ 0, & \text{otherwise}, \end{cases} \\
X_2 w_{ij} &= (q_1 \lambda_{12})^{-i} \cdot \begin{cases} w_{i,j+1}, & (i,j+1) \in I, \\ \mu w_{i,0}, & (i,j+1) \notin I, I_2 = \mathbb{N}_{d_2} \\ & \quad \text{and } \alpha_2 \neq [j]_{q_2}(1+(q_1-1)\alpha_1), \\ 0, & \text{otherwise}, \end{cases}
\end{aligned}
\tag{6.45}
$$

$$
\begin{aligned}
Y_1 w_{ij} &= q_1^{-i+1}(\alpha_1 - [i-1]_{q_1}) \cdot \\
&\cdot \begin{cases} w_{i-1,j}, & (i-1,j) \in I, \\ (\rho \lambda_{12}^{d_1 j})^{-1} w_{d_1-1,j}, & (i-1,j) \notin I, I_1 = \mathbb{N}_{d_1} \text{ and } \alpha_1 \neq [i-1]_{q_1}, \\ 0, & \text{otherwise}, \end{cases} \\
Y_2 w_{ij} &= \lambda_{12}^{-i} q_2^{-j+1}(\alpha_2 - [j-1]_{q_2}(1+(q_1-1)\alpha_1)) \cdot \\
&\cdot \begin{cases} w_{i,j+1}, & (i,j+1) \in I, \\ \mu^{-1} w_{i,d_2-1}, & (i,j+1) \notin I, I_1 = \mathbb{N}_{d_2} \\ & \quad \text{and } \alpha_2 \neq [j-1]_{q_2}(1+(q_1-1)\alpha_1), \\ 0, & \text{otherwise}. \end{cases}
\end{aligned}
\tag{6.46}
$$

Thus we have proved the following result.

Theorem 6.14. *Let A be a quantized Weyl algebra of rank two with arbitrary parameters $q_1, q_2 \in \mathbb{C} \backslash \{0, 1\}$. Then any simple weight A-module with no proper inner breaks is isomorphic to one of the modules defined by formulas (6.6), (6.7), (6.8), (6.9), (6.10), (6.11), (6.21-6.22), (6.23-6.24), (6.41-6.42), (6.43-6.44) or (6.45-6.46).*

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References

[1] V. Bavula, *Finite-dimensionality of* Ext^n *and* Tor_n *of simple modules over a class of algebras.* (Russian) Funktsional. Anal. i Prilozhen. **25** No. 3 (1991) 80–82; translation in Funct. Anal. Appl. **25** No. 3 (1991) 229–230.

[2] V. Bavula, *Generalized Weyl algebras and their representations*, St. Petersb. Math. J., **4** (1993) 71–93.

[3] V. Bavula, V. Bekkert, *Indecomposable representations of generalized Weyl algebras*, Comm. Algebra **28** (2000) 5067–5100.

[4] Brenken, B., *A classification of some noncommutative tori*, The Rocky Mountain Journal of Mathematics, **20** No. 2 (1990) 389–397.

[5] Yu. Drozd, V. Futorny, S. Ovsienko, *Harish-Chandra subalgebras and Gelfand-Zetlin modules*, in: *Finite-dimensional algebras and related topics*, (Ottawa, ON, 1992), NATO Adv. Sci. Inst. Ser. C. Math. Phys. Sci. **424** (1994) 79–93.

[6] Yu. Drozd, B. Guzner, S. Ovsienko, *Weight modules over generalized Weyl algebras*, J. Algebra **184** (1996) 491–504.

[7] Mazorchuk, V., Ponomarenko, M., Turowska, L., *Some associative algebras related to $U(\mathfrak{g})$ and twisted generalized Weyl algebras*, Math. Scand. **92** (2003) 5–30.

[8] Mazorchuk, V., Turowska, L., *Simple weight modules over twisted generalized Weyl algebras*, Comm. Alg. **27** No. 6 (1999) 2613–2625.

[9] Newman, M., *Integral Matrices*, Academic Press, New York, 1972.

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Paper II

Hopf structures on ambiskew polynomial rings

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Abstract

We derive necessary and sufficient conditions for an ambiskew polynomial ring to have a Hopf algebra structure of a certain type. This construction generalizes many known Hopf algebras, for example $U(\mathfrak{sl}_2)$, $U_q(\mathfrak{sl}_2)$ and the enveloping algbra of the 3-dimensional Heisenberg Lie algebra. In a torsion-free case we describe the finite-dimensional simple modules, in particular their dimensions and prove a Clebsch-Gordan decomposition theorem for the tensor product of two simple modules. We construct a Casimir type operator and prove that any finite-dimensional weight module is semisimple.

1 Introduction

In [4], the authors define a four parameter deformation of the Heisenberg (oscillator) Lie algebra $\mathscr{W}_{\alpha,\beta}^{\gamma}(q)$ and study its representations. Moreover by requiring this algebra to be invariant under $q \to q^{-1}$, they define a Hopf algebra structure on $\mathscr{W}_{\alpha,\beta}^{\gamma}(q)$ generalizing several previous results.

The quantum group $U_q(\mathfrak{sl}_2(\mathbb{C}))$ has by definition the structure of a Hopf algebra. In [10], an extension of this quantum group to an associative algebra denoted by $U_q(f(H,K))$ (where f is a Laurent polynomial in two variables) is defined and finite-dimensional representations are studied. The authors show that under certain conditions on f, a Hopf algebra structure can be introduced. Among these Hopf algebras is for example the Drinfeld double $\mathscr{D}(\mathfrak{sl}_2)$.

All of the mentioned algebras fall (after suitable mathematical formalization in the case of $\mathscr{W}_{\alpha,\beta}^{\gamma}(q)$) into the class of so called ambiskew polynomial rings (see Section 2 for the definiton). Motivated by these examples of similar classes of algebras, all of which can be equipped with Hopf algebra structures, we consider a certain type of Hopf structures on a class of ambiskew polynomial rings.

In Section 2, we recall some definitions and fix notation. We present the conditions for a certain Hopf structure on an ambiskew polynomial ring in Section 3, while Section 4 is devoted to examples. In Section 5 we introduce some convenient notation and state some useful formulas for viewing R as an algebra of functions on its set of maximal ideals. Finite-dimensional simple modules are studied in Section 6. Those have already been classified in [6], but we focus on describing the dimensions in terms of the highest weights. The main result is stated in Theorem 6.19. The classical Clebsch-Gordan theorem for $U(\mathfrak{sl}_2)$ is generalized in Section 7 to the present more general setting, using the results of the previous section. Finally, in

Section 8 we first construct a kind of Casimir operator and prove that it can be used to distinguish non-isomorphic simple modules. This is then used to prove that any weight module is semisimple.

2 Preliminaries

Throughout, $\mathbb{K}$ will be an algebraically closed field of characteristic zero. All algebras are associative and unital $\mathbb{K}$-algebras.

By a *Hopf structure* on an algebra A we mean a triple (Δ, ε, S) where the *coproduct* $\Delta : A \to A \otimes A$ is a homomorphism, ($A \otimes A$ is given the tensor product algebra structure) the *counit* $\varepsilon : A \to \mathbb{K}$ is a homomorphism, and the *antipode* $S : A \to A$ is an anti-homomorphism such that

$$(\mathrm{Id} \otimes \Delta)(\Delta(x)) = (\Delta \otimes \mathrm{Id})(\Delta(x)), \qquad \text{(Coassociativity)} \qquad (2.1)$$

$$m\Big((\mathrm{Id} \otimes \varepsilon)(\Delta(x))\Big) = x = m\Big((\varepsilon \otimes \mathrm{Id})(\Delta(x))\Big), \qquad \text{(Counit axiom)} \qquad (2.2)$$

$$m\Big((S \otimes \mathrm{Id})(\Delta(x))\Big) = \varepsilon(x) = m\Big((\mathrm{Id} \otimes S)(\Delta(x))\Big), \qquad \text{(Antipode axiom)} \qquad (2.3)$$

for all $x \in A$. Here $m : A \otimes A \to A$ denotes the multiplication map of A. A *Hopf algebra* is an algebra equipped with a Hopf structure. An element $x \in A$ of a Hopf algebra A is called *grouplike* if $\Delta(x) = x \otimes x$ and *primitive* if $\Delta(x) = x \otimes 1 + 1 \otimes x$. In the former case it follows from the axioms that $\varepsilon(x) = 1$, x is invertible and $S(x) = x^{-1}$ while in the latter $\varepsilon(x) = 0$ and $S(x) = -x$.

If V_i $(i = 1, 2)$ are two modules over a Hopf algebra H, then $V_1 \otimes V_2$ becomes an H-module in the following way

$$a(v_1 \otimes v_2) = \sum_i (a_i' v_1) \otimes (a_i'' v_2) \qquad (2.4)$$

for $v_i \in V_i$ $(i = 1, 2)$ if $a \in H$ with $\Delta(a) = \sum_i a_i' \otimes a_i''$. From (2.1) it follows that if V_i $(i = 1, 2, 3)$ are modules over H then the natural vector space isomorphism $V_1 \otimes (V_2 \otimes V_3) \simeq (V_1 \otimes V_2) \otimes V_3$ is an isomorphism of H-modules. From (2.2) follows that the one-dimensional module $\mathbb{K}_\varepsilon$ associated to the representation ε of H is a tensor unit, i.e. $\mathbb{K}_\varepsilon \otimes V \simeq V \simeq V \otimes \mathbb{K}_\varepsilon$ as H-modules for any H-module V.

Let R be a finitely generated commutative algebra over $\mathbb{K}$. Let σ be a $\mathbb{K}$-algebra automorphism of R, $\mathsf{h} \in R$ and $\xi \in \mathbb{K} \backslash \{0\}$. Then we define the algebra $A = A(R, \sigma, \mathsf{h}, \xi)$ as the associative $\mathbb{K}$-algebra formed by adjoining to R two symbols X_+, X_- subject to the relations

$$X_\pm a = \sigma^{\pm 1}(a) X_\pm \text{ for } a \in R, \qquad (2.5)$$

$$X_+ X_- = \mathsf{h} + \xi X_- X_+. \qquad (2.6)$$

This algebra is called an *ambiskew polynomial ring*. Its structure and representations were studied by Jordan [7] (see also references therein).

We recall the definition of a *generalized Weyl algebra (GW-algebra)* (see [1] and references therein). If B is a ring, σ an automorphism of B, and $t \in B$ a central element, then the generalized Weyl algebra $B(\sigma, t)$ is the ring extension of B generated by two elements x_+, x_- subject to the relations

$$\begin{aligned} x_\pm a &= \sigma^{\pm 1}(a) x_\pm, \quad \text{for } a \in B, \\ x_- x_+ = t, &\quad \text{and} \quad x_+ x_- = \sigma(t). \end{aligned} \tag{2.7}$$

The relation between these two constructions is the following. Let $A = A(R, \sigma, \mathsf{h}, \xi)$ be an ambiskew polynomial ring. Denote by $R[t]$ be the polynomial ring in one variable t with coefficients in R and let us extend the automorphism σ of R to a $\mathbb{K}$-algebra automorphism of $R[t]$ satisfying

$$\sigma(t) = \mathsf{h} + \xi t. \tag{2.8}$$

Then A is isomorphic to the GW-algebra $R[t](\sigma, t)$.

3 The Hopf structure

Let $A = A(R, \sigma, \mathsf{h}, \xi)$ be an ambiskew polynomial ring and assume that R has been equipped with a Hopf structure. In this section we will extend the Hopf structure on R to A. We make the following ansatz, guided by [4] and [10]:

$$\Delta(X_\pm) = X_\pm \otimes r_\pm + l_\pm \otimes X_\pm, \tag{3.1}$$
$$\varepsilon(X_\pm) = 0, \tag{3.2}$$
$$S(X_\pm) = s_\pm X_\pm. \tag{3.3}$$

The elements $r_\pm, l_\pm$ and $s_\pm$ will be assumed to belong to R.

Theorem 3.1. *Formulas (3.1)-(3.3) define a Hopf algebra structure on A which extends that of R iff $r_\pm, l_\pm, s_\pm$ are invertible and*

$$(\sigma \otimes \mathrm{Id}) \circ \Delta|_R = \Delta \circ \sigma|_R = (\mathrm{Id} \otimes \sigma) \circ \Delta|_R, \tag{3.4a}$$
$$S \circ \sigma|_R = \sigma^{-1} \circ S|_R, \tag{3.4b}$$

$$\Delta(\mathsf{h}) = \mathsf{h} \otimes r_+ r_- + l_+ l_- \otimes \mathsf{h}, \tag{3.5a}$$
$$\varepsilon(\mathsf{h}) = 0, \tag{3.5b}$$
$$S(\mathsf{h}) = -(l_+ l_- r_+ r_-)^{-1} \mathsf{h}, \tag{3.5c}$$

$$r_\pm \text{ and } l_\pm \text{ are grouplike, i.e. } \Delta(x) = x \otimes x \text{ for } x \in \{r_\pm, l_\pm\}, \tag{3.6a}$$
$$\sigma(l_\pm) \otimes \sigma(r_\mp) = \xi l_\pm \otimes r_\mp, \tag{3.6b}$$

$$(s_\pm)^{-1} = -l_\pm \sigma^{\pm 1}(r_\pm). \tag{3.7}$$

Proof. From (2.5)-(2.6) we see that ε extends to a homomorphism $A \to \mathbb{K}$ satisfying (3.2) if and only if (3.5b) holds. Assume for a moment that Δ extends to a homomorphism $A \to A \otimes A$. From (3.1)-(3.2) it follows that ε is a counit iff

$$\varepsilon(r_+) = \varepsilon(r_-) = \varepsilon(l_+) = \varepsilon(l_-) = 1. \tag{3.8}$$

Δ is coassociative iff (dropping the $\pm$)

$$(\mathrm{Id} \otimes \Delta)(\Delta(X)) = (\Delta \otimes \mathrm{Id})(\Delta(X))$$

which is equivalent to

$$X \otimes \Delta(r) + l \otimes X \otimes r + l \otimes l \otimes X = X \otimes r \otimes r + l \otimes X \otimes r + \Delta(l) \otimes X,$$

or

$$X \otimes (\Delta(r) - r \otimes r) = (\Delta(l) - l \otimes l) \otimes X. \tag{3.9}$$

From (2.5)-(2.6) follows that A has a $\mathbb{Z}$-gradation defined by requiring that $\deg r = 0$ for $r \in R$, $\deg X_\pm = \pm 1$. This also induces a $\mathbb{Z}^2$-gradation on $A \otimes A$ in a natural way. The left and right hand sides of equation (3.9) are homogenous of different $\mathbb{Z}^2$-degrees, namely $(\pm 1, 0)$ and $(0, \pm 1)$ respectively. Hence, since homogenous elements of different degrees must be linearly independent, (3.9) is equivalent to both sides being zero which holds iff $r_\pm$ and $l_\pm$ are grouplike.

Δ respects (2.5) iff (again dropping $\pm$)

$$\Delta(X)\Delta(a) = \Delta(\sigma(a))\Delta(X),$$
$$(X \otimes r + l \otimes X)\Delta(a) = \Delta(\sigma(a))(X \otimes r + l \otimes X),$$

$$(\sigma \otimes 1)(\Delta(a)) \cdot (X \otimes r) + (1 \otimes \sigma)(\Delta(a)) \cdot (l \otimes X) = \Delta(\sigma(a))(X \otimes r + l \otimes X),$$
$$\big((\sigma \otimes 1)(\Delta(a)) - \Delta(\sigma(a))\big) \cdot (X \otimes r) + \big((1 \otimes \sigma)(\Delta(a)) - \Delta(\sigma(a))\big) \cdot (l \otimes X) = 0.$$

As before the two terms in the last equation have different $\mathbb{Z}^2$-degrees and therefore must be zero. So Δ respects (2.5) iff (3.4a) holds.

It is straightforward to check that Δ respects (2.6) iff

$$\begin{aligned} &\mathsf{h} \otimes r_+ r_- + l_+ l_- \otimes \mathsf{h} - \Delta(\mathsf{h}) + \\ &\qquad + \Big(l_+ \otimes \sigma(r_-) - \xi \sigma^{-1}(l_+) \otimes r_-\Big) X_- \otimes X_+ + \\ &\qquad\qquad + \Big(\sigma(l_-) \otimes r_+ - \xi l_- \otimes \sigma^{-1}(r_+)\Big) X_+ \otimes X_- = 0. \end{aligned} \tag{3.10}$$

Again these three terms have different degrees so each of them must be zero. Hence (3.5a) holds. Multiply the second term by $X_+ \otimes X_-$ from the right:

$$\big(l_+ \otimes \sigma(r_-) - \xi \sigma^{-1}(l_+) \otimes r_-\big) t \otimes \sigma(t) = 0.$$

Here we use the extension (2.8) of σ to $R[t]$ where $t = X_- X_+$. If we apply $e_1 \otimes e_1'$ to this equation, where e_r (e_r') for $r \in R$ is the evaluation homomorphism $R[t] \to R$ which maps t ($\sigma(t)$) to r, we get

$$l_+ \otimes \sigma(r_-) = \xi \sigma^{-1}(l_+) \otimes r_-.$$

Applying $\sigma \otimes 1$ to this we obtain one of the relations in (3.6b). Similarly the vanishing of the third term in (3.10) implies the other.

Assuming that S is an anti-homomorphism $A \to A$ satisfying (3.3), we obtain that S is an antipode on A iff

$$S(X_\pm)r_\pm + S(l_\pm)X_\pm = 0 = X_\pm S(r_\pm) + l_\pm S(X_\pm),$$

which is equivalent to (3.7), using that $r_\pm$ and $l_\pm$ are grouplike. And S extends to a well-defined anti-homomorphism $A \to A$ iff

$$S(a)S(X_\pm) = S(X_\pm)S(\sigma^{\pm 1}(a)), \quad \text{for } a \in R, \tag{3.11}$$

$$S(X_-)S(X_+) = S(\mathsf{h}) + \xi S(X_+)S(X_-). \tag{3.12}$$

Using (3.7) and that $r_\pm, l_\pm$ are invertible, (3.11) holds iff (3.4b) holds. And (3.12) holds iff

$$\begin{aligned} 0 &= s_- X_- s_+ X_+ - S(\mathsf{h}) - \xi s_+ X_+ s_- X_- = \\ &= s_- \sigma^{-1}(s_+) X_- X_+ - S(\mathsf{h}) - s_+ \xi \sigma(s_-) X_+ X_- = \\ &= -S(\mathsf{h}) - s_+ \sigma(s_-) \xi \mathsf{h} + \\ &\quad + \big(s_- \sigma^{-1}(s_+) - s_+ \sigma(s_-) \xi^2\big) t. \end{aligned}$$

Applying e_0 and e_1 we obtain

$$\begin{aligned} S(\mathsf{h}) &= -\xi s_+ \sigma(s_-) \mathsf{h}, \\ s_- \sigma^{-1}(s_+) &= \xi^2 s_+ \sigma(s_-). \end{aligned}$$

Substituting (3.7) in these equations and using (3.6b), the first is equivalent to (3.5c), while the other already holds. ☐

4 Examples

Many Hopf algebras known in the literature can be viewed as one defined in the previous section.

4.1 Heisenberg algebra

Let $R = \mathbb{C}[c]$ with c primitive, and $\sigma(c) = c$. Choose $\mathsf{h} = c$, $\xi = r_+ = r_- = l_+ = l_- = 1$. Then A is the universal enveloping algebra $U(\mathfrak{h}_3)$ of the three-dimensional Heisenberg Lie algebra.

4.2 $U(\mathfrak{sl}_2)$ and its quantizations

4.2.1 $U(\mathfrak{sl}_2)$

Let $R = \mathbb{C}[H]$ with Hopf algebra structure $\Delta(H) = H \otimes 1 + 1 \otimes H$, $\varepsilon(H) = 0$, $S(H) = -H$. Define $\sigma(H) = H - 1$. Choose $\mathsf{h} = H$, $\xi = r_+ = r_- = l_+ = l_- = 1$. Then $A \simeq U(\mathfrak{sl}_2)$ as Hopf algebras.

4.2.2 $U_q(\mathfrak{sl}_2)$

Let $R = \mathbb{C}[K, K^{-1}]$ with Hopf structure defined by requiring that K is grouplike. Define $\sigma(K) = q^{-2}K$, where $q \in \mathbb{C}, q^2 \neq 1$, and choose $\mathsf{h} = \frac{K-K^{-1}}{q-q^{-1}}$, $\xi = r_- = l_+ = 1$ and $r_+ = K$, $l_- = K^{-1}$. Then the equations in Theorem 3.1 are satisfied giving a Hopf algebra A which is isomorphic to $U_q(\mathfrak{sl}_2)$.

4.2.3 $\breve{U}_q(\mathfrak{sl}_2)$

For the definition of this algebra, see for example [9]. Let $q \in \mathbb{C}, q^4 \neq 1$. Let $R = \mathbb{C}[K, K^{-1}]$ with K grouplike. Define $\sigma(K) = q^{-1}K$, $\mathsf{h} = \frac{K^2-K^{-2}}{q-q^{-1}}$, $\xi = 1$, $r_+ = r_- = K$, $l_+ = l_- = K^{-1}$. Then $A = A(R, \sigma, \mathsf{h}, \xi)$ is a Hopf algebra isomorphic to $\breve{U}_q(\mathfrak{sl}_2)$.

4.3 $U_q(f(H, K))$

Let $R = \mathbb{C}[H, H^{-1}, K, K^{-1}]$, $\sigma(H) = q^2H$, $\sigma(K) = q^{-2}K$. Let $\alpha \in \mathbb{C}$ and $M, p, r, s, t, p', r', s', t' \in \mathbb{Z}$ such that $M = m - n = m' - n' = p + t - r - s$, $s - t = s' - t'$ and $p - r = p' - r'$. Set $\mathsf{h} = \alpha(K^m H^n - K^{-m'} H^{-n'})$, $\xi = 1$, $r_+ = K^p H^r$, $l_+ = K^s H^t$, $r_- = K^{-s'} H^{-t'}$, $l_- = K^{-p'} H^{-r'}$. Then A is the Hopf algebra described in [10], Theorem 3.3.

4.4

Let $R = \mathbb{C}[H, K, K^{-1}]$, $\sigma(H) = H - 2$, $\sigma(K) = q^{-2}K$, $\mathsf{h} = \frac{K-K^{-1}}{q-q^{-1}}$, $\xi = 1$ with H primitive and K grouplike. Let $r_- = l_+ = 1$, $r_+ = K$, $l_- = K^{-1}$. Then (3.4)-(3.7) hold and $A(R, \sigma, \mathsf{h}, \xi)$ is equipped with a Hopf algebra structure. The relevance of this example is explained in Remark 6.11

4.5 Down-up algebras

The down-up algebra $A(\alpha, \beta, \gamma)$ where $\alpha, \beta, \gamma \in \mathbb{C}$, was defined in [2] and studied by many authors, see for example [3], [5], [7], [8], and references therein. It is the algebra generated by u, d and relations

$$ddu = \alpha dud + \beta udd + \gamma d,$$
$$duu = \alpha udu + \beta uud + \gamma u.$$

In [7] it is proved that if σ is allowed to be any endomorphism, not necessarily invertible, then any down-up algebra is an ambiskew polynomial ring. Here we consider the down-up algebra $B = A(0, 1, 1)$. Thus B is the $\mathbb{C}$-algebra with generators u, d and relations

$$d^2u = ud^2 + d, \quad du^2 = u^2d + u. \tag{4.1}$$

Let $R = \mathbb{C}[\mathsf{h}]$, $\sigma(\mathsf{h}) = \mathsf{h} + 1$ and $\xi = -1$. Then B is isomorphic to the ambiskew polynomial ring $A(R, \sigma, \mathsf{h}, \xi)$ via $d \mapsto X_+$ and $u \mapsto X_-$.

One can show that B is isomorphic to the enveloping algebra of the Lie super algebra $\mathfrak{osp}(1,2)$ and hence has a *graded* Hopf structure. A question was raised in [8] whether there exists a Hopf structure on B. We do not answer this question here but we show the existence of a Hopf structure on a larger algebra B_q giving us a formula for the tensor product of weight (in particular finite-dimensional) modules over B.

Let $q \in \mathbb{C}^*$ and fix a value of $\log q$. By q^a we always mean $e^{a \log q}$. Let B_q be the ambiskew polynomial ring $B_q = A(R, \sigma, \mathsf{h}, \xi)$ where $R = \mathbb{C}[\mathsf{h}, w, w^{-1}]$, $\sigma(\mathsf{h}) = \mathsf{h}+1$, $\sigma(w) = qw$, and $\xi = -1$.

Theorem 4.1. *For any $\rho, \lambda \in \mathbb{Z}$ such that $q^{\rho-\lambda} = -1$ and $q^{2\rho} = 1$, the algebra B_q has a Hopf algebra structure given by*

$$\Delta(X_\pm) = X_\pm \otimes w^{\pm\rho} + w^{\pm\lambda} \otimes X_\pm, \quad \varepsilon(X_\pm) = 0,$$
$$S(X_\pm) = -w^{\mp\lambda} X_\pm w^{\mp\rho} = -q^\rho w^{\mp(\rho+\lambda)} X_\pm,$$

and

$$\Delta(w) = w \otimes w, \quad \varepsilon(w) = 1, \quad S(w) = w^{-1},$$
$$\Delta(\mathsf{h}) = \mathsf{h} \otimes 1 + 1 \otimes \mathsf{h}, \quad \varepsilon(\mathsf{h}) = 0, \quad S(\mathsf{h}) = -\mathsf{h}.$$

Proof. The subalgebra $\mathbb{C}[\mathsf{h}, w, w^{-1}]$ of B_q has a Hopf structure given by the maps above. We must verify (3.4)-(3.7) with $\xi = -1$, $r_\pm = w^{\pm\rho}$, $l_\pm = w^{\pm\lambda}$, and $s_\pm = -q^\rho w^{\mp(\rho+\lambda)}$. This is straightforward. □

This gives us a tensor structure on the category of modules over B_q. Next aim is to show how using the Hopf structure on B_q one can define a tensor structure on the category of weight modules over B.

In general, if C is a commutative subalgebra of an algebra A, we say that an A-module V is a *weight module* with respect to C if

$$V = \oplus_{\mathfrak{m} \in \mathrm{Max}(C)} V_\mathfrak{m}, \qquad V_\mathfrak{m} = \{v \in V \mid \mathfrak{m} v = 0\},$$

where $\mathrm{Max}(C)$ denotes the set of all maximal ideals of C. When C is finitely generated this is equivalent to V having a basis in which each $c \in C$ acts diagonally.

By weight modules over B (B_q) we mean weight modules with respect to the subalgebra $\mathbb{C}[\mathsf{h}]$ ($\mathbb{C}[\mathsf{h}, w, w^{-1}]$). We need a simple lemma.

Lemma 4.2. *Any finite-dimensional module V over B is a weight module.*

Proof. By Proposition 5.3 in [7], any finite-dimensional B-module is semisimple. Since direct sums of weight modules are weight modules we can assume that V is simple. Since V is finite-dimensional, the commutative subalgebra $\mathbb{C}[\mathsf{h}]$ has a common eigenvector $v \neq 0$, i.e. $\mathfrak{m} v = 0$ for some maximal ideal $\mathfrak{m}$ of $\mathbb{C}[\mathsf{h}]$. Acting on this weight vector by $X_\pm$ produces another weight vector: $\sigma^{\pm 1}(\mathfrak{m}) X_\pm v = X_\pm \mathfrak{m} v = 0$. Since B is generated by $\mathbb{C}[\mathsf{h}]$ and $X_\pm$, any vector in the B-submodule of V generated by v is a sum of weight vectors. But V was simple so $V = \oplus_\mathfrak{m} V_\mathfrak{m}$. □

Let $\mathscr{W}(B)$ denote the category of weight B-modules and similarly for B_q.

Theorem 4.3. *The category of weight modules over B can be embedded into the category of weight modules over B_q, i.e. there exist functors*

$$\mathscr{W}(B) \xrightarrow{\mathscr{E}} \mathscr{W}(B_q) \xrightarrow{\mathscr{R}} \mathscr{W}(B)$$

whose composition is the identity functor. In particular, the category of finite-dimensional B-modules can be embedded in $\mathscr{W}(B_q)$.

Proof. $\mathscr{R}$ is given by restriction. It takes weight modules to weight modules. Next we define $\mathscr{E}$. Let V be a weight module over B and define

$$wv = q^\alpha v \qquad \text{for } v \in V_{(\mathsf{h}-\alpha)} \text{ and } \alpha \in \mathbb{C}. \tag{4.2}$$

It is immediate that w commutes with h. Let $v \in V_{(\mathsf{h}-\alpha)}$ be arbitrary. Then

$$X_+ wv = X_+ q^\alpha v = q^\alpha X_+ v.$$

On the other hand, since $\mathsf{h}X_+v = X_+(\mathsf{h}-1)v = (\alpha-1)X_+v$ which shows that $X_+v \in V_{(\mathsf{h}-(\alpha-1))}$, we have

$$qwX_+v = qq^{\alpha-1}X_+v = q^\alpha X_+ v.$$

Thus $X_+w = qwX_+$. Similarly $X_-w = q^{-1}wX_-$ on V. Thus V becomes a module over B_q. That V is a weight module with respect to $\mathbb{C}[\mathsf{h}, w, w^{-1}]$ is clear. We define $\mathscr{E}(V)$ to be the same space V with additional action (4.2). If $\varphi : V \to W$ is a morphism of weight B-modules then $\varphi(wv) = w\varphi(v)$ for weight vectors v, since $\varphi(V_\mathfrak{m}) \subseteq W_\mathfrak{m}$ for any maximal ideal $\mathfrak{m}$ of $\mathbb{C}[\mathsf{h}]$. But then $\varphi(wv) = w\varphi(v)$ for all $v \in V$ since V is a weight module. Thus φ is automatically a morphism of B_q-modules and we set $\mathscr{E}(\varphi) = \varphi$. It is clear that the composition of the functors is the identity on objects and morphisms. □

Note that

$$\mathscr{E}\big(\mathscr{W}(B)\big) = \big\{V \in \mathscr{W}(B_q) : \operatorname{Supp}(V) \subseteq \{\mathfrak{m} = (\mathsf{h}-\alpha, w - q^\alpha) : \alpha \in \mathbb{C}\}\big\}. \tag{4.3}$$

It is not difficult to see that

$$\mathscr{E}(V_1) \otimes \mathscr{E}(V_2) \in \mathscr{E}\big(\mathscr{W}(B)\big)$$

and hence there is a unique $V_3 \in \mathscr{W}(B)$ such that

$$\mathscr{E}(V_1) \otimes \mathscr{E}(V_2) = \mathscr{E}(V_3).$$

Thus we can define

$$V_1 \otimes V_2 := V_3$$

and this will make $\mathscr{W}(B)$ into a tensor category.

Remark 4.4. Our result that $\mathscr{W}(B)$ is a tensor category shows that to disprove that B has a Hopf structure one cannot only use pure representation theory (of weight modules).

4.6 Non Hopf ambiskew polynomial rings

There are many examples of ambiskew polynomial rings which do not have any Hopf structure. One example is the Weyl algebra $W = \langle a, b | ab - ba = 1\rangle$ which can have no counit ε. Indeed, a counit is in particular a homomorphism $\varepsilon : W \to \mathbb{C}$ so we would have $1 = \varepsilon(1) = \varepsilon(a)\varepsilon(b) - \varepsilon(b)\varepsilon(a) = 0$. Moreover all down-up algebras are ambiskew polynomial rings (see [7]) and [8] contains necessary conditions for the existence of a Hopf structure on a down-up algebra in terms of the parameters α, β, γ. More precisely, they show that if $A = A(\alpha, \beta, \gamma)$ is a Noetherian down-up algebra that is a Hopf algebra, then $\alpha + \beta = 1$. Moreover if $\gamma = 0$, then $(\alpha, \beta) = (2, -1)$ and as algebras, A is isomorphic to the universal enveloping algebra of the three-dimensional Heisenberg Lie algebra, while if $\gamma \neq 0$, then $-\beta$ is not an nth root of unity for $n \geq 3$. It would be of interest to generalize such a result to a more general class of ambiskew polynomial rings and also to other GW-algebras.

5 R as functions on a group

From now on we assume that $A = A(R, \sigma, \mathsf{h}, \xi)$ is an algebra of the form defined in Section 3 and that conditions (3.4)-(3.7) hold so that A becomes a Hopf algebra with R as a Hopf subalgebra. Let G denote the set of all maximal ideals in R. Since $\mathbb{K}$ is algebraically closed and R is finitely generated, the inclusion map $i_{\mathfrak{m}} : \mathbb{K} \to R/\mathfrak{m}$ is onto for any $\mathfrak{m} \in G$ and we let $\varphi_{\mathfrak{m}} : R \to \mathbb{K}$ denote the composition of the projection $R \to R/\mathfrak{m}$ and $i_{\mathfrak{m}}^{-1}$. Thus $\varphi_{\mathfrak{m}}(a)$ is the unique element of $\mathbb{K}$ such that $a - \varphi_{\mathfrak{m}}(a) \in \mathfrak{m}$. We define the *weight sum* of $\mathfrak{m}, \mathfrak{n} \in G$ to be

$$\mathfrak{m} + \mathfrak{n} := \ker(m \circ (\varphi_{\mathfrak{m}} \otimes \varphi_{\mathfrak{n}}) \circ \Delta|_R).$$

This is the kernel of a $\mathbb{K}$-algebra homomorphisms $R \to \mathbb{K}$, hence $\mathfrak{m} + \mathfrak{n} \in G$. We will never use the usual addition of ideals so $+$ should not cause any confusion. Using that Δ is coassociative, ε is a counit and S is an antipode, one easily deduces that $+$ is associative, that $\underline{0} := \ker \varepsilon$ is a unit element and $S(\mathfrak{m})$ is the inverse of $\mathfrak{m}$. Thus G is a group under $+$. If R is cocommutative, G is abelian.

Example 5.1. Let $R = \mathbb{C}[H]$. Then $G = \{(H - \alpha) | \alpha \in \mathbb{C}\}$. Give R the Hopf structure $\Delta(H) = H \otimes 1 + 1 \otimes H$, $\varepsilon(H) = 0$ and $S(H) = -H$. Then the operation $+$ will be

$$(H - \alpha) + (H - \beta) = (H - (\alpha + \beta)),$$

i.e. the correspondence $\mathbb{C} \ni \alpha \mapsto (H - \alpha) \in G$ is an additive group isomorphism.

If $R = \mathbb{C}[K, K^{-1}]$ then $G = \{(K - \alpha) | \alpha \in \mathbb{C}^*\}$. With the Hopf structure $\Delta(K) = K \otimes K$, $\varepsilon(K) = 1$ and $S(K) = K^{-1}$, the operation $+$ will be

$$(K - \alpha) + (K - \beta) = (K - \alpha\beta)$$

for $\alpha, \beta \neq 0$. Thus $G \simeq \langle \mathbb{C}^*, \cdot \rangle$.

We will often think of elements from R as $\mathbb{K}$-valued functions on G and for $x \in R$ and $\mathfrak{m} \in G$ we will use the notation $x(\mathfrak{m})$ for $\varphi_{\mathfrak{m}}(x)$. Note however that different elements $x, y \in R$ can represent the same function. In fact one can check that the map from R to functions on G is a homomorphism of $\mathbb{K}$-algebras with kernel equal to the radical $\mathrm{Rad}(R) := \cap_{\mathfrak{m}\in G}\mathfrak{m}$.

Define a map

$$\zeta : \mathbb{Z} \to G,\ n \mapsto \underline{n} := \sigma^n(\underline{0}). \tag{5.1}$$

Lemma 5.2. *Let $\mathfrak{m}, \mathfrak{n} \in G$. Then for any $a \in R$,*

$$\sigma(a)(\mathfrak{m}) = a\big(\sigma^{-1}(\mathfrak{m})\big), \tag{5.2}$$

$$a(\mathfrak{m}+\mathfrak{n}) = m \circ (\varphi_{\mathfrak{m}} \otimes \varphi_{\mathfrak{n}}) \circ \Delta(a) = \sum_{(a)} a'(\mathfrak{m})a''(\mathfrak{n}), \tag{5.3}$$

$$\mathfrak{m} + \underline{1} = \sigma(\mathfrak{m}) = \underline{1} + \mathfrak{m}. \tag{5.4}$$

Thus ζ is a group homomorphism and its image is contained in the center of G.

Proof. Since for any $a \in R$ we have

$$\sigma(a)(\mathfrak{m}) - a = \sigma^{-1}\big(\sigma(a)(\mathfrak{m}) - \sigma(a)\big) \in \sigma^{-1}(\mathfrak{m}),$$

(5.2) holds. Similarly,

$$a(\mathfrak{m}+\mathfrak{n}) - a \in \mathfrak{m}+\mathfrak{n}$$

so applying the map $m \circ (\varphi_{\mathfrak{m}} \otimes \varphi_{\mathfrak{n}}) \circ \Delta$ to $a(\mathfrak{m}+\mathfrak{n}) - a$ yields zero. This gives (5.3). Finally we have for any $a \in \mathfrak{m}$,

$$\begin{aligned}\sigma(a)(\mathfrak{m}+\underline{1}) = m \circ (\varphi_{\mathfrak{m}} \otimes \varphi_{\underline{1}}) \circ \Delta(\sigma(a)) = m \circ (\varphi_{\mathfrak{m}} \otimes \varphi_{\underline{1}}) \circ (1 \otimes \sigma)\Delta(a) = \\ = m \circ (\varphi_{\mathfrak{m}} \otimes \varphi_{\underline{0}}) \circ \Delta(a) = a(\mathfrak{m}+\underline{0}) = a(\mathfrak{m}) = 0.\end{aligned}$$

Here we used (5.3) in the first and the fourth equality, (3.4a) in the second and (5.2) in the third. Thus $\sigma(\mathfrak{m}) \subseteq \mathfrak{m} + \underline{1}$ and then equality holds since both sides are maximal ideals. The proof of the other equality in (5.4) is symmetric. □

Example 5.3. If $R = \mathbb{C}[K, K^{-1}]$ with $\Delta(K) = K \otimes K, \varepsilon(K) = 1, S(K) = K^{-1}$ and $\sigma(K) = q^{-2}K$, then $\ker \varepsilon = (K-1)$ so

$$\underline{n} = \sigma^n(\underline{0}) = \sigma^n((K-1)) = (q^{-2n}K - 1) = (K - q^{2n}).$$

From (5.3) follows that if $x \in R$ is grouplike, then viewed as a function $G \to \mathbb{K}$ it is a multiplicative homomorphism. Using (5.3) and (3.5a)-(3.5c), the following formulas are satisfied by h as a function on G.

$$\begin{aligned}\mathsf{h}(\mathfrak{m}+\mathfrak{n}) &= \mathsf{h}(\mathfrak{m})r(\mathfrak{n}) + l(\mathfrak{m})\mathsf{h}(\mathfrak{n}),\\ \mathsf{h}(\underline{0}) &= 0,\\ \mathsf{h}(-\mathfrak{m}) &= -r^{-1}l^{-1}\mathsf{h}(\mathfrak{m}),\end{aligned} \tag{5.5}$$

where $r = r_+r_-$ and $l = l_+l_-$.

6 Finite-dimensional simple modules

In this section we consider finite-dimensional simple modules over the algebra A. The main theorem is Theorem 6.19 where we, under the torsion-free assumption (6.1), characterize the finite-dimensional simple modules of a given dimension in terms of their highest weights. This result will be used in Section 7 to prove a Clebsch-Gordan decomposition theorem.

Throughout the rest of the paper we will assume that

$$\sigma^n(\mathfrak{m}) \neq \mathfrak{m} \text{ for any } n \in \mathbb{Z}\backslash\{0\} \text{ and any } \mathfrak{m} \in G. \tag{6.1}$$

By (5.4), this condition holds iff $\underline{1}$ has infinite order in G.

Remark 6.1. Condition (6.1) does not hold for $U(\mathfrak{h}_3)$. For $U(\mathfrak{sl}_2)$, the algebra B_q in Section 4.5 and the algebra in Section 4.4, condition (6.1) always holds. For the other examples in Section 4, (6.1) holds iff q is not a root of unity.

6.1 Weight modules, Verma modules and their finite-dimensional simple quotients

In this section we define weight modules, Verma modules and derive an equation for the dimensions of the finite-dimensional simple quotients of Verma modules.

Let V be an A-module. We call $\mathfrak{m} \in G$ a *weight* of V if $\mathfrak{m}v = 0$ for some nonzero $v \in V$. The *support* of V, denoted $\mathrm{Supp}(V)$, is the set of weights of V. To a weight $\mathfrak{m}$ we associate its *weight space*

$$V_\mathfrak{m} = \{v \in V : \mathfrak{m}v = 0\}.$$

Elements of $V_\mathfrak{m}$ are called *weight vectors of weight* $\mathfrak{m}$. A module V is a *weight module* if $V = \oplus_\mathfrak{m} V_\mathfrak{m}$. A *highest weight vector* $v \in V$ of weight $\mathfrak{m}$ is a weight vector of weight $\mathfrak{m}$ such that $X_+ v = 0$. A module V is called a *highest weight module* if it is generated by a highest weight vector. From the defining relations of A it follows that

$$X_\pm V_\mathfrak{m} \subseteq V_{\sigma^{\pm 1}(\mathfrak{m})}. \tag{6.2}$$

Equation (6.2) implies that a highest weight module is a weight module.

Let $\mathfrak{m} \in G$. The *Verma module* $M(\mathfrak{m})$ is defined as the left A-module $A/I(\mathfrak{m})$ where $I(\mathfrak{m})$ is the left ideal $AX_+ + A\mathfrak{m} \subseteq A$. From relations (2.5),(2.6) follows that

$$\{v_n := X_-^n + I(\mathfrak{m}) : n \geq 0\}$$

is a basis for $M(\mathfrak{m})$. It is clear that $M(\mathfrak{m})$ is a highest weight module generated by v_0. We also see that the vectors v_n ($n \geq 0$) are weight vectors of weights $\sigma^n(\mathfrak{m})$ respectively. By (6.1) we conclude $\dim M(\mathfrak{m})_\mathfrak{m} = 1$. Therefore the sum of all its proper submodules is proper and equals the unique maximal submodule $N(\mathfrak{m})$ of $M(\mathfrak{m})$. Thus $M(\mathfrak{m})$ has a unique simple quotient $L(\mathfrak{m})$. Since it is easy to see that any highest weight module over A of highest weight $\mathfrak{m}$ is a quotient of $M(\mathfrak{m})$ we deduce that $L(\mathfrak{m})$ is the unique irreducible highest weight module over A with given highest weight $\mathfrak{m} \in G$. We set

$$G_f := \{\mathfrak{m} \in G \mid \dim L(\mathfrak{m}) < \infty\}.$$

Proposition 6.2. *Any finite-dimensional simple module over A is isomorphic to* $L(\mathfrak{m})$ *for some* $\mathfrak{m} \in G_f$.

Proof. Let V be a finite-dimensional simple A-module. Since $\mathbb{K}$ is algebraically closed, R has a common eigenvector $v \neq 0$, i.e. there exists $\mathfrak{n} \in G$ such that $\mathfrak{n}v = 0$. From (2.5) it follows that $\sigma^n(\mathfrak{n})(X_+)^n v = 0$ for any $n \geq 0$. By (6.1), the set $\{X_+^n v : n \geq 0\}$ is a set of weight vectors of different weights. Since V is finite-dimensional it follows that $(X_+)^n v = 0$ for some $n > 0$. This proves the existence of a highest weight vector of weight $\mathfrak{m}$ in V for some weight $\mathfrak{m}$. Thus $V = L(\mathfrak{m})$. ☐

Corollary 6.3. *Let V be a finite-dimensional weight module over A. Then* $\mathrm{Supp}(V) \subseteq G_f + \underline{\mathbb{Z}} = \{\mathfrak{m} + \underline{n} : \mathfrak{m} \in G_f, n \in \mathbb{Z}\}$.

Proof. Let $\mathfrak{m} \in \mathrm{Supp}(V)$ and let $0 \neq v \in V_{\mathfrak{m}}$. Then $(X_+)^n v = 0$ for some smallest $n > 0$. But then $(X_+)^{n-1} v$ is a highest weight vector so its weight $\sigma^{n-1}(\mathfrak{m}) = \mathfrak{m} + \underline{n-1}$ must belong to G_f. Thus $\mathfrak{m} = \mathfrak{m} + \underline{n-1} - \underline{n-1} \in G_f + \underline{\mathbb{Z}}$. ☐

The following lemma was essentially proved in [6], Proposition 2.3, and the general result was mentioned in [7]. We give a proof for completeness.

Proposition 6.4. *The dimension of* $L(\mathfrak{m})$ *is the smallest positive integer n such that*

$$\sum_{k=0}^{n-1} \xi^{n-1-k} \mathsf{h}(\mathfrak{m} - \underline{k}) = 0.$$

Proof. Let $e^{\mathfrak{m}}$ be a highest weight vector in $L(\mathfrak{m})$. Let $n > 0$ be the smallest positive integer such that $X_-^n e^{\mathfrak{m}} = 0$. Then the set spanned by the vectors $X_-^j e^{\mathfrak{m}}$, $0 \leq j < n$, is invariant under X_-, under R using (2.5), and under X_+, using (2.6). Hence it is a nonzero submodule and so coincides with $L(\mathfrak{m})$ since the latter is simple. Therefore $n = \dim L(\mathfrak{m})$. Let $k > 0$. Then $X_-^k e^{\mathfrak{m}} = 0$ implies that $X_+^k X_-^k e^{\mathfrak{m}} = 0$. Conversely, suppose $X_+^k X_-^k e^{\mathfrak{m}} = 0$. Then $X_+^{k-1} X_-^k e^{\mathfrak{m}}$ generates a proper submodule and thus is zero. Repeating this argument we obtain $X_-^k e^{\mathfrak{m}} = 0$. Hence $\dim L(\mathfrak{m})$ is the smallest positive integer n such that $X_+^n X_-^n e^{\mathfrak{m}} = 0$. Using induction it is easy to deduce the formulas

$$X_+ X_-^n = X_-^{n-1}\Big(\xi^n X_- X_+ + \sum_{k=0}^{n-1} \xi^{n-1-k} \sigma^k(\mathsf{h})\Big),$$

$$X_+^n X_-^n = \prod_{m=1}^{n} \Big(\xi^m X_- X_+ + \sum_{k=0}^{m-1} \xi^{m-1-k} \sigma^k(\mathsf{h})\Big). \tag{6.3}$$

Applying both sides of this equality to the vector $e^{\mathfrak{m}}$ gives

$$X_+^n X_-^n e^{\mathfrak{m}} = \prod_{m=1}^{n} \sum_{k=0}^{m-1} \xi^{m-1-k} \sigma^k(h) e^{\mathfrak{m}}. \tag{6.4}$$

Using that $e^{\mathfrak{m}}$ is a weight vector of weight $\mathfrak{m}$ and formula (5.2) we have

$$\sigma^k(\mathsf{h}) e^{\mathfrak{m}} = \sigma^k(\mathsf{h})(\mathfrak{m}) e^{\mathfrak{m}} = \mathsf{h}(\mathfrak{m} - \underline{k}) e^{\mathfrak{m}}.$$

Substituting this into (6.4) we obtain

$$X_+^n X_-^n e^{\mathfrak{m}} = \prod_{m=1}^{n} \sum_{k=0}^{m-1} \xi^{m-1-k} \mathsf{h}(\mathfrak{m} - \underline{k}) e^{\mathfrak{m}}.$$

The smallest positive n such that this is zero must be the one such that the last factor is zero. The claim is proved. □

Corollary 6.5. *If* $\mathfrak{m}, \mathfrak{m}_0 \in G$ *where* $\mathsf{h}(\mathfrak{m}_0) = 0$, *then*

$$\dim L(\mathfrak{m}_0 + \mathfrak{m}) = \dim L(\mathfrak{m}) = \dim L(\mathfrak{m} + \mathfrak{m}_0).$$

Proof. Note that (5.5) implies that $\mathsf{h}(\mathfrak{n}+\mathfrak{m}_0) = \mathsf{h}(\mathfrak{n}) r(\mathfrak{m}_0)$ and $\mathsf{h}(\mathfrak{m}_0+\mathfrak{n}) = l(\mathfrak{m}_0)\mathsf{h}(\mathfrak{n})$ for any $\mathfrak{n} \in G$, recall that r and l are invertible and use Proposition 6.4. □

6.2 Dimension and highest weights

The goal in this subsection is to prove Theorem 6.19 which describes in detail the relationship between the dimension of a finite-dimensional simple module and its highest weight.

We begin with a few useful lemmas. Recall that $r = r_+ r_-$ and $l = l_+ l_-$. For brevity we set $r_1 = r(\underline{1})$ and $l_1 = l(\underline{1})$. Since $r_\pm, l_\pm$ are grouplike so are r and l and thus r_1, l_1 are nonzero scalars.

Lemma 6.6. *We have*
a) $\xi^2 r_1 l_1 = 1$,
b) $\mathsf{h}(-\underline{k}) = -r_1^{-k} l_1^{-k} \mathsf{h}(\underline{k})$ *for any* $k \in \mathbb{Z}$,
c) for any $k \in \mathbb{Z}$ *and* $\mathfrak{m} \in G$ *we have*

$$\xi^k \mathsf{h}(\mathfrak{m} + \underline{k}) + \xi^{-k} \mathsf{h}(\mathfrak{m} - \underline{k}) = \big((\xi r_1)^k + (\xi r_1)^{-k}\big) \mathsf{h}(\mathfrak{m}). \tag{6.5}$$

Proof. For a), multiply the two equations in (3.6b) and apply the multiplication map to both sides to obtain

$$\sigma(l_+ l_- r_+ r_-) = \xi^2 l_+ l_- r_+ r_-.$$

Evaluate both sides at $\underline{1}$ to get

$$1 = lr(\underline{0}) = lr(\sigma^{-1}(\underline{1})) = \sigma(lr)(\underline{1}) = \xi^2 lr(\underline{1}) = \xi^2 l_1 r_1.$$

Next (5.5) gives for any $k \in \mathbb{Z}$,

$$0 = \mathsf{h}(\underline{k} - \underline{k}) = \mathsf{h}(\underline{k}) r_1^{-k} + l_1^k \mathsf{h}(-\underline{k}),$$

hence b) follows. Finally, using (5.5) again, we have

$$\begin{aligned} \xi^k \mathsf{h}(\mathfrak{m} + \underline{k}) + \xi^{-k} \mathsf{h}(\mathfrak{m} - \underline{k}) &= \xi^k \mathsf{h}(\mathfrak{m}) r_1^k + \xi^k l(\mathfrak{m}) \mathsf{h}(\underline{k}) + \\ &\quad + \xi^{-k} \mathsf{h}(\mathfrak{m}) r_1^{-k} + \xi^{-k} l(\mathfrak{m}) \mathsf{h}(-\underline{k}) = \\ &= \mathsf{h}(\mathfrak{m}) \big((\xi r_1)^k + (\xi r_1)^{-k} \big) + \\ &\quad + l(\mathfrak{m}) \mathsf{h}(\underline{k}) (\xi^k - \xi^{-k} r_1^{-k} l_1^{-k}). \end{aligned}$$

In the last equality we used part b). Now the second term in the last expression vanishes due to part a). Thus c) follows. □

In what follows, we will treat the two cases when $\mathsf{h}(\underline{1}) = 0$ and $\mathsf{h}(\underline{1}) \neq 0$ separately. The algebras satisfying the former condition have a representation theory which reminds one of that of the enveloping algebra $U(\mathfrak{h}_3)$ of the three-dimensional Heisenberg Lie algebra, while the latter case includes $U(\mathfrak{sl}_2)$ and other algebras with similar structure of representations.

6.2.1 The case $\mathsf{h}(\underline{1}) = 0$.

If $\mathsf{h} = 0$, then, by Proposition 6.4, any finite-dimensional simple module is one-dimensional. If $\mathsf{h} \neq 0$ we have the following result.

Proposition 6.7. *If* $\mathsf{h} \neq 0$ *and* $\mathsf{h}(\underline{1}) = 0$*, then*

$$\xi^2 = r_1^2 = 1, \ \sigma(\mathsf{h}) = r_1 \mathsf{h}, \ \sigma(r) = r_1 r, \ \text{and} \ \sigma(l) = r_1 l. \tag{6.6}$$

In particular, $\langle X_+, X_-, \mathsf{h} \rangle$ *is a subalgebra of* A *with relations*

$$\begin{array}{lll} [X_+, X_-] = h, & [h, X_\pm] = 0, & \text{if } \xi = 1, \ r_1 = 1, \\ [X_+, X_-] = h, & \{h, X_\pm\} = 0, & \text{if } \xi = 1, \ r_1 = -1, \\ \{X_+, X_-\} = h, & [h, X_\pm] = 0, & \text{if } \xi = -1, \ r_1 = 1, \\ \{X_+, X_-\} = h, & \{h, X_\pm\} = 0, & \text{if } \xi = -1, \ r_1 = -1, \end{array}$$

respectively, where $\{\cdot, \cdot\}$ *denotes anti-commutator.*

Proof. Suppose $\mathsf{h}(\underline{1}) = 0$. Then, by Lemma 6.6b), $\mathsf{h}(-\underline{1}) = 0$. This means that $\mathsf{h} \in \underline{-1} = \sigma^{-1}(\underline{0}) = \sigma^{-1}(\ker \varepsilon)$. Thus $\varepsilon(\sigma(\mathsf{h})) = 0$. Using (2.2), (3.4a) and (3.5a) we deduce

$$\begin{aligned} \sigma(\mathsf{h}) &= m(\varepsilon \otimes 1)(\Delta(\sigma(\mathsf{h}))) = m(\varepsilon \otimes 1)(\sigma \otimes 1)(\Delta(\mathsf{h})) = \\ &= m\Big(\varepsilon(\sigma(\mathsf{h})) \otimes r + \varepsilon(\sigma(l)) \otimes \mathsf{h} \Big) = \varepsilon(\sigma(l)) \mathsf{h}. \end{aligned}$$

Analogously one proves $\sigma(\mathsf{h}) = \varepsilon(\sigma(r))\mathsf{h}$. Hence $\varepsilon(\sigma(r)) = \varepsilon(\sigma(l))$. But

$$\varepsilon(\sigma(r)) = \sigma(r)(\ker \varepsilon) = \sigma(r)(\underline{0}) = r(-\underline{1}) = r_1^{-1}$$

and similarly for l. So $r_1 = l_1$. From Lemma 6.6a) we obtain $(\xi r_1)^2 = 1$. Now

$$S(\sigma(\mathsf{h})) = S(r_1^{-1}\mathsf{h}) = -r_1^{-1}\mathsf{h}, \quad \text{and} \quad \sigma^{-1}(S(\mathsf{h})) = \sigma^{-1}(-h) = -r_1\mathsf{h},$$

so (3.4b) implies that $r_1^2 = 1$. A similar calculation as above shows that $\sigma(r) = r_1^{-1}r = r_1 r$ and $\sigma(l) = l_1^{-1}l = r_1 l$. $\square$

We leave it to the reader to prove the following statement.

Proposition 6.8. *All finite-dimensional simple modules over an algebra $A(R,\sigma,\mathsf{h},\xi)$ satisfying* (6.6) *and one of the commutation relations above are either one- or two-dimensional.*

Remark 6.9. The algebra $U(\mathfrak{h}_3)$ is an ambiskew polynomial ring, as shown in Section 4.1. For this algebra we have $\mathsf{h}(\underline{1}) = 0$ and $\xi = r_1 = 1$.

6.2.2 The case $\mathsf{h}(\underline{1}) \neq 0$.

In this section, we consider the more complicated case when $\mathsf{h}(\underline{1}) \neq 0$. We prove Theorem 6.19 which describes the dimensions of $L(\mathfrak{m})$ in terms of $\mathfrak{m}$. The following two subsets of G will play a vital role:

$$G_0 = \{\mathfrak{m} \in G \mid \mathsf{h}(\mathfrak{m}) = 0\}, \tag{6.7}$$
$$G_{1/2} = \{\mathfrak{m} \in G \mid \mathsf{h}(\mathfrak{m} - \underline{1}) + \xi\mathsf{h}(\mathfrak{m}) = 0\}. \tag{6.8}$$

The reason for this notation is that when $A = U(\mathfrak{sl}_2)$ as in Section 4.2.1 then we have $G_0 = \{(H-0)\}$ and $G_{1/2} = \{(H - \frac{1}{2})\}$. From (5.5) it is immediate that G_0 is a subgroup of G. By Proposition 6.4 we have

$$G_0 = \{\mathfrak{m} \in G \mid \dim L(\mathfrak{m}) = 1\}. \tag{6.9}$$

The following analogous result holds for $G_{1/2}$.

Proposition 6.10.

$$G_{1/2} = \{\mathfrak{m} \in G \mid \dim L(\mathfrak{m}) = 2\}. \tag{6.10}$$

Proof. If $\mathfrak{m} \in G_{1/2}$, then by Proposition 6.4, $\dim L(\mathfrak{m}) \leq 2$. But if $\dim L(\mathfrak{m}) = 1$, then $\mathsf{h}(\mathfrak{m}) = 0$ so using $\mathfrak{m} \in G_{1/2}$ we get $\mathsf{h}(\mathfrak{m} - \underline{1}) = 0$ also. Since G_0 is a group we deduce that $\underline{1} \in G_0$, i.e. $\mathsf{h}(\underline{1}) = 0$ which is a contradiction. So $\dim L(\mathfrak{m}) = 2$. The converse inclusion is immediate from Proposition 6.4. $\square$

Set

$$N = \begin{cases} \text{order of } \xi r_1 & \text{if } (\xi r_1)^2 \neq 1 \text{ and } \xi r_1 \text{ is a root of unity,} \\ \infty & \text{otherwise.} \end{cases} \tag{6.11}$$

Remark 6.11. The algebra $A(R,\sigma,\mathsf{h},\xi)$ from Section 4.4 satisfies condition (6.1) while $N < \infty$ iff q is a root of unity. In all the other examples from Section 4 where (6.1) holds, we also have $N = \infty$.

We also set

$$N' = \begin{cases} N, & \text{if } N \text{ is odd,} \\ N/2, & \text{if } N \text{ is even,} \\ \infty, & \text{if } N = \infty. \end{cases}$$

The next statement describes the intersection of G_0 and $G_{1/2}$ with $\underline{\mathbb{Z}}$.

Proposition 6.12. *We have*

$$G_0 \cap \underline{\mathbb{Z}} = \begin{cases} \{\underline{0}\}, & \text{if } N = \infty, \\ \underline{N'\mathbb{Z}}, & \text{otherwise,} \end{cases} \tag{6.12}$$

and

$$G_{1/2} \cap \underline{\mathbb{Z}_{>0}} = \begin{cases} \emptyset, & \text{if } N = \infty, \\ \{\underline{n} \in \underline{\mathbb{Z}_{>0}} : N \big| 2n-1\}, & \text{otherwise.} \end{cases} \tag{6.13}$$

Remark 6.13. The set $G_{1/2} \cap \underline{\mathbb{Z}_{\leq 0}}$ can be understood using (6.13) and Lemma 6.16a).

Proof. We first prove (6.12). Let $n \in \mathbb{Z}$. The right hand side of (6.12) is invariant under $n \mapsto -n$. By Lemma 6.6b) so is the left hand side. Moreover since $\mathsf{h}(\underline{0}) = 0$, the ideal $\underline{0}$ belongs to both sides of the equality. Thus we can assume $n > 0$.

Using (5.5) and that r and l, viewed as functions $G \to \mathbb{K}$, are multiplicative homomorphisms it follows by induction that

$$\mathsf{h}(\underline{n}) = \mathsf{h}(\underline{1}) \sum_{i=0}^{n-1} r_1^i l_1^{n-1-i}.$$

By Lemma 6.6a), $r_1/l_1 = (\xi r_1)^2/(\xi^2 r_1 l_1) = (\xi r_1)^2$, so we can rewrite this as

$$\mathsf{h}(\underline{n}) = \mathsf{h}(\underline{1}) l_1^{n-1} \sum_{i=0}^{n-1} (\xi r_1)^{2i}. \tag{6.14}$$

If $N = \infty$ and $(\xi r_1)^2 \neq 1$ then by (6.14) we have $\underline{n} \in G_0 \cap \underline{\mathbb{Z}}$ iff $(\xi r_1)^{2n} = 1$, which is false. If $(\xi r_1)^2 = 1$, then (6.14) implies that $\underline{n} \notin G_0 \cap \underline{\mathbb{Z}}$. If $N < \infty$, then $(\xi r_1)^2 \neq 1$ so by (6.14), $\mathsf{h}(\underline{n}) = 0$ iff $(\xi r_1)^{2n} = 1$ i.e. iff $N \big| 2n$. This is equivalent to $N' \big| n$.

Next we prove (6.13). Suppose $n \in \mathbb{Z}_{>0}$. By definition, $n \in G_{1/2}$ iff

$$\mathsf{h}(\underline{n} - \underline{1}) + \xi \mathsf{h}(\underline{n}) = 0.$$

Using (6.14) on both terms and dividing by $\mathsf{h}(\underline{1}) \xi l_1^{n-1}$, this is equivalent to

$$\xi^{-1} l_1^{-1} \sum_{k=0}^{n-2} (\xi r_1)^{2k} + \sum_{k=0}^{n-1} (\xi r_1)^{2k} = 0.$$

But $\xi^{-1} l_1^{-1} = \xi r_1$ by Lemma 6.6a) so this can be rewritten as

$$\sum_{k=0}^{2n-2} (\xi r_1)^k = 0. \tag{6.15}$$

Thus $(\xi r_1)^2 \neq 1$ and multiplying by $\xi r_1 - 1$ we get $(\xi r_1)^{2n-1} = 1$. Therefore $N < \infty$ and $N \big| 2n-1$. Conversely, if $N < \infty$ and $N \big| 2n-1$ then $(\xi r_1)^2 \neq 1$ and $(\xi r_1)^{2n-1} = 1$ which implies (6.15). This proves (6.13). □

Proposition 6.14. *Suppose* $\mathsf{h}(\underline{1}) \neq 0$ *and* $G_{1/2} \neq \emptyset$. *Then*
a) $\xi r_1 \neq -1$, *and*
b) $G_{1/2}$ *is a left and right coset of* G_0 *in* G.

Proof. Let $\mathfrak{m}_{1/2} \in G_{1/2}$. To prove a), suppose that $\xi r_1 = -1$. Then

$$\begin{aligned}0 &= \mathsf{h}(\mathfrak{m}_{1/2} - \underline{1}) + \xi \mathsf{h}(\mathfrak{m}_{1/2}) = \\ &= \mathsf{h}(\mathfrak{m}_{1/2}) r(-\underline{1}) + l(\mathfrak{m}_{1/2}) \mathsf{h}(-\underline{1}) + \xi \mathsf{h}(\mathfrak{m}_{1/2}) = \\ &= \mathsf{h}(\mathfrak{m}_{1/2})(r_1^{-1} + \xi) + l(\mathfrak{m}_{1/2}) \mathsf{h}(-\underline{1}) = \\ &= -l(\mathfrak{m}_{1/2}) r_1^{-1} l_1^{-1} \mathsf{h}(\underline{1}),\end{aligned}$$

where we used Lemma 6.6b) in the last equality. Since l is invertible we deduce that $\mathsf{h}(\underline{1}) = 0$ which is a contradiction.

To prove part b), we will show that

$$G_{1/2} = G_0 + \mathfrak{m}_{1/2}.$$

One proves $G_{1/2} = \mathfrak{m}_{1/2} + G_0$ in an analagous way. Let $\mathfrak{m} \in G_0$ be arbitrary. Then using (5.5) twice,

$$\mathsf{h}(\mathfrak{m} + \mathfrak{m}_{1/2} - \underline{1}) + \xi \mathsf{h}(\mathfrak{m} + \mathfrak{m}_{1/2}) = l(\mathfrak{m}) \big(\mathsf{h}(\mathfrak{m}_{1/2} - \underline{1}) + \xi \mathsf{h}(\mathfrak{m}_{1/2}) \big) = 0.$$

Since l is invertible we get $\mathfrak{m} + \mathfrak{m}_{1/2} \in G_{1/2}$.

Conversely, suppose $\mathfrak{m} \in G_{1/2}$. Then

$$\begin{aligned}\mathsf{h}(\mathfrak{m} - \underline{1}) + \xi \mathsf{h}(\mathfrak{m}) &= 0, \\ \mathsf{h}(\mathfrak{m}_{1/2} - \underline{1}) + \xi \mathsf{h}(\mathfrak{m}_{1/2}) &= 0.\end{aligned}$$

Multiply the first equation by $r(-\mathfrak{m}_{1/2})$ and the second by $-r(-\mathfrak{m}_{1/2}) l(-\mathfrak{m}_{1/2}) l(\mathfrak{m})$ and add them together. Then we get

$$\begin{aligned}&\big((\mathsf{h}(\mathfrak{m}) r_1^{-1} + l(\mathfrak{m}) \mathsf{h}(-\underline{1}) \big) r(-\mathfrak{m}_{1/2}) - \\ &\qquad r(-\mathfrak{m}_{1/2}) l(-\mathfrak{m}_{1/2}) l(\mathfrak{m}) \big(\mathsf{h}(\mathfrak{m}_{1/2} r_1^{-1} + l(\mathfrak{m}_{1/2}) \mathsf{h}(-\underline{1}) \big) + \xi \mathsf{h}(\mathfrak{m} - \mathfrak{m}_{1/2}) = 0,\end{aligned}$$

or equivalently,

$$\mathsf{h}(\mathfrak{m}) r_1^{-1} r(-\mathfrak{m}_{1/2}) - r(-\mathfrak{m}_{1/2}) l(-\mathfrak{m}_{1/2}) l(\mathfrak{m}) \mathsf{h}(\mathfrak{m}_{1/2}) r_1^{-1} + \xi \mathsf{h}(\mathfrak{m} - \mathfrak{m}_{1/2}) = 0.$$

Using (5.5) this can be written

$$r_1^{-1}(1 + \xi r_1) \mathsf{h}(\mathfrak{m} - \mathfrak{m}_{1/2}) = 0.$$

Since $\xi r_1 \neq -1$ by part a), we conclude that $\mathsf{h}(\mathfrak{m} - \mathfrak{m}_{1/2}) = 0$. This shows that $\mathfrak{m} \in G_0 + \mathfrak{m}_{1/2}$. □

The following lemma will be useful.

Lemma 6.15. *Let $j \in \mathbb{Z}$. If $\mathfrak{m}_0 \in G_0$, then*

$$\mathfrak{m}_0 + \underline{j} \in G_0 \Longleftrightarrow \underline{j} \in G_0, \tag{6.16}$$

and if $\mathsf{h}(\underline{1}) \neq 0$ *and* $\mathfrak{m}_{1/2} \in G_{1/2}$, *then*

$$\mathfrak{m}_{1/2} + \underline{j} \in G_{1/2} \Longleftrightarrow \underline{j} \in G_0. \tag{6.17}$$

Proof. (6.16) is immediate since G_0 is a subgroup of G. If $\underline{j} \in G_0$, then $\mathfrak{m}_{1/2} + \underline{j} \in G_{1/2}$ by Proposition 6.14. Conversely, if $\mathfrak{m}_{1/2} + \underline{j} \in G_{1/2}$ then by Proposition 6.14, $G_0 \ni \mathfrak{m}_{1/2} + \underline{j} - \mathfrak{m}_{1/2} = \underline{j}$. □

The next statements will be needed in Section 8.

Lemma 6.16. *Suppose* $\mathsf{h}(\underline{1}) \neq 0$ *and let* $\mathfrak{m}, \mathfrak{n} \in G_{1/2}$. *Then*

a) $\underline{1} - \mathfrak{m} \in G_{1/2}$, *and*

b) $\mathfrak{m} + \mathfrak{n} - \underline{1} \in G_0$.

Proof. Part a) follows from the calculation

$$\begin{aligned}\mathsf{h}(\underline{1} - \mathfrak{m} - \underline{1}) + \xi \mathsf{h}(\underline{1} - \mathfrak{m}) &= -l(-\mathfrak{m})r(-\mathfrak{m})\mathsf{h}(\mathfrak{m}) - \xi l(\underline{1} - \mathfrak{m})(r(\underline{1} - \mathfrak{m})\mathsf{h}(\mathfrak{m} - \underline{1}) = \\ &= -l(-\mathfrak{m})r(-\mathfrak{m})\big(\mathsf{h}(\mathfrak{m}) + \xi r_1 l_1 \mathsf{h}(\mathfrak{m} - \underline{1})\big) = \\ &= -l(-\mathfrak{m})r(-\mathfrak{m})\xi^{-1}\big(\xi \mathsf{h}(\mathfrak{m}) + \mathsf{h}(\mathfrak{m} - \underline{1})\big) = 0.\end{aligned}$$

For part b), use that $\dim L(\underline{1} - \mathfrak{n}) = 2$ by part a), and thus $\mathfrak{m} + \mathfrak{n} - \underline{1} = \mathfrak{m} - (\underline{1} - \mathfrak{n}) \in G_0$ by Proposition 6.14b). □

The formulas provided by the following technical lemma are the key to proving our main theorem.

Lemma 6.17. *Let $\mathfrak{m} \in G$ and $j \in \mathbb{Z}_{\geq 0}$. If $n = 2j + 1$ then*

$$\sum_{k=0}^{n-1} \xi^{n-1-k} \mathsf{h}(\mathfrak{m} - \underline{k}) = r_1^{-j} \mathsf{h}(\mathfrak{m} - \underline{j}) \sum_{k=0}^{n-1} (\xi r_1)^k \tag{6.18}$$

and if $n = 2j + 2$ then

$$\sum_{k=0}^{n-1} \xi^{n-1-k} \mathsf{h}(\mathfrak{m} - \underline{k}) = r_1^{-j} \big(\mathsf{h}(\mathfrak{m} - \underline{j} - \underline{1}) + \xi \mathsf{h}(\mathfrak{m} - \underline{j})\big) \sum_{k=0}^{n/2-1} (\xi r_1)^{2k}. \tag{6.19}$$

Proof. If $n = 2j+1$, we make the change of index $k \mapsto j - k$, then factor out ξ^j and apply formula (6.5):

$$\sum_{k=0}^{2j} \xi^{2j-k} \mathsf{h}(\mathfrak{m} - \underline{k}) = \sum_{k=-j}^{j} \xi^{j+k} \mathsf{h}(\mathfrak{m} - \underline{j} + \underline{k}) = \xi^j \mathsf{h}(\mathfrak{m} - \underline{j}) \sum_{k=-j}^{j} (\xi r_1)^k.$$

Factoring out $(\xi r_1)^{-j}$ and changing index from k to $k-j$ yields (6.18).

For the $n=2j+2$ case we first split the sum in the left hand side of (6.19) into two sums corresponding to odd and even k:

$$\sum_{k=0}^{j} \xi^{2j-2k}\mathsf{h}(\mathfrak{m}-\underline{2k}-\underline{1})+\sum_{k=0}^{j}\xi^{2j+1-2k}\mathsf{h}(\mathfrak{m}-\underline{2k})$$

Then we make the change of summation index $k\mapsto -k+j/2$ in both sums

$$\xi^{j}\sum_{k=-j/2}^{j/2}\xi^{2k}\mathsf{h}(\mathfrak{m}-\underline{j}-\underline{1}+\underline{2k})+\xi^{j+1}\sum_{k=-j/2}^{j/2}\xi^{2k}\mathsf{h}(\mathfrak{m}-\underline{j}+\underline{2k})$$

and use (6.5) on each of them to get

$$\big(\mathsf{h}(\mathfrak{m}-\underline{j}-\underline{1})+\xi\mathsf{h}(\mathfrak{m}-\underline{j})\big)\xi^{j}\sum_{k=-j/2}^{j/2}(\xi r_1)^{2k}.$$

If we factor out $(\xi r_1)^{-j}$ and change summation index from k to $k-j/2$ we obtain (6.19). $\square$

We now come to the main results in this section.

Main Lemma 6.18. *Assume that* $\mathsf{h}(\underline{1})\neq 0$ *and let* $\mathfrak{m}\in G$. *Then*

a) $\dim L(\mathfrak{m})\leq N$,

b) if $\dim L(\mathfrak{m})=n<N$ *then* $\mathfrak{m}\in G_{\frac{i-1}{2}}+\underline{j}$ *where* $n=2j+i$, $i\in\{1,2\}$, $j\in\mathbb{Z}_{\geq 0}$, *and*

c) if $i\in\{1,2\}$, $j\in\mathbb{Z}_{\geq 0}$, $2j+i\leq N$ *and* $\mathfrak{m}\in G_{\frac{i-1}{2}}$ *then*

$$\dim L(\mathfrak{m}+\underline{j})=2j+i. \tag{6.20}$$

d) If $N'<\infty$ *then* $\dim L(\mathfrak{m}+\underline{N'j})=\dim L(\mathfrak{m})$ *for any* $j\in\mathbb{Z}$.

Proof. Part a) is trivial when $N=\infty$. If N is finite and odd, Proposition 6.4 and (6.18) imply that $\dim L(\mathfrak{m})\leq N$. If N is finite and even, then $(\xi r_1)^N=1$ and $(\xi r_1)^2\neq 1$ so $\sum_{k=0}^{N/2-1}(\xi r_1)^{2k}=0$. Hence Proposition 6.4 and (6.19) implies $\dim L(\mathfrak{m})\leq N$ in this case as well.

Next we turn to part b). Suppose first that $\dim L(\mathfrak{m})=n=2j+1<N$. Then by Proposition 6.4 and (6.18) the right hand side of (6.18) is zero. The definition of N implies that $\mathsf{h}(\mathfrak{m}-\underline{j})=0$, i.e. $\mathfrak{m}\in G_0+\underline{j}$. If instead $\dim L(\mathfrak{m})=2j+2<N$, Proposition 6.4 and (6.19) similarly implies that $\mathfrak{m}\in G_{1/2}+\underline{j}$.

To prove (6.20), we proceed by induction on j. For $j=0$ it follows from (6.9) and (6.10). Suppose it holds for $j=0,1,\ldots,k-1$, where $k>0$ and $2k+i\leq N$. We first show that $\dim L(\mathfrak{m}+\underline{k})\leq 2k+i$. If $i=1$ then by (6.18),

$$\sum_{l=0}^{2k}\xi^{2k-l}\mathsf{h}(\mathfrak{m}+\underline{k}-\underline{l})=r_1^{-k}\mathsf{h}(\mathfrak{m})\sum_{l=0}^{2k}(\xi r_1)^{l}=0$$

since $\mathfrak{m} \in G_0$. Similarly, if $i = 2$, then (6.19) gives

$$\sum_{l=0}^{2k+1} \xi^{2k+1-l}\mathsf{h}(\mathfrak{m} + \underline{k} - \underline{l}) = r_1^{-k}\big(\mathsf{h}(\mathfrak{m} - \underline{1}) + \xi\mathsf{h}(\mathfrak{m})\big)\sum_{l=0}^{k}(\xi r_1)^{2l} = 0$$

since $\mathfrak{m} \in G_{1/2}$ in this case. Thus $\dim L(\mathfrak{m} + \underline{j}) \leq 2j + i$ by Proposition 6.4. Write $\dim L(\mathfrak{m}+\underline{k}) = 2k'+i'$ where $k' \geq 0$, $i' \in \{1,2\}$ and assume that $2k'+i' < 2k+i$. By part b) we have $\mathfrak{m}+\underline{k} \in G_{\frac{i'-1}{2}} + \underline{k'}$ which implies that $\dim L(\mathfrak{m}+\underline{k}-\underline{k'}) = i'$ by (6.9) and (6.10). This contradicts the induction hypothesis unless $k' = 0$. Assuming $k' = 0$ we get $\mathfrak{m} + \underline{k} \in G_{\frac{i'-1}{2}}$. If $i = i'$ then from Lemma 6.15 follows that $\underline{k} \in G_0$. Since $0 < k < \frac{2k+i}{2} \leq N/2 \leq N'$ this contradicts 6.12. We now show that $i \neq i'$ is also impossible. If $i = 1$ and $i' = 2$, then $\mathfrak{m} \in G_0$ and $\mathfrak{m} + \underline{k} \in G_{1/2}$ so by Proposition 6.14b), $\underline{k} \in G_{1/2} \cap \underline{\mathbb{Z}_{>0}}$. By (6.13) we get $N \big| 2k - 1$ which is absurd because $0 < 2k-1 < 2k+1 \leq N$. If $i = 2$ and $i' = 1$ then $\mathfrak{m} \in G_{1/2}$ and $\mathfrak{m}+\underline{k} \in G_0$. By Proposition 6.14b) we have $\underline{-k} = \mathfrak{m} - (\mathfrak{m} + \underline{k}) \in G_{1/2}$. By Lemma 6.16a), $\underline{1+k} \in G_{1/2}$ so (6.13) implies that $N \big| 2(1+k) - 1 = 2k+1$. This is impossible since $0 < 2k+1 < 2k+2 \leq N$. We have proved that the assumption $2k'+i' < 2k+i$ is false and hence that $\dim L(\mathfrak{m} + \underline{k}) = 2k + i$, which proves the induction step.

Finally, part d) follows from Corollary 6.5 and Proposition 6.12. □

Theorem 6.19. *Assume* $\mathsf{h}(\underline{1}) \neq 0$ *and let* $\mathfrak{m} \in G$.

- *If* $N = \infty$, *then*

$$\dim L(\mathfrak{m}) < \infty \Longleftrightarrow \mathfrak{m} \in (G_0 + \underline{\mathbb{Z}_{\geq 0}}) \cup (G_{1/2} + \underline{\mathbb{Z}_{\geq 0}}) \tag{6.21}$$

and

$$\dim L(\mathfrak{m}_0 + \underline{j}) = 2j + 1, \quad \textit{for } \mathfrak{m}_0 \in G_0 \textit{ and } j \in \mathbb{Z}_{\geq 0}, \tag{6.22}$$

$$\dim L(\mathfrak{m}_{1/2} + \underline{j}) = 2j + 2, \quad \textit{for } \mathfrak{m}_{1/2} \in G_{1/2} \textit{ and } j \in \mathbb{Z}_{\geq 0}. \tag{6.23}$$

- *If* $N < \infty$ *and* N *is even, then*

$$\dim L(\mathfrak{m}) < \infty \Longleftrightarrow \mathfrak{m} \in (G_0 + \underline{\mathbb{Z}}) \cup (G_{1/2} + \underline{\mathbb{Z}}) \tag{6.24}$$

and

$$\dim L(\mathfrak{m} + \underline{(N/2)j}) = \dim L(\mathfrak{m}), \quad \textit{for any } \mathfrak{m} \in G \textit{ and } j \in \mathbb{Z}, \tag{6.25}$$

and for $\mathfrak{m}_0 \in G_0$ *and* $\mathfrak{m}_{1/2} \in G_{1/2}$ *we have*

$$\dim L(\mathfrak{m}_0 + \underline{j}) = 2j + 1, \quad \textit{if } 0 \leq j < N/2, \tag{6.26}$$

$$\dim L(\mathfrak{m}_{1/2} + \underline{j}) = 2j + 2, \quad \textit{if } 0 \leq j < N/2. \tag{6.27}$$

- *If $N < \infty$ and N is odd, then*

$$\dim L(\mathfrak{m}) < \infty \Longleftrightarrow \mathfrak{m} \in G_0 + \underline{\mathbb{Z}} = G_{1/2} + \underline{\mathbb{Z}} \tag{6.28}$$

and

$$\dim L(\mathfrak{m} + \underline{Nj}) = \dim L(\mathfrak{m}), \quad \textit{for any } \mathfrak{m} \in G \textit{ and } j \in \mathbb{Z}, \tag{6.29}$$

and for $\mathfrak{m}_0 \in G_0$ and $\mathfrak{m}_{1/2} \in G_{1/2}$ we have

$$\dim L(\mathfrak{m}_0 + \underline{j}) = \begin{cases} 2j+1, & \textit{if } 0 \le j < \frac{N+1}{2}, \\ 2j+1-N, & \textit{if } \frac{N+1}{2} \le j < N, \end{cases} \tag{6.30}$$

$$\dim L(\mathfrak{m}_{1/2} + \underline{j}) = \begin{cases} 2j+2, & \textit{if } 0 \le j < \frac{N-1}{2}, \\ 2j+2-N, & \textit{if } \frac{N-1}{2} \le j < N. \end{cases} \tag{6.31}$$

Proof. When $N = \infty$, relations (6.21)-(6.23) are immediate from Lemma 6.18b) and c).

Suppose N is finite and even. The $\Rightarrow$ implication in (6.24) holds by Lemma 6.18b). And (6.25) follows from (6.12) and Corollary 6.5. Assume that $\mathfrak{m} \in (G_0 + \underline{\mathbb{Z}}) \cup (G_{1/2} + \underline{\mathbb{Z}})$. Using (6.25) we can assume that $\mathfrak{m} = \mathfrak{m}' + \underline{j}$ where $\mathfrak{m}' \in G_0 \cup G_{1/2}$ and $0 \le j < N/2$. Then, if $i \in \{1,2\}$ we have $2j + i \le N$ and Lemma 6.18c) implies (6.26)-(6.27) and therefore $\dim L(\mathfrak{m}) < \infty$ so (6.24) is also proved.

Assume that N is finite and odd. By (6.13) we have $\underline{(N+1)/2} \in G_{1/2}$. Therefore $G_0 + \underline{\mathbb{Z}} = G_0 + \underline{(N+1)/2} + \underline{\mathbb{Z}} = G_{1/2} + \underline{\mathbb{Z}}$ since $G_{1/2}$ is a right coset of G_0 in G by Proposition 6.14. As before, Lemma 6.18b) implies the $\Rightarrow$ case in (6.28) and (6.29) holds by virtue of (6.12) and Corollary 6.5. If $\mathfrak{m} \in G_0 + \underline{\mathbb{Z}}$ we can assume by (6.29) that $\mathfrak{m} \in G_0 + \underline{j}$ where $0 \le j < N$. If $j < \frac{N+1}{2}$, then $2j + 1 < N + 2$ so since N is odd we have $2j+1 \le N$. By Lemma 6.18c) we deduce that $\dim L(\mathfrak{m}) = 2j+1$. If instead $j \ge \frac{N+1}{2}$, then $\mathfrak{m} = \underline{(N+1)/2} + \mathfrak{m} - \underline{(N+1)/2} \in G_{1/2} + \underline{k}$ where $k = j - \frac{N+1}{2}$ so $0 \le k < \frac{N-1}{2}$. Thus $2k + 2 \le N$ so Lemma 6.18c) implies that $\dim L(\mathfrak{m}) = 2k + 2 = 2j + 1 - N$. This proves (6.30) and the $\Leftarrow$ implication in (6.28). Finally (6.31) is equivalent to (6.30) in the following sense. Let $0 \le j < N$ and $\mathfrak{m}_{1/2} \in G_{1/2}$. Then

$$\dim L(\mathfrak{m}_{1/2} + \underline{j}) = \dim L(\mathfrak{m}_0 + \underline{j'}),$$

where $j' = j + (N+1)/2$ and $\mathfrak{m}_0 = \mathfrak{m}_{1/2} - \underline{(N+1)/2}$. Now $\mathfrak{m}_0 \in G_0$ since $G_{1/2}$ is a coset of G_0 in G. If $0 \le j < \frac{N-1}{2}$, then $\frac{N+1}{2} \le j' < N$ so by (6.30) we have

$$\dim L(\mathfrak{m}_{1/2} + \underline{j}) = \dim L(\mathfrak{m}_0 + \underline{j'}) = 2j' + 1 - N = 2j + 2.$$

And if $\frac{N-1}{2} \le j < N$, then $0 \le j' - N < \frac{N+1}{2}$ and hence

$$\dim L(\mathfrak{m}_{1/2} + \underline{j}) = \dim L(\mathfrak{m}_0 + \underline{j' - N}) = 2(j' - N) + 1 = 2j + 1 - N.$$

The proof is finished. $\square$

Corollary 6.20. *If $N = \infty$ and $\mathfrak{m} \in G_0 \cup G_{1/2}$, then $L(\mathfrak{m} + \underline{j})$ is infinite-dimensional for any $j \in \mathbb{Z}_{<0}$.*

Proof. If the dimension of $L(\mathfrak{m} + \underline{j})$ were finite and odd (even), then $\dim L(\mathfrak{m} + \underline{j - k}) = 1$ (2) for some $k \geq 0$ by Lemma 6.18b). By Lemma 6.18c), $L(\mathfrak{m})$ has then dimension $2(j-k)+1$ $(2(j-k)+2)$ and thus $j = k$ which is absurd. □

Corollary 6.21. *Suppose $N = \infty$ and let $\mathfrak{m} \in G_f$. Then $L(\mathfrak{m})$ is the unique finite-dimensional quotient of $M(\mathfrak{m})$.*

Proof. It is enough to prove that the unique maximal proper submodule $N(\mathfrak{m})$ of $M(\mathfrak{m})$ is simple. By Theorem 6.19 we can write $\mathfrak{m} = \mathfrak{n} + \underline{j}$ where $\mathfrak{n} \in G_0 \cup G_{1/2}$ and $j \in \mathbb{Z}_{\geq 0}$. From the proof of Proposition 6.4 we have

$$\operatorname{Supp}(L(\mathfrak{m})) = \{\mathfrak{n} + \underline{j}, \mathfrak{n} + \underline{j} - \underline{1}, \ldots, \mathfrak{n} - \underline{j}\}.$$

Thus $N(\mathfrak{m})$ is a highest weight module of highest weight $\mathfrak{n} - \underline{j} - \underline{1}$. So $N(\mathfrak{m})$ is a quotient of $M(\mathfrak{n} - \underline{j} - \underline{1})$. But $M(\mathfrak{n} - \underline{j} - \underline{1})$ is simple, otherwise it would have a finite-dimensional simple quotient, i.e. $L(\mathfrak{n} - \underline{j} - \underline{1})$ would be finite-dimensional, contradicting Corollary 6.20. Thus $N(\mathfrak{m})$ is also simple. □

Remark 6.22. We finish this section by remarking that there exist algebras in the class studied in this paper which do not have even-dimensional simple modules as for example the algebra B_q from Section 4.5. Indeed, in this case we have $\xi r_1 = -1$ and so $N = \infty$ by definition. By Proposition 6.14, $G_{1/2} = \emptyset$ so by Theorem 6.19, there can exist no even-dimensional simple modules.

7 Tensor products and a Clebsch-Gordan formula

As we have seen in Section 2 the existence of a Hopf structure on an algebra allows one to define tensor product of its representations by (2.4). The aim of this section is to prove a formula which decomposes the tensor product of two simple A-modules into a direct sum of simple modules. It generalizes the classical Clebsch-Gordan formula for modules over $U(\mathfrak{sl}_2)$. We will assume that $A = A(R, \sigma, \mathsf{h}, \xi)$ is an ambiskew polynomial ring and that it carries a Hopf structure of the type considered in Section 3. We will also assume (6.1) and that $N = \infty$.

Lemma 7.1. *Let V and W be two A-modules. Then*

$$V_{\mathfrak{m}} \otimes W_{\mathfrak{n}} \subseteq (V \otimes W)_{\mathfrak{m}+\mathfrak{n}} \tag{7.1}$$

for any $\mathfrak{m}, \mathfrak{n} \in G$. Hence if V and W are weight modules, then so is $V \otimes W$ and

$$\operatorname{Supp}(V \otimes W) = \{\mathfrak{m} + \mathfrak{n} : \mathfrak{m} \in \operatorname{Supp}(V), \mathfrak{n} \in \operatorname{Supp}(W)\}.$$

Proof. Let $v \in V_{\mathfrak{m}}, w \in W_{\mathfrak{n}}$. Then for any $r \in R$,

$$r(v \otimes w) = \sum_{(r)} r'v \otimes r''w = \sum_{(r)} r'(\mathfrak{m})v \otimes r''(\mathfrak{n})w =$$
$$= \sum_{(r)} r'(\mathfrak{m})r''(\mathfrak{n})v \otimes w = r(\mathfrak{m}+\mathfrak{n})v \otimes w$$

by (5.3), proving (7.1). Thus if V, W are weight modules,

$$V \otimes W = (\oplus_{\mathfrak{m}} V_{\mathfrak{m}}) \otimes (\oplus_{\mathfrak{n}} W_{\mathfrak{n}}) = \oplus_{\mathfrak{m},\mathfrak{n}} V_{\mathfrak{m}} \otimes W_{\mathfrak{n}} = \oplus_{\mathfrak{m}} \big(\oplus_{\mathfrak{m}_1+\mathfrak{m}_2=\mathfrak{m}} V_{\mathfrak{m}_1} \otimes W_{\mathfrak{m}_2} \big).$$

□

Theorem 7.2. *Let* $\mathfrak{m}, \mathfrak{n} \in G_f$. *We have the following isomorphism*

$$L(\mathfrak{m}) \otimes L(\mathfrak{n}) \simeq L(\mathfrak{m}+\mathfrak{n}) \oplus L(\mathfrak{m}+\mathfrak{n}-\underline{1}) \oplus \ldots \oplus L(\mathfrak{m}+\mathfrak{n}-\underline{s}+\underline{1}) \tag{7.2}$$

where $s = \min\{\dim L(\mathfrak{m}), \dim L(\mathfrak{n})\}$.

Proof. Let $e^{\mathfrak{m}}, e^{\mathfrak{n}}$ denote highest weight vectors in $L(\mathfrak{m})$, $L(\mathfrak{n})$ respectively and set $e^{\mathfrak{m}}_j := (X_-)^j e^{\mathfrak{m}}$ for $j \in \mathbb{Z}_{\geq 0}$ and similarly for $\mathfrak{n}$. Set $V = L(\mathfrak{m}) \otimes L(\mathfrak{n})$. By Lemma 7.1 we have

$$V_{\mathfrak{m}+\mathfrak{n}-\underline{k}} = \oplus_{i+j=k} \mathbb{K} e^{\mathfrak{m}}_i \otimes e^{\mathfrak{n}}_j$$

for $k \in \mathbb{Z}_{\geq 0}$. Fix $0 \leq k \leq s-1$. We will prove that

$$\dim \ker X_+|_{V_{\mathfrak{m}+\mathfrak{n}-\underline{k}}} = 1. \tag{7.3}$$

It follows from the calculations in the proof of Proposition 6.4 that when $j > 0$, $X_+ e^{\mathfrak{m}}_j$ is a nonzero multiple of $e^{\mathfrak{m}}_{j-1}$. Let $v^{\mathfrak{m}}_j$ denote this multiple. Let

$$u = \sum_{i=0}^{k} \lambda_i e^{\mathfrak{m}}_i \otimes e^{\mathfrak{n}}_{k-i}$$

be an arbitrary vector in $V_{\mathfrak{m}+\mathfrak{n}-\underline{k}}$. Then

$$X_+ u = \sum_{i=0}^{k} \lambda_i (X_+ e^{\mathfrak{m}}_i \otimes r_+ e^{\mathfrak{n}}_{k-i} + l_+ e^{\mathfrak{m}}_i \otimes X_+ e^{\mathfrak{n}}_{k-i}) =$$
$$= \sum_{i=0}^{k-1} \big[\lambda_{i+1} v^{\mathfrak{m}}_{i+1} r_+(\mathfrak{n}-\underline{k}+\underline{i}+\underline{1}) + \lambda_i l_+(\mathfrak{m}-\underline{i}) v^{\mathfrak{n}}_{k-i} \big] e^{\mathfrak{m}}_i \otimes e^{\mathfrak{n}}_{k-1-i}.$$

Setting

$$c_i = l_+(\mathfrak{m}-\underline{i}) v^{\mathfrak{n}}_{k-i},$$
$$c'_i = v^{\mathfrak{m}}_i r_+(\mathfrak{n}-\underline{k}+\underline{i}),$$

the condition for u to be a highest weight vector can hence be written as

$$\begin{bmatrix} c_0 & c_1' & & & \\ & c_1 & c_2' & & \\ & & \ddots & \ddots & \\ & & & c_{k-1} & c_k' \end{bmatrix} \begin{bmatrix} \lambda_0 \\ \lambda_1 \\ \vdots \\ \lambda_k \end{bmatrix} = 0. \tag{7.4}$$

Since r_+ and l_+ are grouplike, they are invertible and hence $c_i \neq 0 \neq c_{i+1}'$ for any $i = 0, 1, \ldots, k-1$. Therefore the space of solutions to (7.4) is one-dimensional. Thus (7.3) is proved.

From the definition of Verma modules, it follows that for $k = 0, 1, \ldots, s-1$, there is a nonzero A-module morphism

$$M(\mathfrak{m} + \mathfrak{n} - \underline{k}) \to L(\mathfrak{m}) \otimes L(\mathfrak{n})$$

which maps a highest weight vector in $M(\mathfrak{m} + \mathfrak{n} - \underline{k})$ to a highest weight vector in $L(\mathfrak{m}) \otimes L(\mathfrak{n})$ of weight $\mathfrak{m} + \mathfrak{n} - \underline{k}$. But $L(\mathfrak{m}) \otimes L(\mathfrak{n})$ is finite-dimensional so this morphism must factor through $L(\mathfrak{m} + \mathfrak{n} - \underline{k})$ by Corollary 6.21. Taking direct sums of these morphisms we obtain an A-module morphism

$$\varphi : L(\mathfrak{m} + \mathfrak{n}) \oplus L(\mathfrak{m} + \mathfrak{n} - \underline{1}) \oplus \ldots \oplus L(\mathfrak{m} + \mathfrak{n} - \underline{s} + \underline{1}) \to L(\mathfrak{m}) \otimes L(\mathfrak{n}).$$

We claim it is injective. Indeed, the projection of the kernel of φ to any term $L(\mathfrak{m} + \mathfrak{n} - \underline{i})$ must be zero, because it is a proper submodule of the simple module $L(\mathfrak{m} + \mathfrak{n} - \underline{i})$.

To conclude we now calculate the dimensions of both sides. Write $\dim L(\mathfrak{m}) = 2j_1 + i_1$ and $\dim L(\mathfrak{n}) = 2j_2 + i_2$ where $j_1, j_2 \in \mathbb{Z}_{\geq 0}$ and $i_1, i_2 \in \{1, 2\}$. By Lemma 6.18b), $\dim L(\mathfrak{m} - \underline{j_1}) = i_1$ and $\dim L(\mathfrak{n} - \underline{j_2}) = i_2$. First note that

$$\dim L(\mathfrak{m} - \underline{j_1} + \mathfrak{n} - \underline{j_2}) = i_1 + i_2 - 1.$$

When $i_1 = i_2 = 1$, this is true because G_0 is a subgroup of G. When one of i_1, i_2 is 1 and the other 2, it follows from Proposition 6.14b). And if $i_1 = i_2 = 2$, it follows from Lemma 6.16b) and Theorem 6.19.

From Theorem 6.19 also follows that $\dim L(\mathfrak{m} + \underline{k}) = \dim L(\mathfrak{m}) + 2k$ if $\dim L(\mathfrak{m}) < \infty$ and $k \in \mathbb{Z}_{\geq 0}$. Hence, recalling that $s = \min\{\dim L(\mathfrak{m}), \dim L(\mathfrak{n})\}$, we have

$$\begin{aligned} \sum_{k=0}^{s-1} \dim L(\mathfrak{m} + \mathfrak{n} - \underline{k}) &= \sum_{k=0}^{s-1} \dim L(\mathfrak{m} - \underline{j_1} + \mathfrak{n} - \underline{j_2} + \underline{j_1} + \underline{j_2} - \underline{k}) = \\ &= \sum_{k=0}^{s-1} \big(i_1 + i_2 - 1 + 2(j_1 + j_2 - k)\big) = \\ &= s(i_1 + i_2 - 1 + 2j_1 + 2j_2) - s(s-1) = \\ &= s(\dim L(\mathfrak{m}) + \dim L(\mathfrak{n}) - s) = \\ &= \dim L(\mathfrak{m}) \dim L(\mathfrak{n}) = \dim \big(L(\mathfrak{m}) \otimes L(\mathfrak{n})\big). \end{aligned}$$

This completes the proof of the theorem. $\square$

Under some conditions it is possible to introduce a $*$-structure on A. In this connection it would be interesting to study Clebsch-Gordan coefficients and the relation with special functions. This will be a subject for future investigation.

8 Casimir operators and semisimplicity

Arguing as in the proof of Lemma 4.2, it is easy to see that any finite-dimensional semisimple module over $A = A(R, \sigma, \mathsf{h}, \xi)$ is a weight module. In this section we will prove the converse, that any finite-dimensional weight module over A is semisimple. Note that in general not all finite-dimensional modules over our algebra A are semisimple. The corresponding example is constructed in [10] for the algebra from Section 4.3. A necessary and sufficient condition for all finite-dimensional modules over an ambiskew polynomial ring to be semisimple was given in [7], Theorem 5.1.

In this section we assume that $A = A(R, \sigma, \mathsf{h}, \xi)$ is an ambiskew polynomial ring with a Hopf structure of the type introduced in Section 3 such that (6.1) holds. We also assume that $N = \infty$.

Let V be a finite-dimensional weight module over A. We will first treat the case when $\operatorname{Supp}(V) \subseteq \mathfrak{m} + \underline{\mathbb{Z}}$ where $\mathfrak{m} \in G_0$ is fixed. Define a linear map

$$C_V : V \to V$$

by requiring

$$C_V v = \sigma^j(t)v, \quad \text{for } v \in V_{\mathfrak{m}+\underline{j}} \text{ and } j \in \mathbb{Z}.$$

Here σ denotes the extended automorphism (2.8). More explicitly we have (if $j \geq 0$)

$$C_V v = \sigma^j(t)v = \Big(\xi^j t + \sum_{k=0}^{j-1} \xi^k \sigma^{j-1-k}(\mathsf{h})\Big)v = \xi^j t v + \sum_{k=0}^{j-1} \xi^k \mathsf{h}(\mathfrak{m} + \underline{k+1})v$$

and similarly when $j < 0$. It is easy to check that C_V is a morphism of A-modules. Hence it is constant on each finite-dimensional simple module V by Schur's Lemma. Moreover if $\varphi : V \to W$ is a morphism of weight A-modules with support in $\mathfrak{m} + \underline{\mathbb{Z}}$, then $\varphi C_V = C_W \varphi$.

Proposition 8.1. *Let $j_1, j_2 \in \mathbb{Z}_{\geq 0}$. If $C_{L(\mathfrak{m}+\underline{j_1})} = C_{L(\mathfrak{m}+\underline{j_2})}$, then $j_1 = j_2$.*

Proof. By applying $C_{L(\mathfrak{m}+\underline{j})}$ to the highest weight vector of $L(\mathfrak{m} + \underline{j})$, ($j \in \mathbb{Z}_{\geq 0}$) we get

$$C_{L(\mathfrak{m}+\underline{j})} = \sum_{k=0}^{j-1} \xi^k \mathsf{h}(\mathfrak{m} + \underline{k+1}). \tag{8.1}$$

We can assume $j_1 < j_2$. By assumption we have

$$0 = \sum_{k=0}^{j_2-1} \xi^k \mathsf{h}(\mathfrak{m} + \underline{k+1}) - \sum_{k=0}^{j_1-1} \xi^k \mathsf{h}(\mathfrak{m} + \underline{k+1}) = \sum_{k=j_1}^{j_2-1} \xi^k \mathsf{h}(\mathfrak{m} + \underline{k+1}) =$$
$$= \xi^{j_1} \sum_{k=0}^{j_2-j_1-1} \xi^k \mathsf{h}(\mathfrak{m} + \underline{j_2} - \underline{(j_2-j_1)} + \underline{k+1}).$$

By Proposition 6.4 this means that $\dim L(\mathfrak{m} + \underline{j_2}) \leq j_2 - j_1$. But this contradicts Theorem 6.19 which says that $\dim L(\mathfrak{m} + \underline{j_2}) = 2j_2 + 1$. ☐

Theorem 8.2. *Let V be a finite-dimensional weight module over A with support in $G_0 + \underline{\mathbb{Z}}$. Then V is semisimple.*

Proof. We follow the idea of the proof of Proposition 12 in [9], Chapter 3. Writing

$$V = \oplus_{\mathfrak{m} \in G_0} \big(\oplus_{j \in \mathbb{Z}} V_{\mathfrak{m}+\underline{j}} \big)$$

and noting that $\oplus_{j\in\mathbb{Z}} V_{\mathfrak{m}+\underline{j}}$ are submodules, we can reduce to the case when $\mathrm{Supp}(V) \subseteq \mathfrak{m} + \underline{\mathbb{Z}}$ for a fixed $\mathfrak{m} \in G_0$.

Let $\lambda_1, \dots, \lambda_k$ be the generalized eigenvalues of the Casimir operator C_V, i.e. the elements of the set

$$\{\lambda \in \mathbb{K} : \ker(C_V - \lambda\,\mathrm{Id})^p \neq 0 \text{ for some } p > 0\}.$$

Then each generalized eigenspace $\sum_p \ker(C_V - \lambda_i\,\mathrm{Id})^p$ is invariant under A, hence they are submodules. It suffices to prove that each such submodule is semisimple. Let V be one of them. Let $V_1 = \{v \in V : X_+ v = 0\}$. Then V_1 is invariant under R and since V is a weight module, $V_1 = \oplus_{\mathfrak{n}\in G}(V_1 \cap V_\mathfrak{n})$. Now if $0 \neq v \in V_1 \cap V_\mathfrak{n}$, then v is a highest weight vector of V and generates a submodule isomorphic to $L(\mathfrak{n})$. Hence if $V_1 \cap V_\mathfrak{n} \neq 0$ for more than one $\mathfrak{n} \in G$, C_V will have two different eigenvalues by Proposition 8.1 which is impossible. Here we used that the restriction of C_V to a submodule W coincides with C_W. Hence V_1 is contained in a single weight space, say $V_\mathfrak{n}$. Let $v_1, \dots, v_k$ be a basis for V_1. Then each v_i generates a simple submodule isomorphic to $L(\mathfrak{n})$. We will show that the sum of these submodules is direct. Vectors of different weights are linearly lindependent so it suffices to show that if

$$\sum_{i=1}^{k} \lambda_i (X_-)^m v_i = 0$$

then all $\lambda_i = 0$. Assume the sum was nonzero and act by X_+ m times. In each step we get a nonzero result because we have not reached the highest weight $\mathfrak{n}$ yet. But then, using (6.3), we have a linear relation among the v_k – a contradiction. We have shown that V contains the direct sum V' of k copies of $L(\mathfrak{n})$. Now X_+ acts injectively on V/V'. This is only possible in a torsion-free finite-dimensional weight A-module if it is 0-dimensional. Thus V is semisimple. ☐

We now turn to the general case. Assume now that A has an even-dimensional irreducible representation. By Lemma 6.18b), $G_{1/2} \neq \emptyset$. We fix $\mathfrak{m}_{1/2} \in G$. Then $G_{1/2} = G_0 + \mathfrak{m}_{1/2}$ by Proposition 6.14.

Theorem 8.3. *Any finite-dimensional weight module V over A is semisimple.*

Proof. By Corollary 6.3 and Theorem 6.19,

$$\mathrm{Supp}(V) \subseteq (G_0 + \underline{\mathbb{Z}}) \cup (G_{1/2} + \underline{\mathbb{Z}})$$

Thus we have a decomposition

$$V = \Big(\bigoplus_{\mathfrak{m} \in G_0} V_{\mathfrak{m}+\underline{\mathbb{Z}}} \Big) \oplus \Big(\bigoplus_{\mathfrak{m} \in G_0} V_{\mathfrak{m}+\mathfrak{m}_{1/2}+\underline{\mathbb{Z}}} \Big)$$

where $V_{\mathfrak{n}+\underline{\mathbb{Z}}} := \oplus_{j\in\mathbb{Z}} V_{\mathfrak{n}+\underline{j}}$ for $\mathfrak{n} \in G$ are submodules. It remains to prove that a weight module V with support in $\mathfrak{m} + \mathfrak{m}_{1/2} + \underline{\mathbb{Z}}$ is semisimple. By Lemma 7.1,

$$\mathrm{Supp}\big(V \otimes L(\mathfrak{m}_{1/2})\big) \subseteq \mathfrak{m} + \mathfrak{m}_{1/2} + \mathfrak{m}_{1/2} + \underline{\mathbb{Z}} = \mathfrak{m}' + \underline{\mathbb{Z}}$$

where $\mathfrak{m}' := \mathfrak{m} + \mathfrak{m}_{1/2} + \mathfrak{m}_{1/2} - \underline{1} \in G_0$ by Lemma 6.16b). Hence $V \otimes L(\mathfrak{m}_{1/2})$ is semisimple by Theorem 8.2. By the Clebsch-Gordan formula (7.2), the tensor product of two semisimple modules is semisimple again. Therefore $V \otimes L(\mathfrak{m}_{1/2}) \otimes L(\underline{1} - \mathfrak{m}_{1/2})$ is semisimple, where $\dim L(\underline{1} - \mathfrak{m}_{1/2}) = 2$ by Lemma 6.16a). On the other hand, by (7.2) again we have

$$V \otimes L(\mathfrak{m}_{1/2}) \otimes L(\underline{1} - \mathfrak{m}_{1/2}) \simeq V \otimes \big(L(\underline{0}) \oplus L(\mathfrak{m})\big) \simeq \big(V \otimes L(\underline{0})\big) \oplus \big(V \otimes L(\mathfrak{m})\big).$$

Finally, it is easy to verify the isomorphism $V \simeq V \otimes L(\underline{0})$, $v \mapsto v \otimes e$ where $0 \neq e \in L(\underline{0})$ is fixed. Thus V is isomorphic to a submodule of the semisimple module $V \otimes L(\mathfrak{m}_{1/2}) \otimes L(\underline{1} - \mathfrak{m}_{1/2})$ and is therefore itself semisimple. □

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References

[1] V. V. Bavula, *Generalized Weyl algebras and their representations*, St. Petersburg Math. J. **4** (1993) 71–93.

[2] G. Benkart, T. Roby, *Down-up algebras*, J. Algebra **209** No. 1 (1998) 305–344; Addendum, J. Algebra **213** No. 1 (1999) 378.

[3] G. Benkart, S. Witherspoon, *A Hopf structure for down-up algebras*, Math. Z. **238** No. 3 (2001) 523–553.

[4] V. V. Borzov, E. V. Damaskinski, S. B. Egorov, *Representations of the deformed oscillator algebra under different choices of generators* (Russian) Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **245** (1997), Vopr. Kvant. Teor. Polya i Stat. Fiz. **14** 80–106, 284; translation in J. Math. Sci. (New York) **100** No. 2 (2000) 2061–2076.

[5] P. A. A. B. Carvalho, I. M. Musson, *Down-up algebras and their representation theory*, J. Algebra **228** (2000) 286–310.

[6] D. A. Jordan, *Finite-dimensional simple modules over certain iterated skew polynomial rings*, J. Pure Appl. Algebra **98** (1995) 45–55.

[7] D. A. Jordan, *Down-up algebras and ambiskew polynomial rings*, J. Algebra **228** (2000) 311–346.

[8] E. E. Kirkman, I. M. Musson, *Hopf down-up algebras*, J. Algebra **262** (2003) 42–53.

[9] Klimyk, A., Schmüdgen, K., *Quantum groups and their representations*, Springer-Verlag, Berlin, 1997.

[10] D. Wang, Q. Ji, S. Yang, *Finite-dimensional representations of quantum group* $U_q(f(H,K))$, Comm. Alg. **30** No. 5 (2002) 2191–2211.

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Paper III

Unitarizable weight modules over generalized Weyl algebras

Jonas T. Hartwig

Abstract

We define a notion of unitarizability for weight modules over a generalized Weyl algebra (of rank one, with commutative coeffiecient ring R), which is assumed to carry an involution of the form $X^* = Y$, $R^* \subseteq R$. We prove that a weight module V is unitarizable iff it is isomorphic to its finitistic dual $V^\sharp$. Using the classification of weight modules by Drozd, Guzner and Ovsienko, we obtain necessary and sufficient conditions for an indecomposable weight module to be isomorphic to its finitistic dual, and thus to be unitarizable. Some examples are given, including $U_q(\mathfrak{sl}_2)$ for q a root of unity.

1 Introduction

For a $*$-algebra A over $\mathbb{C}$ and an A-module V, a basic question is whether V is unitarizable. That is, can V be equipped with an inner product which is A-admissable, i.e. $(av, w) = (v, a^*w)$ for $a \in A, v, w \in V$? This is so in many well-behaved examples, like simple finite-dimensional modules over a finite-dimensional group-algebra, but unfortunately false in general. However, the modules for which this is false might still be unitarizable in the weaker sense of having an admissable inner product which is non-degenerate but not necessarily positive definite.

A new feature for this broadened notion of unitarizability is that there may exist unitarizable indecomposable modules which are not simple.

Such indefinite inner product spaces have been thoroughly studied in the analytical setting of operator algebras, see [KS]. There are also many applications to areas in physics, for example quantum field theory. See [MS] and references therein.

On the algebraic side, existence and uniqueness questions of such indefinite inner products was considered in [MT1] in the general situation of A being a $*$-algebra over an algebraically closed field and M being a finite-dimensional A-module, or a weight A-module with finite-dimensional weight spaces. Among other things, it was shown that an A-module M has a non-degenerate admissable form iff M is isomorphic to its finitistic dual $M^\sharp$. A description of all simple weight (with respect to a Cartan subalgebra) modules with finite-dimensional weight spaces over a complex finite-dimensional semisimple Lie algebra which are unitarizable with a non-degenerate symmetric form was given in [MT2] and independently in [G].

In this paper we consider generalized Weyl algebras (GWAs). These are certain noncommutative rings, first introduced in [B], and studied since in many different papers (see [BB], [BO], [BL] and references therein). The class contains a wide range of examples such as ambiskew polynomial rings [J], which includes Noetherian generalized down-up algebras [CS]; $U(\mathfrak{sl}_2)$ and its various deformations and generalizations (see for example [BO]) as well as the first Weyl algebra and quantum Weyl algebra.

We will consider GWAs of rank one, $A = R(\sigma, t)$, and assume that R is a commutative ring. For such GWAs, all indecomposable weight modules with finite-dimensional weight spaces were classified in [DGO], up to indecomposable elements in a skew polynomial ring over a field. There are five families of modules, some of them depending on many parameters. It is interesting, therefore, to ask if some of these modules possess extra structure.

The purpose of this paper is two-fold:

1) To define an appropriate notion of unitarizability for weight modules over a generalized Weyl algebra equipped with an involution satisfying $X^* = Y$, $Y^* = X$, $R^* \subseteq R$. See Definition 3.1.

2) To find conditions on the parameters of the indecomposable weight modules V over a generalized Weyl algebra, which are necessary and sufficient for the modules to be unitarizable with a non-degenerate admissable form. The main results here are Theorems 5.2, 5.3, 5.6, 5.8, and 5.13 which completely answers this question in the case of real orbit ω, i.e. $\mathfrak{m}^* = \mathfrak{m}\ \forall \mathfrak{m} \in \omega$.

$*$-Representations of generalized Weyl algebras (i.e. representations unitarizable with a positive definite form) and more general algebras were considered in [MT3].

After recalling some basic definitions in Section 2, we give in Section 3 the definition of admissable form and of the finitistic dual $V^\sharp$. We prove analogs of some results from [MT1] such as Proposition 3.18 on the correspondence between forms and morphisms.

In Section 4 we recall the classification theorem from [DGO]. We have collected all notation necessary in Section 4.1.

In Section 5 we consider in turn each type of indecomposable weight module and give necessary and sufficient conditions for the existence of a non-degenerate admissable form.

We end by considering some examples in Section 6. In particular we obtain in Section 6.3 conditions for indecomposable non-simple modules over $U_q(\mathfrak{sl}_2)$ (q a root of unity), to have non-degenerate admissable forms.

2 Setup

Let

- R be a commutative ring with 1,

- $* : R \to R$ an automorphism of order 1 or 2,
- $\sigma : R \to R$ an automorphism commuting with $*$, and
- $t \in R$ be selfadjoint, i.e. $t^* = t$.

Let $A = R(\sigma, t)$ be the associated *generalized Weyl algebra* (GWA) [B]. Thus A is the ring generated by the set $R \cup \{X, Y\}$, where X, Y are two new symbols, with the relations that R is a subring of A and

$$YX = t, \quad XY = \sigma(t), \quad Xr = \sigma(r)X, \quad Yr = \sigma^{-1}(r)Y \quad \forall r \in R. \tag{2.1}$$

By (2.1), $*$ extends to an involution on A (i.e. $(a+b)^* = a^* + b^*, (ab)^* = b^*a^*$, $a^{**} = a$, $\forall a, b \in A$) by requiring

$$X^* = Y, \qquad Y^* = X.$$

Relations (2.1) also imply that A is a $\mathbb{Z}$-graded ring $A = \oplus_{n\in\mathbb{Z}} A_n$ with gradation given by $\deg X = 1, \deg Y = -1, \deg r = 0\ \forall r \in R$. Let Ω be the set of orbits for the action of σ on the set $\mathrm{Max}(R)$ of maximal ideals of R. For $\omega \in \Omega$ we let R_ω denote the direct sum of all the R-modules $R/\mathfrak{m}$ for $\mathfrak{m} \in \omega$:

$$R_\omega = \bigoplus_{\mathfrak{m}\in\omega} R/\mathfrak{m}. \tag{2.2}$$

The R-module R_ω will be used as a subtitute for a ground field, when defining admissable forms in Section 3.2. The automorphism σ induces isomorphisms $R/\mathfrak{m} \to R/\sigma(\mathfrak{m})$, $\mathfrak{m} \in \mathrm{Max}(R)$, which we also denote by σ. Extending additively, we get a map $\sigma : R_\omega \to R_\omega$. The automorphism $*$ of R induces a map $R/\mathfrak{m} \to R/\mathfrak{m}^*$, and hence a map $R_\omega \to R_{\omega^*}$ which will be denoted by conjugation.

Remark 2.1. Let $A = R(\sigma, t)$ be a GWA and $*$ an anti-involution on A satisfying $R^* \subseteq R$ and $X^* = \varepsilon Y$, where $\varepsilon \in R$ is invertible. Then, after a change of generators, we can assume $\varepsilon = 1$ and thus that $t^* = t$. Indeed, set $X_1 = X$, $Y_1 = \varepsilon Y$ and $t_1 = Y_1X_1 = \varepsilon t$. Then $X_1Y_1 = X\varepsilon Y = \sigma(\varepsilon)\sigma(t) = \sigma(t_1)$. Clearly $X_1 r = \sigma(r)X_1$ and $Y_1 r = \sigma^{-1}(r)Y_1$, $\forall r \in R$. Moreover $X_1^* = Y_1$ so that $t_1^* = t_1$.

Definition 2.2. A module V over a ring, which contains R as a subring, will be called a *weight module* if $V = \oplus_{\mathfrak{m}\in\mathrm{Max}(R)} V_\mathfrak{m}$, where $V_\mathfrak{m} = \{v \in V : \mathfrak{m}v = 0\}$. The R-submodules $V_\mathfrak{m}$ of V are called *weight spaces* and elements of $V_\mathfrak{m}$ are *weight vectors of weight* $\mathfrak{m}$. The *support of* V, denoted $\mathrm{Supp}(V)$, is defined as the set $\{\mathfrak{m} \in \mathrm{Max}(R) : V_\mathfrak{m} \neq 0\}$.

3 Admissable forms and the finitistic dual

3.1 Motivation of definition

In section 3.2 we will define an admissable form on a weight A-module V to be a certain biadditive form on V with values in the R-module R_ω. To motivate this definition, let us first consider another, at first sight more natural, attempt at a definiton.

As we will see, a problem appears when ω is finite. Suppose therefore that $\omega \in \Omega$ is a finite orbit. Let $p = |\omega|$. Let $\omega \in \Omega$ and let V be a weight module over A with $\mathrm{Supp}(V) \subseteq \omega$. If we choose and fix an element $\mathfrak{m} \in \omega$, we can define a $R/\mathfrak{m}$-vector space structure on V by $(r+\mathfrak{m})v = \sigma^k(r)v$ if $v \in V_{\sigma^k(\mathfrak{m})}$ and $0 \le k < p$. Then, for $v \in V_{\sigma^k(\mathfrak{m})}$ and $\lambda = r + \mathfrak{m} \in R/\mathfrak{m}$,

$$X^p \lambda v = X^p \sigma^k(r) v = \sigma^{p+k}(r) X^p v = \sigma^p(\lambda) X^p v.$$

It would perhaps seem natural to define V to be unitarizable if there is a nonzero *admissable $R/\mathfrak{m}$-form* on V, i.e. a map $G : V \times V \to R/\mathfrak{m}$ satisfying

$$G \text{ is additive in each argument,} \tag{3.1a}$$

$$G(\lambda v, w) = \lambda G(v,w) \qquad \text{for all } v,w \in V,\ \lambda \in R/\mathfrak{m}, \tag{3.1b}$$

$$G(av, w) = G(v, a^* w) \qquad \text{for all } v,w \in V,\ a \in A. \tag{3.1c}$$

However, then, for $v,w \in V$ and $\lambda \in R/\mathfrak{m}$,

$$G(X^p \lambda v, w) = G(\lambda v, Y^p w) = \lambda G(v, Y^p w) = \lambda G(X^p v, w),$$

while on the other hand,

$$G(X^p \lambda v, w) = G(\sigma^p(\lambda) X^p v, w) = \sigma^p(\lambda) G(X^p v, w).$$

Thus, any weight module V with $\mathrm{Supp}(V) \subseteq \omega$ on which $X^p \neq 0$ (or $Y^p \neq 0$ for analogous reasons) would automatically be excluded from the possibility of being unitarizable (at least with a non-degenerate form), unless $\sigma^p : R/\mathfrak{m} \to R/\mathfrak{m}$ is the identity map for some (hence all) $\mathfrak{m} \in \omega$.

Although $\sigma^p : R/\mathfrak{m} \to R/\mathfrak{m}$ is the identity in many important examples (for example, if R is a finitely generated algebra over an algebraically closed field k and σ is a k-algebra automorphism, then $\sigma^p : R/\mathfrak{n} \to R/\mathfrak{n}$ is the identity for any $\mathfrak{n} \in \mathrm{Max}(R)$ with $\sigma^p(\mathfrak{n}) = \mathfrak{n}$), we feel that this notion of admissable form is too restrictive.

To remedy this situation we introduce in Section 3.2 a modified definition of unitarizability which has three advantages. First, no unnecessary restrictions applies as to which modules can be unitarizable when $\sigma^p : R/\mathfrak{m} \to R/\mathfrak{m}$ is nontrivial. Secondly, the definition does not depend on any unnatural choice of maximal ideal in the orbit. And thirdly, in the special case when $\sigma^p : R/\mathfrak{m} \to R/\mathfrak{m}$ really is the identity map (and also when the orbit ω is infinite), the definition is equivalent to the one above in the sense that one form can be obtained from the other in a bijective manner, as described in Proposition 3.4.

3.2 Admissable forms and unitarizability

Let $\omega \in \Omega$ and V be a weight module over A with $\mathrm{Supp}(V) \subseteq \omega$.

Definition 3.1. An *admissable form* F on V is a map

$$F : V \times V \to R_\omega$$

such that

$$F \text{ is additive in each argument,} \tag{3.2a}$$

$$F(rv, w) = rF(v, w) \qquad \text{for all } v, w \in V,\ r \in R, \tag{3.2b}$$

$$F(av, w) = \sigma^{\deg a}\big(F(v, a^* w)\big) \qquad \text{for all } v, w \in V,\ a \in \cup_{n\in\mathbb{Z}} A_n. \tag{3.2c}$$

An admissable form F is called *non-degenerate* if for any nonzero $v \in V$ there exist $w_1, w_2 \in V$ such that $F(w_1, v) \neq 0 \neq F(v, w_2)$.

Definition 3.2. A weight module V over A, whose support is contained in an orbit, is *unitarizable* if there exists a nonzero admissable form on V.

Note that, since $\deg a^* = -\deg a$ for homogenous $a \in A$, relation (3.2c) is equivalent to $F(v, aw) = \sigma^{\deg a}\big(F(a^* v, w)\big)$.

3.3 Relation to admissable $R/\mathfrak{m}$-forms

In view of the discussion in Section 3.1 we make the following definition.

Definition 3.3. We call $\omega \in \Omega$ *torsion trivial* if whenever $\mathfrak{m} \in \omega$, $n \in \mathbb{Z}$ and $\sigma^n(\mathfrak{m}) = \mathfrak{m}$ then the induced map $\sigma^n : R/\mathfrak{m} \to R/\mathfrak{m}$ is the identity.

Assume that $\omega \in \Omega$ is torsion trivial. For $\mathfrak{m}_1, \mathfrak{m}_2 \in \omega$, say $\mathfrak{m}_2 = \sigma^n(\mathfrak{m}_1)$, define $\sigma_{\mathfrak{m}_1,\mathfrak{m}_2} = \sigma^n : R/\mathfrak{m}_1 \to R/\mathfrak{m}_2$. Then $\sigma_{\mathfrak{m}_1,\mathfrak{m}_2}$ is independent of the choice (if any) of n, since ω is torsion trivial. Fix $\mathfrak{m} \in \omega$. Let V be a weight A-module with $\mathrm{Supp}(V) \subseteq \omega$. Give V the structure of an $R/\mathfrak{m}$-vector space by $(r + \mathfrak{m})v = \sigma_{\mathfrak{m},\sigma^k(\mathfrak{m})}(r + \mathfrak{m})v = \sigma^k(r)v$ for $v \in V_{\sigma^k(\mathfrak{m})}$ and $r + \mathfrak{m} \in R/\mathfrak{m}$.

Proposition 3.4. *When ω is torsion trivial, there is a bijective correspondence between admissable forms F and admissable $R/\mathfrak{m}$-forms G on V.*

Proof. Given F, define G by $G = \pi \circ F$, where $\pi : R_\omega \to R/\mathfrak{m}$ is given by

$$\pi\big((\lambda_\mathfrak{n})_{\mathfrak{n}\in\omega}\big) = \sum_{\mathfrak{n}\in\omega} \sigma_{\mathfrak{n},\mathfrak{m}}(\lambda_\mathfrak{n}).$$

Since F is biadditive, so is G. To verify (3.1b), let $\mathfrak{n} = \sigma^k(\mathfrak{m}) \in \omega$ be arbitrary, $v \in V_{\sigma^k(\mathfrak{m})}$, $w \in V$ and $\lambda = r + \mathfrak{m} \in R/\mathfrak{m}$. Then, using that $F(V_\mathfrak{n}, V) \subseteq R/\mathfrak{n}$, which follows from (3.2b), we have

$$\begin{aligned} G(\lambda v, w) &= \pi(F(\sigma^k(r)v, w)) = \sigma^{-k}\big(\sigma^k(r)F(v, w)\big) = r\sigma^{-k}\big(F(v, w)\big) = \\ &= \lambda G(v, w). \end{aligned}$$

To show (3.1c), let $\mathfrak{n} \in \omega, v \in V_\mathfrak{n}, a \in A_k$. Then $av \in V_{\sigma^k(\mathfrak{n})}$ so

$$\begin{aligned} G(av, w) &= \sigma_{\sigma^k(\mathfrak{n}),\mathfrak{m}}\big(F(av, w)\big) = \sigma_{\sigma^k(\mathfrak{n}),\mathfrak{m}}\sigma^k\big(F(v, a^* w)\big) = \sigma_{\mathfrak{n},\mathfrak{m}}\big(F(v, a^* w)\big) = \\ &= G(v, a^* w). \end{aligned}$$

This proves that G is an admissable $R/\mathfrak{m}$-form on V.

Conversely, given G, define F by

$$F(v,w) = \sigma_{\mathfrak{m},\mathfrak{n}}\big(G(v,w)\big) \quad \text{for } v \in V_{\mathfrak{n}},\ w \in V.$$

Then F is biadditive. To prove (3.2b), let $\mathfrak{n} = \sigma^k(\mathfrak{m}) \in \omega, v \in V_{\mathfrak{n}}, w \in V$ and $r \in R$. Put $\lambda = r + \mathfrak{m}$. We have

$$\begin{aligned} F(\sigma^k(r)v,w) = \sigma^k\big(G(\sigma^k(r)v,w)\big) = \sigma^k\big(G(\lambda v,w)\big) = \sigma^k\big(\lambda G(v,w)\big) = \\ = \sigma^k(r)\sigma^k\big(G(v,w)\big) = \sigma^k(r)F(v,w). \end{aligned}$$

Since r was arbitrary, (3.2b) is proved. It remains to show that F satisfies (3.2c). Let $v \in V_{\mathfrak{n}}, a \in A_k$. Then

$$F(av,w) = \sigma_{\mathfrak{m},\sigma^k(\mathfrak{n})}\big(G(av,w)\big) = \sigma^k \circ \sigma_{\mathfrak{m},\mathfrak{n}}\big(G(v,a^*w)\big) = \sigma^k\big(F(v,a^*w)\big).$$

Thus F is an admissable form on V. □

3.4 Symmetric and real orbits

Definition 3.5. An orbit $\omega \in \Omega$ is called *symmetric* if $\mathfrak{m}^* \in \omega$ for any $\mathfrak{m} \in \omega$, and *real* if $\mathfrak{m}^* = \mathfrak{m}$ for any $\mathfrak{m} \in \omega$.

Proposition 3.6. *If ω is symmetric but not real, then $|\omega|$ is finite, even, $|\omega| \geq 4$, and $\mathfrak{m}^* = \sigma^{|\omega|/2}(\mathfrak{m})$ for any $\mathfrak{m} \in \omega$.*

Proof. Since ω is symmetric but not real, there is some $\mathfrak{n} \in \omega$ such that $\mathfrak{n}^* = \sigma^N(\mathfrak{n})$ for some $N \neq 0$. Then

$$\mathfrak{n} = \mathfrak{n}^{**} = \sigma^N(\mathfrak{n})^* = \sigma^N(\mathfrak{n}^*) = \sigma^{2N}(\mathfrak{n}).$$

Hence $|\omega| = p < \infty$ and $2N$ is a multiple of p. Without loss of generality we can assume $0 < N < p$. Then $2N = p$ is the only possibility. Thus $|\omega| \geq 4$ and $\mathfrak{n}^* = \sigma^{|\omega|/2}(\mathfrak{n})$. Since any $\mathfrak{m} \in \omega$ has the form $\sigma^k(\mathfrak{n})$, and σ and $*$ commute, it follows that $\mathfrak{m}^* = \sigma^{|\omega|/2}(\mathfrak{m})$ for any $\mathfrak{m} \in \omega$. □

3.5 Orthogonality of weight spaces

Proposition 3.7. *Let $\omega \in \Omega$ and let V be a weight A-module with* $\mathrm{Supp}(V) \subseteq \omega$*. If F is an admissable form on V, then $F(V_{\mathfrak{m}}, V_{\mathfrak{n}}) = 0$ for any $\mathfrak{m}, \mathfrak{n} \in \omega$ with $\mathfrak{m} \neq \mathfrak{n}^*$.*

Proof. By (3.2b) and (3.2c),

$$(\mathfrak{m} + \mathfrak{n}^*)F(V_{\mathfrak{m}}, V_{\mathfrak{n}}) = F(\mathfrak{m}V_{\mathfrak{m}}, V_{\mathfrak{n}}) + F(V_{\mathfrak{m}}, \mathfrak{n}V_{\mathfrak{n}}) = 0.$$

If $\mathfrak{m} \neq \mathfrak{n}^*$ then $\mathfrak{m} + \mathfrak{n}^* = R \ni 1$ so $F(V_{\mathfrak{m}}, V_{\mathfrak{n}}) = 0$. □

Corollary 3.8. *Let $\omega \in \Omega$ be an orbit. If there exists a unitarizable weight A-module V with* $\mathrm{Supp}(V) \subseteq \omega$*, then ω is symmetric.*

Proof. If V is unitarizable, it has a nonzero admissable form F. Since F is nonzero and V is a weight module, $F(V_\mathfrak{m}, V_\mathfrak{n}) \neq 0$ for some $\mathfrak{m}, \mathfrak{n} \in \mathrm{Supp}(V) \subseteq \omega$. By Proposition 3.7, $\mathfrak{m}^* = \mathfrak{n} \in \omega$. If $\mathfrak{m}_1 \in \omega$ is arbitrary, then $\mathfrak{m}_1 = \sigma^n(\mathfrak{m})$ for some n and $\mathfrak{m}_1^* = \sigma^n(\mathfrak{m})^* = \sigma^n(\mathfrak{m}^*) = \sigma^n(\mathfrak{n}) \in \omega$. This proves that ω is symmetric. □

Corollary 3.9. *If $\omega \in \Omega$ is real and V is a weight A-module with* $\mathrm{Supp}(V) \subseteq \omega$*, then the weight spaces of V are pairwise orthogonal with respect to any admissable form.*

Proof. This is immediate from Proposition 3.7. □

3.6 The finitistic dual $V^\sharp$

Let $\omega \in \Omega$ and V be a weight module over A with $\mathrm{Supp}(V) \subseteq \omega$. Suppose F is an admissable form on V. Let $u \in V$. Define $\tilde{F}_u : V \to R_\omega$ by $\tilde{F}_u(v) = F(u, v)$.

Proposition 3.10. *The map $\tilde{F}_u$ has the following properties:*

$$\tilde{F}_u(v_1 + v_2) = \tilde{F}_u(v_1) + \tilde{F}_u(v_2) \qquad \forall v_1, v_2 \in V, \tag{3.3a}$$
$$\tilde{F}_u(rv) = r^* \tilde{F}_u(v) \qquad \forall r \in R, v \in V, \tag{3.3b}$$
$$\tilde{F}_u(V_\mathfrak{m}) = 0 \quad \textit{for all but finitely many } \mathfrak{m} \in \omega. \tag{3.3c}$$

Proof. (3.3a), (3.3b) follow from (3.2a)-(3.2c). For (3.3c), write $u = \sum_{i=1}^n u_i$, where $u_i \in V_{\mathfrak{m}_i}$. Then if $\mathfrak{n} \in \omega \setminus \{\mathfrak{m}_1^*, \dots, \mathfrak{m}_n^*\}$ we get

$$\tilde{F}_u(V_\mathfrak{n}) = F(u_1, V_\mathfrak{n}) + \cdots + F(u_n, V_\mathfrak{n}) = 0$$

by Proposition 3.7. □

Definition 3.11. Let $\omega \in \Omega$ and V be a weight A-module with $\mathrm{Supp}(V) \subseteq \omega$. The *finitistic dual $V^\sharp$* of V is the set of all maps $\varphi : V \to R_\omega$ satisfying the properties of Proposition 3.10, i.e.

$$\varphi(v_1 + v_2) = \varphi(v_1) + \varphi(v_2) \qquad \forall v_1, v_2 \in V, \tag{3.4a}$$
$$\varphi(rv) = r^* \varphi(v) \qquad \forall r \in R, v \in V, \tag{3.4b}$$
$$\varphi(V_\mathfrak{m}) = 0 \quad \text{for all but finitely many } \mathfrak{m} \in \omega. \tag{3.4c}$$

Proposition 3.12. *$V^\sharp$ carries an A-module structure defined as follows. Let $\varphi \in V^\sharp$ and $r \in R$. Define $r\varphi, X\varphi, Y\varphi : V \to R_\omega$ by*

$$(r\varphi)(v) = \varphi(r^* v) = r\varphi(v), \tag{3.5a}$$
$$(X\varphi)(v) = \sigma\big(\varphi(Yv)\big), \tag{3.5b}$$
$$(Y\varphi)(v) = \sigma^{-1}\big(\varphi(Xv)\big), \tag{3.5c}$$

for any $v \in V$.

Proof. First we must prove that $r\varphi, X\varphi, Y\varphi \in V^\sharp$. It is clear that $r\varphi$ satisfies (3.4a),(3.4b),(3.4c) since φ does. Also $X\varphi$ and $Y\varphi$ satisfies (3.4a),(3.4c). We show (3.4b) for $X\varphi$:

$$(X\varphi)(rv) \overset{(3.5b)}{=} \sigma\big(\varphi(Yrv)\big) = \sigma\big(\varphi(\sigma^{-1}(r)Yv)\big) \overset{(3.4b)}{=} \sigma\big(\sigma^{-1}(r)^*\big)\sigma\big(\varphi(Yv)\big) = \overset{(3.5b)}{=} r^*(X\varphi)(v).$$

Analogously, $Y\varphi$ satisfies (3.4b).

We must also show that the relations in A are preserved. For any $\varphi \in V^\sharp$ we have

$$(YX\varphi)(v) \overset{(3.5c)}{=} \sigma^{-1}\big((X\varphi)(Xv)\big) \overset{(3.5b)}{=} \varphi(YXv) = \varphi(tv) \overset{(3.5a)}{=} (t\varphi)(v) \quad \forall v \in V$$

so $YX\varphi = t\varphi$. Similarly, $XY\varphi = \sigma(t)\varphi$ for any $\varphi \in V^\sharp$. Also, for any $r \in R$ and $\varphi \in V^\sharp$,

$$(Xr\varphi)(v) \overset{(3.5b)}{=} \sigma\big((r\varphi)(Yv)\big) \overset{(3.5a)}{=} \sigma\big(\varphi(r^*Yv)\big) = \sigma\big(\varphi(Y\sigma(r^*)v)\big) = \overset{(3.5b)}{=} (X\varphi)(\sigma(r)^*v) \overset{(3.5a)}{=} \big(\sigma(r)X\varphi)(v) \quad \forall v \in V.$$

Analogously one proves that $Yr\varphi = \sigma^{-1}(r)Y\varphi$ for any $r \in R, \varphi \in V^\sharp$. Thus the relations of A are preserved, so (3.5a)-(3.5c) extends to an action of A on $V^\sharp$. $\square$

Proposition 3.13. *$V^\sharp$ is a weight A-module with*

$$(V^\sharp)_\mathfrak{m} = \{\varphi \in V^\sharp \;:\; \varphi|_{V_\mathfrak{n}} = 0 \text{ for all } \mathfrak{n} \in \omega \text{ except possibly for } \mathfrak{n} = \mathfrak{m}^*\} \tag{3.6}$$
$$= \{\varphi \in V^\sharp \;:\; \varphi(V) \subseteq R/\mathfrak{m}\}. \tag{3.7}$$

Proof. Let $\varphi \in V^\sharp$. Then $\mathfrak{m}\varphi = 0 \Leftrightarrow \varphi(\mathfrak{m}^*v) = 0 \;\; \forall v \in V \Leftrightarrow \varphi|_{V_\mathfrak{n}} = 0$ for all $\mathfrak{n} \in \omega$ except possibly for $\mathfrak{n} = \mathfrak{m}^*$, proving (3.6). The second equality holds since $\mathfrak{m}\varphi = 0 \Leftrightarrow \mathfrak{m}\varphi(V) = 0 \Leftrightarrow \varphi(V) \subseteq (R_\omega)_\mathfrak{m} = R/\mathfrak{m}$. Since any φ is the sum of its corestrictions $\varphi_\mathfrak{m} = \pi_\mathfrak{m} \circ \varphi$, where $\pi_\mathfrak{m} : R_\omega \to R/\mathfrak{m}$, $V^\sharp$ is a weight module. $\square$

Proposition 3.14. *Let $\omega \in \Omega$ and let V be a weight A-module with* $\mathrm{Supp}(V) \subseteq \omega$. *Then* $\mathrm{Supp}(V^\sharp) = \mathrm{Supp}(V)^* = \{\mathfrak{m}^* \;:\; \mathfrak{m} \in \mathrm{Supp}(V)\}$.

Proof. Assume $\mathfrak{m} \in \mathrm{Supp}(V^\sharp)$ and let $0 \neq \varphi \in (V^\sharp)_\mathfrak{m}$. Then, by (3.6), $\varphi(v) \neq 0$ for some $v \in V_{\mathfrak{m}^*}$. This implies that $\mathfrak{m}^* \in \mathrm{Supp}(V)$, i.e. $\mathfrak{m} \in \mathrm{Supp}(V)^*$. Conversely, if $\mathfrak{m} \in \mathrm{Supp}(V)^*$ and $0 \neq v \in V_{\mathfrak{m}^*}$ we can extend v to an $R/\mathfrak{m}^*$-basis of $V_{\mathfrak{m}^*}$ and define $\varphi \in V^\sharp$ by requiring that $\varphi(V_\mathfrak{n}) = 0$, $\mathfrak{n} \neq \mathfrak{m}^*$, $\varphi(v) = 1 + \mathfrak{m}$ and $\varphi(w) = 0$ for all other basis vectors w in $V_{\mathfrak{m}^*}$. Then, by (3.6), $\varphi \in (V^\sharp)_\mathfrak{m}$ so that $\mathfrak{m} \in \mathrm{Supp}(V^\sharp)$. $\square$

Proposition 3.15. *If* $\dim_{R/\mathfrak{m}} V_\mathfrak{m} < \infty$ *for all* $\mathfrak{m} \in \mathrm{Supp}(V)$ *then $V^{\sharp\sharp}$ and V are isomorphic as A-modules.*

Proof. Define $\Psi : V \to V^{\sharp\sharp}$ by $\Psi(v)(\varphi) = \varphi(v)$ for $v \in V$, $\varphi \in V^\sharp$. Then

$$\Psi(Xv)(\varphi) = \varphi(Xv) \overset{(3.5c)}{=} \sigma\big((Y\varphi)(v)\big) = \sigma\big(\Psi(v)(Y\varphi)\big) \overset{(3.5b)}{=} (X\Psi(v))(\varphi)$$

for any $v \in V, \varphi \in V^\sharp$. Similarly, $\Psi(Yv) = Y\Psi(v)$ and $\Psi(rv) = r\Psi(v)$ for any $r \in R$, proving that Ψ is an A-module homomorphism. Let $v \in V$, $v \neq 0$ and write v as a finite sum of weight vectors $v_\mathfrak{m} \neq 0$. Then there exists $\varphi \in (V^\sharp)_{\mathfrak{m}^*}$ such that $\varphi(v) \neq 0$, i.e. $\Psi(v)(\varphi) \neq 0$ so $\Psi(v) \neq 0$. Thus Ψ is injective. Also, by considering dual bases, $\dim V_\mathfrak{m} = \dim(V^\sharp)_\mathfrak{m}$. Since $\Psi(V_\mathfrak{m}) \subseteq (V^{\sharp\sharp})_\mathfrak{m}$ we conclude that Ψ is an isomorphism. □

Let $\omega \in \Omega$. If $\Psi : V \to W$ is a homomorphism of weight A-modules with support in ω, we define $\Psi^\sharp : W^\sharp \to V^\sharp$ by

$$\big(\Psi^\sharp(\varphi)\big)(v) = \varphi\big(\Psi(v)\big) \quad \forall v \in V, \forall \varphi \in W^\sharp \tag{3.8}$$

Proposition 3.16. *$\Psi^\sharp$ is also an A-module homomorhpism. Moreover, $\sharp$ is a contravariant endofunctor on the category of weight A-modules with support in ω.*

Proof. For any $v \in V, \varphi \in W^\sharp, r \in R$, we have

$$\begin{aligned}
\big(\Psi^\sharp(r\varphi)\big)(v) &= (r\varphi)\big(\Psi(v)\big) && \text{by definition of } \Psi^\sharp \\
&= \varphi\big(r^*\Psi(v)\big) && \text{by } A\text{-module structure on } W^\sharp \\
&= \varphi\big(\Psi(r^*v)\big) && \text{since } \Psi \text{ is an } A\text{-module morphism} \\
&= \big(\Psi^\sharp(\varphi)\big)(r^*v) && \text{by definition of } \Psi^\sharp \\
&= \big(r\Psi^\sharp(\varphi)\big)(v) && \text{by } A\text{-module structure on } V^\sharp
\end{aligned}$$

In the same way one shows that $\Psi^\sharp$ commutes with the actions of X and Y. That $\sharp$ is a functor is easy to check. □

3.7 The bijection between forms and morphisms

Let $\omega \in \Omega$ and V be a weight A-module with $\mathrm{Supp}(V) \subseteq \omega$. Assume F is an admissable form on V. For $u \in V$, recall that $\tilde{F}_u \in V^\sharp$ by Proposition 3.10.

Proposition 3.17. *The map $\tilde{F} : V \to V^\sharp$ defined by $u \mapsto \tilde{F}_u$ is an A-module homomorphism.*

Proof. For any $r \in R, u, v \in V$ we have

$$\tilde{F}_{ru}(v) = F(ru, v) = F(u, r^*v) = \tilde{F}_u(r^*v) = (r\tilde{F}_u)(v)$$

and

$$\tilde{F}_{Xu}(v) = F(Xu, v) = \sigma\big(F(u, Yv)\big) = \sigma\big(\tilde{F}_u(Yv)\big) = (X\tilde{F}_u)(v).$$

Similarly, $\tilde{F}_{Yu} = Y\tilde{F}_u$ for any $u \in V$. Thus $\tilde{F}$ is an A-module homomorphism. □

The following proposition is analogous the corresponding result proved in [MT1] for finite-dimensional modules over algebras.

Proposition 3.18. *The map $F \mapsto \tilde{F}$ is an isomorphism of abelian groups between the space of admissable forms on V and $\mathrm{Hom}_A(V, V^\sharp)$. Moreover, non-degenerate forms correspond to isomorphisms.*

Proof. Given $\Phi \in \mathrm{Hom}_A(V, V^\sharp)$, define $\hat{\Phi} : V \times V \to R$ by $\hat{\Phi}(v, w) = \Phi(v)(w)$. Then $\hat{\Phi}$ is an admissable form on V and the maps $F \mapsto \tilde{F}$ and $\Phi \mapsto \hat{\Phi}$ are inverses to each other. If $\hat{\Phi}(v, w) = 0\ \forall w$ implies that $v = 0$, then Φ is injective. If $\hat{\Phi}(v, w) = 0\ \forall v$ implies that $w = 0$, then Φ is surjective. This proves the last claim. □

3.8 A semi-simplicity condition

Proposition 3.19. *Let V be a weight A-module, with* $\mathrm{Supp}(V)$ *contained in a real orbit, such that* $\dim_{R/\mathfrak{m}} V_\mathfrak{m} = 1\ \forall \mathfrak{m} \in \mathrm{Supp}(V)$*. If $V^\sharp \simeq V$ then V is semi-simple.*

Proof. If $V^\sharp \simeq V$, then, by Proposition 3.18, V has a non-degenerate admissable form F. Let U be any submodule of V. Then U is itself a weight module and, since $\dim_{R/\mathfrak{m}} V_\mathfrak{m} = 1$ for all $\mathfrak{m} \in \mathrm{Supp}(V)$, we have $U = \oplus_{\mathfrak{m} \in S} V_\mathfrak{m}$ for some subset $S \subseteq \mathrm{Supp}(V)$. Let $U^\perp = \{v \in V\ :\ F(v, u) = 0\ \forall u \in U\}$. By the defining properties of an admissable form (3.1), $U^\perp$ is an A-submodule of V. On the other hand, by Corollary 3.9 and the non-degeneracy of F, we have $F(V_\mathfrak{m}, V_\mathfrak{n}) = 0$ iff $\mathfrak{m} \neq \mathfrak{n}$ for $\mathfrak{m}, \mathfrak{n} \in \mathrm{Supp}(V)$. Thus $U^\perp = \oplus_{\mathfrak{m} \in \mathrm{Supp}(V) \setminus S} V_\mathfrak{m}$. This proves that $U \oplus U^\perp = V$. Hence, any submodule has an invariant complement so V is semi-simple. □

3.9 Symmetric forms

Recall that the map $R_\omega \to R_{\omega^*}$ induced by $* : R \to R$ is denoted by conjugation.

Definition 3.20. Let ω be a symmetric orbit and F an admissable form on a weight A-module V with $\mathrm{Supp}(V) \subseteq \omega$. The *adjoint form* $F^\sharp : V \times V \to R_\omega$ of F is defined by

$$F^\sharp(v, w) = \overline{F(w, v)}, \quad v, w \in V. \tag{3.9}$$

It is easy to check that $F^\sharp$ is also an admissable form on V. If $F = F^\sharp$, then F is called *symmetric*.

If ω is torsion trivial, we call an admissable $\mathbb{K}_\omega$-form F symmetric if the corresponding admissable form is symmetric.

Proposition 3.21. *Suppose that $\omega \in \Omega$ is symmetric and torsion trivial. Fix $\mathfrak{m} \in \omega$ and put $\mathbb{K}_\omega = R/\mathfrak{m}$. Assume that conjugation on $\mathbb{K}_\omega$ is non-trivial, and that the fixed field under conjugation of $\mathbb{K}_\omega$ is infinite, of characteristic not two.*

Let V be a finite-dimensional weight A-module with support in ω. If V has a non-degenerate admissable $\mathbb{K}_\omega$-form, then it has a symmetric non-degenerate admissable $\mathbb{K}_\omega$-form.

The proof is exactly as in [MT1], but we provide it for convenience.

Proof. Let $F : V \times V \to \mathbb{K}_\omega$ be a non-degenerate admissable $\mathbb{K}_\omega$-form on V. Since conjugation is nontrivial, there is an $s \in \mathbb{K}_\omega$ with $\bar{s} = -s$. Then $F_1 = F + F^\sharp$ and $F_2 = s(F - F^\sharp)$ are both symmetric admissable $\mathbb{K}_\omega$-forms. Define $f \in \mathbb{K}_\omega[x]$ by $f(x) = \det(F_1' + xF_2')$. Here F_i' denotes the matrix of F_i relative some $\mathbb{K}_\omega$-linear basis of V. Since $f(s^{-1}) = \det(2F') \neq 0$, f is a nonzero polynomial. Among the infinitely many $r \in \mathbb{K}_\omega$ with $\overline{r} = r$, pick one which is not a zero of f. Then $F_1 + rF_2$ is a symmetric non-degenerate admissable $\mathbb{K}_\omega$-form on V. $\square$

Remark 3.22. Assume R is a finitely generated algebra over an algebraically closed field $\mathbb{K}$ of characteristic zero and assume that σ is a $\mathbb{K}$-automorphism of R. Let V be an indecomposable weight module over A with support in a real orbit ω. Call two $\mathbb{K}$-forms F_1, F_2 on V are equivalent if there is an automorphism φ of V and an element $\lambda \in \mathbb{K}, \lambda \neq 0$ such that $F_1(v,w) = \lambda F_2\big((\varphi(v), \varphi(w)\big)$ for all $v, w \in V$.

The following statements follow directly from Theorems 2,4 in [MT1].

1) If V is simple and $V \simeq V^\sharp$, then there is a unique up to equivalence non-degenerate admissable $\mathbb{K}$-form on V. If conjugation is nontrivial on $\mathbb{K}$ this form can be chosen to be symmetric, and if conjugation is trivial on $\mathbb{K}$, the form can be chosen to be symmetric or skew-symmetric.

2) If there is a symmetric non-degenerate admissable $\mathbb{K}$-form on V, then it is unique up to equivalence.

4 The classification of weight modules

In this section we review the classification of indecomposable weight modules with finite-dimensional weight spaces over a generalized Weyl algebra, obtained by Drozd, Guzner and Ovsienko in [DGO].

4.1 Notation

A maximal ideal $\mathfrak{m}$ of R is called a *break* if $t \in \mathfrak{m}$. For $\omega \in \Omega$, let B_ω be the set of all breaks in ω: $B_\omega = \{\mathfrak{m} \in \omega : t \in \mathfrak{m}\}$. Often we put $p = |\omega|$, $m = |B_\omega|$. Let $\mathbb{K}_\mathfrak{m} = R/\mathfrak{m}$. For $r \in R$ we define $r_\mathfrak{m} = r + \mathfrak{m} \in \mathbb{K}_\mathfrak{m}$. For each $\omega \in \Omega$, fix an $\mathfrak{m}(\omega) \in \omega$ and put $\mathbb{K}_\omega = \mathbb{K}_{\mathfrak{m}(\omega)}$.

If $\omega \in \Omega$ is infinite, it is naturally ordered by defining $\mathfrak{m} < \mathfrak{n}$ iff $\mathfrak{n} = \sigma^k(\mathfrak{m})$ for some $k > 0$.

If $|\omega| = p < \infty$, define a ternary relation on ω by $\mathfrak{m} < \mathfrak{m}' < \mathfrak{m}''$ if $\mathfrak{m}' = \sigma^i(\mathfrak{m}), \mathfrak{m}'' = \sigma^k(\mathfrak{m})$ for some $0 < i < k < p$. Let $m = |B_\omega|$ and define a bijective corresponence $\mathbb{Z}_m \to B_\omega$, $i \mapsto \mathfrak{m}_i$ such that $i < j < k$ in $\mathbb{Z}_\mathfrak{m}$ implies $\mathfrak{m}_i < \mathfrak{m}_j < \mathfrak{m}_k$ in ω and $\mathfrak{m}_0 = \mathfrak{m}(\omega)$. For $\mathfrak{m} \in \omega$, let $j(\mathfrak{m})$ denote the only $j \in \mathbb{Z}_m$ such that $\mathfrak{m}_{j-1} < \mathfrak{m} \leq \mathfrak{m}_j$. Let $p_1, p_2, \dots, p_m \in \mathbb{Z}_{>0}$ be minimal such that $\sigma^{p_j}(\mathfrak{m}_{j-1}) = \mathfrak{m}_j$. Equivalently, p_i is the number of $\mathfrak{m} \in \omega$ with $j(\mathfrak{m}) = i$. Note that $p_1 + p_2 + \cdots + p_m = p$. Furthermore, we put $\tau = \tau_\omega = \sigma^p$. Let $\mathbb{K}_\omega[x, x^{-1}; \tau]$ be the skew Laurent polynomial ring over $\mathbb{K}_\omega$ with automorphism τ: $xa = \tau(a)x$ for $a \in \mathbb{K}_\omega$. Similarly, $\mathbb{K}_\omega[x; \tau^k]$ is the skew polynomial ring over $\mathbb{K}_\omega$ with automorphism τ^k ($k \in \mathbb{Z}_{\geq 0}$). An element f of such a skew (Laurent) polynomial ring P is called *indecomposable* if the left P-module P/Pf is indecomposable. Two elements $f, g \in P$ are called *similar* if $P/Pf \simeq P/Pg$ as left P-modules.

Let $\mathbf{D}$ denote the free monoid on two letters x, y. Thus $\mathbf{D}$ is the set of words $w = z_1 z_2 \cdots z_n$, where $z_i \in \{x, y\}$, with associative multiplication given by concatenation, and neutral element being the empty word ε of zero length. A word w is an *m-word* if its length n is a multiple of $m \in \mathbb{Z}_{>0}$. An m-word is *non-periodic* if it is not a power of another m-word. We will let $\sharp : \mathbf{D} \to \mathbf{D}$, $w \mapsto w^\sharp$, denote the automorphism given by $x^\sharp = y$, $y^\sharp = x$. We also equip $\mathbf{D}$ with a $\mathbb{Z}$-action given by

$$1.z_1 z_2 \cdots z_n = z_2 z_3 \cdots z_n z_1.$$

for $z_1 z_2 \cdots z_n \in \mathbf{D}$. Following [DGO], we use the notation $w(k)$ for $k.w$.

When ω is symmetric, we will denote the map $\mathbb{K}_\omega \to \mathbb{K}_\omega$, which is induced by the involution $*$ on R, by conjugation $a \mapsto \overline{a}$.

4.2 The different kinds of modules

4.2.1 Infinite orbit without breaks

Define $V(\omega)$, where $\omega \in \Omega$, $|\omega| = \infty$ and $B_\omega = \emptyset$, as the space $V(\omega) = \oplus_{\mathfrak{m}\in\omega} \mathbb{K}_\mathfrak{m}$ with A-module structure given by $Xv = \sigma(t_\mathfrak{m} v)$ and $Yv = \sigma^{-1}(v)$ for $v \in \mathbb{K}_\mathfrak{m}$.

4.2.2 Infinite orbit with breaks

We use an alternative parametrization of these modules, which is more convenient for our purposes. It is easily seen to be equivalent to that of [DGO]. First we need some terminology. Recall the order on infinite orbits ω defined in Section 4.1. An interval S in an infinite orbit ω will be called *supportive* if it satisfies the following property: if S contains a minimal element $\mathfrak{n}_0$, then $\sigma^{-1}(\mathfrak{n}_0) \in B_\omega$ and if S has a maximal element $\mathfrak{n}_1$, then $\mathfrak{n}_1 \in B_\omega$. Let $I(S)$ be the set of *inner breaks* of S:

$$I(S) = \{\mathfrak{m} \in S \cap B_\omega : \sigma(\mathfrak{m}) \in S\}.$$

Now let $\omega \in \Omega$ be infinite with $B_\omega \neq \emptyset$. Let $S \subseteq \omega$ be a supportive interval and let I_X be any subset of $I(S)$. Define $V(\omega, S, I_X) = \oplus_{\mathfrak{m}\in S} \mathbb{K}_\mathfrak{m}$ with, for $v \in \mathbb{K}_\mathfrak{m}$,

$$Xv = \begin{cases} \sigma(t_\mathfrak{m} v), & \text{if } \mathfrak{m} \notin B_\omega, \\ \sigma(v), & \text{if } \mathfrak{m} \in I_X, \\ 0, & \text{otherwise,} \end{cases} \qquad Yv = \begin{cases} \sigma^{-1}(v), & \text{if } \sigma^{-1}(\mathfrak{m}) \notin B_\omega, \\ \sigma^{-1}(v), & \text{if } \sigma^{-1}(\mathfrak{m}) \in I(S) \backslash I_X, \\ 0, & \text{otherwise.} \end{cases} \tag{4.1}$$

Note that if $V = V(\omega, S, I_X)$ then $S = \mathrm{Supp}(V)$ and $I_X = \{\mathfrak{m} \in I(S) : XV_\mathfrak{m} \neq 0\}$.

4.2.3 Finite orbit without breaks

Given an orbit ω, with $|\omega| = p < \infty$ and $B_\omega = \emptyset$, and an indecomposable polynomial $f = \alpha_0 + \alpha_1 x + \cdots + \alpha_d x^d \in \mathbb{K}_\omega[x, x^{-1}; \tau]$ with $\alpha_0 \neq 0 \neq \alpha_d$, define $V(\omega, f) = \oplus_{\mathfrak{m}\in\omega} (\mathbb{K}_\mathfrak{m})^d$ with A-module structure given by defining for $v \in (\mathbb{K}_\mathfrak{m})^d$

$$Xv = \begin{cases} \sigma(t_{\mathfrak{m}}v), & \text{if } \mathfrak{m} \neq \mathfrak{m}(\omega), \\ \sigma(F_f t_{\mathfrak{m}}v), & \text{if } \mathfrak{m} = \mathfrak{m}(\omega), \end{cases} \tag{4.2a}$$

$$Yv = \begin{cases} \sigma^{-1}(v), & \text{if } \sigma^{-1}(\mathfrak{m}) \neq \mathfrak{m}(\omega), \\ F_f^{-1}\sigma^{-1}(v), & \text{if } \sigma^{-1}(\mathfrak{m}) = \mathfrak{m}(\omega), \end{cases} \tag{4.2b}$$

where

$$F_f = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & -\alpha_0/\alpha_d \\ 1 & 0 & 0 & \cdots & 0 & -\alpha_1/\alpha_d \\ 0 & 1 & 0 & \cdots & 0 & -\alpha_2/\alpha_d \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -\alpha_{d-1}/\alpha_d \end{bmatrix}.$$

4.2.4 Finite orbit with breaks, first kind

Let $\omega \in \Omega$, $|\omega| = p < \infty$ and $B_\omega \neq \emptyset$. Let $i \in \mathbb{Z}_m$ and $w = z_1z_2\cdots z_n \in \mathbf{D}$. Consider $n+1$ symbols $e_0, e_1, \ldots, e_n$. For $\mathfrak{m} \in \omega$, let $V_{\mathfrak{m}}$ be the vector space over $\mathbb{K}_{\mathfrak{m}}$ with basis consisting of all pairs $[\mathfrak{m}, e_k]$ such that $i+k = j(\mathfrak{m})$ in $\mathbb{Z}_m$. Put $V(\omega, i, w) = \oplus_{\mathfrak{m}\in\omega} V_{\mathfrak{m}}$ and supply it with A-module structure by

$$X[\mathfrak{m}, e_k] = \begin{cases} \sigma(t_{\mathfrak{m}})[\sigma(\mathfrak{m}), e_k], & \text{if } \mathfrak{m} \notin B_\omega, \\ [\sigma(\mathfrak{m}), e_{k+1}], & \text{if } \mathfrak{m} \in B_\omega \text{ and } z_{k+1} = x, \\ 0, & \text{otherwise,} \end{cases}$$

$$Y[\mathfrak{m}, e_k] = \begin{cases} [\sigma^{-1}(\mathfrak{m}), e_k], & \text{if } \sigma^{-1}(\mathfrak{m}) \notin B_\omega, \\ [\sigma^{-1}(\mathfrak{m}), e_{k-1}], & \text{if } \sigma^{-1}(\mathfrak{m}) \in B_\omega \text{ and } z_k = y, \\ 0, & \text{otherwise.} \end{cases}$$

4.2.5 Finite orbit with breaks, second kind

Define $V(\omega, w, f)$, where $\omega \in \Omega$, $|\omega| = p < \infty$ and $|B_\omega| = m > 0$, $w = z_1z_2\cdots z_n \in \mathbf{D}\backslash\{\varepsilon\}$ is a non-periodic m-word, and $f = a_1 + a_2x + \cdots + a_dx^{d-1} + x^d \neq x^d$ is an indecomposable element of $\mathbb{K}_\omega[x; \tau^{n/m}]$ (it should be $\tau^{n/m}$ and not just τ as stated in [DGO]), as follows. Consider dn symbols e_{ks} ($k = 1, \ldots, n$, $s = 1, \ldots, d$). For $\mathfrak{m} \in \omega$, let $V_{\mathfrak{m}}$ be the vector space over $\mathbb{K}_{\mathfrak{m}}$ with basis consisting of all pairs $[\mathfrak{m}, e_{ks}]$ such that $k \equiv j(\mathfrak{m}) \pmod m$. Define $V(\omega, w, f) = \oplus_{\mathfrak{m}\in\omega} V_{\mathfrak{m}}$ and supply it with A-module structure by

$$X[\mathfrak{m}, e_{ks}] = \begin{cases} \sigma(t_{\mathfrak{m}})[\sigma(\mathfrak{m}), e_{ks}], & \text{if } \mathfrak{m} \notin B_\omega, \\ [\sigma(\mathfrak{m}), e_{k+1,s}], & \text{if } \mathfrak{m} \in B_\omega,\ k < n,\ z_{k+1} = x, \\ [\sigma(\mathfrak{m}), e_{1,s+1}], & \text{if } \mathfrak{m} \in B_\omega,\ k = n,\ z_1 = x,\ s < d, \\ -\sum_{r=1}^{d} \sigma(a_r)[\sigma(\mathfrak{m}), e_{1r}], & \text{if } \mathfrak{m} \in B_\omega,\ k = n,\ z_1 = x,\ s = d, \\ 0, & \text{otherwise,} \end{cases} \tag{4.3}$$

$$Y[\mathfrak{m}, e_{ks}] = \begin{cases} [\sigma^{-1}(\mathfrak{m}), e_{ks}], & \text{if } \sigma^{-1}(\mathfrak{m}) \notin B_\omega, \\ [\sigma^{-1}(\mathfrak{m}), e_{k-1,s}], & \text{if } \sigma^{-1}(\mathfrak{m}) \in B_\omega,\ k > 1,\ z_k = y, \\ [\sigma^{-1}(\mathfrak{m}), e_{n,s-1}], & \text{if } \sigma^{-1}(\mathfrak{m}) \in B_\omega,\ k = 1,\ z_1 = y,\ s > 1, \\ -\sum_{r=1}^{d} a_r^\circ[\sigma^{-1}(\mathfrak{m}), e_{nr}], & \text{if } \sigma^{-1}(\mathfrak{m}) \in B_\omega,\ k = 1,\ z_1 = y,\ s = 1, \\ 0, & \text{otherwise.} \end{cases} \tag{4.4}$$

Here $a^\circ_{d+1-r} = \tau^{r-1}(a_r)$, i.e. $a^\circ_r = \tau^{d-r}(a_{d+1-r})$. As compared to [DGO], we changed notation from e_{ks} to $e_{k,d+1-s}$ in the case when $z_1 = y$.

The weight diagram of a module of the form $V = V(\omega, w, f)$, where the first letter of w is $z_1 = x$, is illustrated in Figure 1. Each dot $\overset{\mathfrak{m}}{\bullet}$ is a one-dimensional (over $R/\mathfrak{m}$) subspace of the weight space $V_\mathfrak{m}$. Arrows going in the right direction correspond to X while left arrows correspond to Y. The diagram $\overset{\mathfrak{m}}{\bullet} \rightleftarrows \overset{\sigma(\mathfrak{m})}{\bullet}$ means that X and Y act bijectively on the corresponding one-dimensional subspaces. We shall write

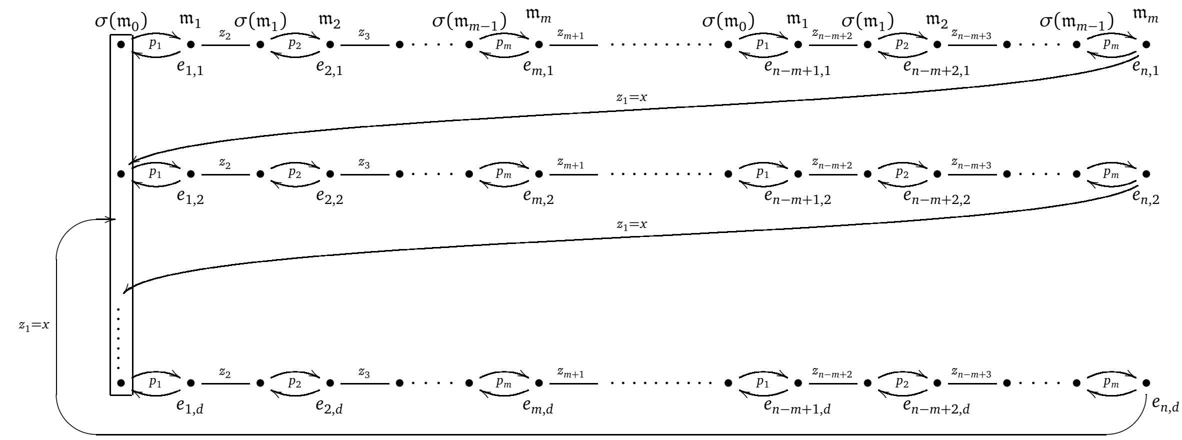
$$\overset{\sigma(\mathfrak{m})}{\bullet} \overset{n}{\rightleftarrows} \overset{\sigma^n(\mathfrak{m})}{\bullet}$$

to denote the weight diagram

$$\overset{\sigma(\mathfrak{m})}{\bullet} \rightleftarrows \overset{\sigma^2(\mathfrak{m})}{\bullet} \rightleftarrows \bullet \cdots\cdots \overset{\sigma^{n-1}(\mathfrak{m})}{\bullet} \rightleftarrows \overset{\sigma^n(\mathfrak{m})}{\bullet} .$$

The diagram $\overset{\mathfrak{m}}{\bullet} \overset{z}{\text{——}} \overset{\sigma(\mathfrak{m})}{\bullet}$ where $z \in \{x, y\}$, means that if $z = x$ then X acts bijectively from $\overset{\mathfrak{m}}{\bullet}$ to $\overset{\sigma(\mathfrak{m})}{\bullet}$ and Y acts as zero on $\overset{\sigma(\mathfrak{m})}{\bullet}$ while if $z = y$, then Y is bijective as a map from $\overset{\sigma(\mathfrak{m})}{\bullet}$ to $\overset{\mathfrak{m}}{\bullet}$ and X acts as zero on $\overset{\mathfrak{m}}{\bullet}$. Often, in weight diagrams each weight space is depicted as a column of dots. In Figure 1, however, for clarity, each column is only a subspace of a certain weight space, and each weight is repeated n/m times horizontally. Recall that, by convention, $p_m = p_0$ and $\mathfrak{m}_m = \mathfrak{m}_0$.

Figure 1: Weight diagram for $V(\omega, w, f)$ when $z_1 = x$.



4.3 The classification theorem

Theorem 4.1 ([DGO], Theorem 5.7).

(i) The A-modules $V(\omega), V(\omega, f), V(\omega, S, I_X), V(\omega, i, w)$, and $V(\omega, w, f)$ are indecomposable weight A-modules.

(ii) Every weight A-module V such that $\dim_{\mathbb{K}_{\mathfrak{m}}} V_{\mathfrak{m}} < \infty$ whenever $\mathfrak{m}$ belong to a finite orbit, decomposes uniquely into a direct sum of modules isomorphic to those listed in (i).

(iii) The only isomorphisms between the listed modules are the following:

- *If f and g are similar in $\mathbb{K}_\omega[x, x^{-1}; \tau]$, then*
$$V(\omega, f) \simeq V(\omega, g). \tag{4.5}$$
- *If f and g are similar in $\mathbb{K}_\omega[x; \tau^{n/m}]$, and $i \in \mathbb{Z}$, then*
$$V(\omega, w, f) \simeq V(\omega, w(mi), \tau^i(g)), \tag{4.6}$$
where $m = |B_\omega|$ and $n = |w|$.

Remark 4.2. In [DGO], τ^i is uncorrectly missing from (4.6). In general, if i is not a multiple of n/m, then f is not similar to $\tau^i(f)$ in $\mathbb{K}_\omega[x; \tau^{n/m}]$. But for $g = f$, one can construct an isomorphism $\varphi : V(\omega, w(m), \tau(f)) \to V(\omega, w, f)$ determined by the conditions

$$1) \quad \varphi\big([\sigma(\mathfrak{m}_0), e_{1,1}]\big) = [\sigma(\mathfrak{m}_0), e_{m+1,1}], \tag{4.7}$$

$$2) \quad \varphi([\mathfrak{m}, e_{k,s}]) \in \begin{cases} \oplus_{r=1}^d \mathbb{K}_{\mathfrak{m}}[\mathfrak{m}, e_{k+m,r}] & k+m \le n, \\ \oplus_{r=1}^d \mathbb{K}_{\mathfrak{m}}[\mathfrak{m}, e_{k+m-n,r}] & k+m > n. \end{cases} \tag{4.8}$$

Remark 4.3. Taking $i = n/m$ in (4.6) we deduce that f is similar $\tau^{n/m}(f)$ in $P := \mathbb{K}_\omega[x; \tau^{n/m}]$. This isomorphism is explicitly given by

$$\begin{aligned} \varphi : P/P\tau^{n/m}(f) &\to P/Pf \\ g + P\tau^{n/m}(f) &\mapsto gx + Pf. \end{aligned}$$

This map is well defined since $\tau^{n/m}(f)x = xf$. It is a homomorphism of left P-modules. Moreover, since $f \neq x^d$ and is indecomposable, its constant term is nonzero. Therefore φ is surjective. Since dimensions agree, φ is an isomorphism as claimed.

The following description of the simple weight A-modules was also given.

Theorem 4.4 ([DGO], Theorem 5.8). *The weight A-modules $V(\omega), V(\omega, f)$ for irreducible $f \in \mathbb{K}_\omega[x, x^{-1}; \tau]$, $V(\omega, S, \emptyset)$ for supportive interval $S \subseteq \omega$ with $I(S) = \emptyset$, $V(\omega, i, \varepsilon)$ and $V(\omega, w, f)$ for irreducible $f \in \mathbb{K}_\omega[x; \tau^{n/m}]$ and $w = x^m$ or $w = y^m$ where $m = |B_\omega|$, are simple and each simple weight A-module is isomorphic to one from this list.*

5 Description of indecomposable weight modules having a non-degenerate admissable form

In this section we consider in turn each of the five types of indecomposable modules from the DGO classification in Section 4 and determine necessary and sufficient conditions, in terms of the parameters, for the modules to be isomorphic to their finitistic dual which, by Proposition 3.18, is equivalent to having a non-degenerate admissable form. We will only consider the case when $\mathrm{Supp}(V)$ is contained in a real orbit ω. The case of symmetric nonreal orbit will be left for future studies.

The following lemma will be useful.

Lemma 5.1. *If V is indecomposable, then so is $V^\sharp$.*

Proof. We prove that if V is decomposable, then so is $V^\sharp$. Then the result follows since $V^{\sharp\sharp} \simeq V$, by Proposition 3.15. Assume V is decomposable and let $i_j : U_j \to V$, $j = 1, 2$, be the inclusions of two submodules U_j whose direct sum is V. Let $W_j = \ker(i_j^\sharp) \subseteq V^\sharp$, $j = 1, 2$. Let $\varphi \in W_1 \cap W_2$. Then $i_1^\sharp(\varphi) = 0 = i_2^\sharp(\varphi)$. Thus $\varphi(i_j(u)) = 0\ \forall u \in U_j$, $j = 1, 2$. Since $V = i_1(U_1) + i_2(U_2)$ we deduce $\varphi = 0$. Hence $W_1 \cap W_2 = 0$. Let $\varphi \in V^\sharp$ be arbitrary. Then $\varphi p_1 + \varphi p_2 = \varphi$, where $p_j : V \to U_j$ are the projections. Also $i_1^\sharp(\varphi p_2)(v) = (\varphi p_2)(i_1(v)) = 0 \forall v \in U_1$, and similarly $i_2^\sharp(\varphi p_1) = 0$. This proves that $V^\sharp = W_1 + W_2$. □

5.1 Infinite orbit without breaks

Theorem 5.2. *Let $V = V(\omega)$, where ω is an infinite real orbit with $B_\omega = \emptyset$. Then $V^\sharp \simeq V$.*

Proof. We have $\mathrm{Supp}(V) = \omega$. By the classification theorem, there is only one indecomposable module whose support is contained in ω. By Lemma 5.1, $V^\sharp$ is indecomposable and by Proposition 3.14, $\mathrm{Supp}(V^\sharp) = \mathrm{Supp}(V) = \omega$. Hence we conclude that $V^\sharp \simeq V$. □

Let ω be infinite real, $B_\omega = \emptyset$, $V = V(\omega)$. We now determine all non-degenerate admissable forms on V, and their index in the symmetric complex case. Let $e_0 \in V_{\mathfrak{m}(\omega)}$, $e_0 \neq 0$. Let $e_0^\sharp \in V^\sharp$ be defined by $e_0^\sharp(e_0) = 1_{\mathfrak{m}(\omega)}$ and $e_0^\sharp(V_\mathfrak{m}) = 0\ \forall \mathfrak{m} \in \omega, \mathfrak{m} \neq \mathfrak{m}(\omega)$. Then $e_0^\sharp$ spans $(V^\sharp)_{\mathfrak{m}(\omega)}$ over $\mathbb{K}_\omega$ so any isomorphism $\Phi : V \to V^\sharp$ must satisfy $\Phi(e_0) = \lambda e_0^\sharp$ for some nonzero $\lambda \in \mathbb{K}_\omega$. Conversely, it is easy to see that for any nonzero $\lambda \in \mathbb{K}_\omega$ there exists a unique isomorphism $\Phi_\lambda : V \to V^\sharp$ satisfying $\Phi_\lambda(e_0) = \lambda e_0^\sharp$. The set $\{e_n := X^n e_0,\ e_{-n-1} := Y^{n+1} e_0 \mid n \in \mathbb{Z}_{\geq 0}\}$ is a basis for V over $\mathbb{K}_\omega$ and the corresponding $\mathbb{K}_\omega$-form Ψ_λ (which is obtained using the bijections between $\mathrm{Hom}_A(V, V^\sharp)$ and admissable forms in Proposition 3.18 and between admissable forms and $\mathbb{K}_\omega$-forms in Proposition 3.4) satisfies

$$\Psi_\lambda(e_n, e_m) = 0, \quad m \neq n,$$

$$\Psi_\lambda(e_n, e_n) = \begin{cases} t\sigma^{-1}(t) \cdots \sigma^{-n+1}(t)\lambda, & n \geq 0, \\ \sigma(t)\sigma^2(t) \cdots \sigma^{-n}(t)\lambda, & n < 0. \end{cases} \tag{5.1}$$

To simplify notation we use here the natural R-module action on $\mathbb{K}_\omega$. For example $t\lambda$ equals the product $(t+\mathfrak{m}(\omega))\lambda$ in $\mathbb{K}_\omega$. From the formula (5.1), and that $t^*=t$, we see that the adjoint form $\Psi_\lambda^\sharp$ is equal to $\Psi_{\overline{\lambda}}$.

In the case when $\mathbb{K}_\omega \simeq \mathbb{C}$ and conjugation is ordinary complex conjugation, we associate to a symmetric form Ψ_λ, $\lambda \in \mathbb{R}$, a scalar product on V defined by $(e_k,e_l) = \operatorname{sgn}\big(\Psi_\lambda(e_k,e_l)\big)\Psi_\lambda(e_k,e_l)$. Then $\Psi_\lambda(v,w) = (Jv,w)$ $\forall v,w \in V$, where $Je_k = \operatorname{sgn}\big(\Psi_\lambda(e_k,e_k)\big)e_k$. J is an involution operator in the sense that $J^2 = \mathrm{Id}_V$ and that it is self-adjoint with respect to the scalar product on V. Therefore, (the completion of) V together with Ψ_λ is a Krein space (see [KS]). Let $V_\pm = \{v \in V : Jv = \pm v\}$. Then $V = V_+ \oplus V_-$. We claim that any pair $(\dim V_+, \dim V_-)$ can occur. In fact, consider the sequence $(i_n)_{n\in\mathbb{Z}}$ where $i_n = \operatorname{sgn}\big(\Psi_\lambda(e_n,e_n)\big)$. Then any sequence $(i_n)_{n\in\mathbb{Z}} \in \{1,-1\}^{\mathbb{Z}}$ can occur. Indeed, let $R = \mathbb{C}[t_n \mid n \in \mathbb{Z}]$ be a polynomial algebra in infinitely many indeterminates t_n. Let $t = t_0$, define $t_n^* = t_n$, $i^* = -i$ and extend $*$ to an $\mathbb{R}$-algebra automorphism of R. Let $\sigma(t_n) = t_{n+1}$ and let $\mathfrak{m}$ be the maximal ideal generated by $t_n - a_n$, $n \in \mathbb{Z}$, where $a_n \in \mathbb{R}$ are given by $a_n = i_{-n}i_{-n+1}$, $n \in \mathbb{Z}$. Let ω be the orbit containing $\mathfrak{m}$ and set $\mathfrak{m}(\omega) = \mathfrak{m}$. The orbit ω is infinite, real, and $B_\omega = \emptyset$. Then the sequence associated to the form Ψ_{i_0} on $V(\omega)$ equals $(i_n)_{n\in\mathbb{Z}}$.

5.2 Infinite orbit with breaks

Theorem 5.3. *Let $V = V(\omega,S,I_X)$, where $\omega \in \Omega$ is infinite and real, $|B_\omega| > 0$, $S \subseteq \omega$ is a supportive interval, and $I_X \subseteq I(S)$. Then $V^\sharp \simeq V(\omega,S,I(S)\backslash I_X)$. In particular V has a non-degenerate admissable form iff $I(S) = \emptyset$ which is equivalent to V being simple.*

Proof. If $V^\sharp \simeq V$, then Proposition 3.19 and that V is indecomposable imply that V must be simple. The converse follows when we prove the more general statement that $V^\sharp \simeq V(\omega,S,I(S)\backslash I_X)$.

By Lemma 5.1, $V^\sharp$ is indecomposable and by Proposition 3.14 and that ω is real, $\mathrm{Supp}(V^\sharp) = \mathrm{Supp}(V) = S$. So by the classification theorem, Theorem 4.1, we deduce that $V^\sharp \simeq V(\omega,S,J)$ for some subset J of $I(S)$. It remains to prove that, for $\mathfrak{m} \in I(S)$, $X(V^\sharp)_\mathfrak{m} \neq 0$ iff $XV_\mathfrak{m} = 0$.

Suppose $\mathfrak{m} \in I(S)$ with $X(V^\sharp)_\mathfrak{m} = 0$. Let $\varphi \in (V^\sharp)_\mathfrak{m}$ be nonzero. Then, by Proposition 3.13, $\varphi|_{V_\mathfrak{n}} = 0$ if $\mathfrak{n} \neq \mathfrak{m}$ and $\varphi(v) = 1_\mathfrak{m}$ for some $v \in V_\mathfrak{m}$. Let $u \in V_{\sigma(\mathfrak{m})}$ be nonzero. We have $0 = (X\varphi)(u) = \sigma\big(\varphi(Yu)\big)$. Thus $Yu = 0$. Thus $u = Xv$ for some nonzero $v \in V_\mathfrak{m}$, otherwise V would be decomposable into $\big(\oplus_{n\le 0} V_{\sigma^n(\mathfrak{m})}\big) \oplus \big(\oplus_{n>0} V_{\sigma^n(\mathfrak{m})}\big)$. This proves that $\mathfrak{m} \in I_X$, i.e. $XV_\mathfrak{m} \neq 0$. The converse is similar.

We conclude that indeed $V^\sharp \simeq V$ iff $I(S) = \emptyset$. By Theorem 4.4, $V(\omega,S,I_X)$ is simple iff $I(S) = \emptyset$. □

Let $\omega \in \Omega$ be real, infinite, $|B_\omega| > 0$. In this case ω is torsion trivial and thus there is a bijection between admissable forms and admissable $\mathbb{K}_\omega$-forms. We now determine all possible non-degenerate admissable $\mathbb{K}_\omega$-forms on $V(\omega,S,\emptyset)$ where S is a supportive interval in ω with $I(S) = \emptyset$.

The subset $S \subseteq \omega$ has either a maximal or a minimal element (otherwise it would contain an inner break). Assume S has a maximal element $\mathfrak{n}_1$. It is a break since S is supportive. We can assume that $\mathfrak{m}(\omega) = \mathfrak{n}_1$. Let $e_0 \in V_{\mathfrak{m}(\omega)}$ be nonzero and $e_0^\sharp \in (V^\sharp)_{\mathfrak{m}(\omega)}$ be such that $e_0^\sharp(e_0) = 1_{\mathfrak{m}(\omega)}$. For $\lambda \in \mathbb{K}_\omega$ there is a unique isomorphism $\Phi_\lambda : V \to V^\sharp$ given by $\Phi_\lambda(e_0) = \lambda e_0^\sharp$. If S has no minimal element, V has a basis $\{e_{-n} := Y^n e_0 \mid n \geq 0\}$. If S has a minimal element $\mathfrak{n}_0$, then $\sigma^{-1}(\mathfrak{n}_0) \in B_\omega$ and V has a basis $\{e_{-n} := Y^n e_0 \mid 0 \leq n \leq N-1\}$ where $\sigma^{-N}(\mathfrak{m}(\omega)) = \sigma^{-1}(\mathfrak{n}_0)$. The corresponding $\mathbb{K}_\omega$-form Ψ_λ calculated on the basis vectors gives

$$\Phi_\lambda(e_{-n}, e_{-m}) = \sigma(t)\sigma^2(t)\cdots\sigma^n(t)\lambda\delta_{n,m} \tag{5.2}$$

for $n, m \geq 0$. If S has no maximal element, but a minimal element $\mathfrak{n}_0$, then $\sigma^{-1}(\mathfrak{n}_0) \in B_\omega$. We choose $\mathfrak{m}(\omega) = \mathfrak{n}_0$ in this case. Then V has a basis $\{e_n := X^n e_0 \mid n \geq 0\}$ and the corresponding $\mathbb{K}_\omega$-form Ψ_λ satisfies

$$\Psi_\lambda(e_n, e_m) = t\sigma^{-1}(t)\cdots\sigma^{-n+1}(t)\lambda\delta_{n,m} \tag{5.3}$$

for $n, m \geq 0$. We see that Ψ_λ is symmetric iff $\overline{\lambda} = \lambda$.

5.3 Finite orbit without breaks

In this section we fix a finite orbit $\omega \in \Omega$ with $B_\omega = \emptyset$. In Theorem 5.6 we will describe the dual modules $V(\omega, f)^\sharp$ for indecomposable $f \in \mathbb{K}_\omega[x, x^{-1}; \tau]$. First we make some preliminary observations. Let $p = |\omega|$ and put $P = \mathbb{K}_\omega[x, x^{-1}; \tau]$.

Proposition 5.4. *Let B be the subalgebra of A generated by X^p, Y^p and all $r \in R$. Let $I = B\mathfrak{m}(\omega)B$ be the ideal in B generated by $\mathfrak{m}(\omega)$. Then there is a ring isomorphism*

$$\psi : B/I \to P$$

given by

$$\psi(X^p + I) = \xi \cdot x, \quad \psi(Y^p + I) = x^{-1}, \quad \psi(r + I) = r_{\mathfrak{m}(\omega)} \quad \text{for } r \in R,$$

where

$$\xi = \big(\sigma(t)\sigma^2(t)\cdots\sigma^p(t)\big)_{\mathfrak{m}(\omega)}. \tag{5.4}$$

Proof. The map ψ is a well-defined ring homomorphism, using the relations (2.1) in A. Assume $b + I \in B/I$ is in the kernel of ψ. Since both rings involved, and ψ, are $\mathbb{Z}$-graded in a natural way, we can assume $b = rX^{pk}$ or $b = rY^{pk}$, $k \geq 0$. We immediately get $k = 0$, $r \in \mathfrak{m}(\omega)$. So ψ is injective. That ψ is surjective is easy to see. $\square$

Let $V = V(\omega, f)$, where $f = \alpha_0 + \alpha_1 x + \cdots + \alpha_d x^d \in P$, $(\alpha_0 \neq 0, \alpha_d \neq 0)$, is indecomposable. Since ω is an orbit of length p, we have $BV_{\mathfrak{m}(\omega)} \subseteq V_{\mathfrak{m}(\omega)}$. Also $IV_{\mathfrak{m}(\omega)} = 0$. Thus $V_{\mathfrak{m}(\omega)}$ becomes a module over B/I and, via the isomorphism in Proposition 5.4, a P-module. The following proposition describes this P-module.

Proposition 5.5.

$$V_{\mathfrak{m}(\omega)} \simeq P/Pf$$

as P-modules.

Proof. Let $e_i = (0, \dots, \overset{i}{1}, \dots, 0) \in V_{\mathfrak{m}(\omega)} = (\mathbb{K}_\omega)^d$. By (4.2a), if $1 \le i < d$,

$$\begin{aligned} X^p e_i &= X^{p-1}\sigma(F_f t_{\mathfrak{m}(\omega)} e_i) = \sigma^p(t_{\mathfrak{m}(\omega)}) X^{p-1}\sigma(e_{i+1}) = \\ &= \sigma^p(t_{\mathfrak{m}(\omega)})\sigma^{p-1}(t_{\sigma(\mathfrak{m}(\omega))}) X^{p-2}\sigma^2(e_{i+1}) = \cdots = \\ &= \xi \cdot e_{i+1}. \end{aligned}$$

Thus

$$(\xi^{-1}X^p)^k e_1 = e_{k+1} \quad \text{for } k = 0, 1, \dots, d-1. \tag{5.5}$$

Also we have, by (4.2a),

$$\xi^{-1}X^p e_d = \sum_{k=0}^{d-1} \tau(-\alpha_k/\alpha_d) e_{k+1}. \tag{5.6}$$

Using (5.5) and (5.6) we get

$$\begin{aligned} \tau(f).e_1 &= \sum_{k=0}^{d} \tau(\alpha_k) x^k.e_1 = \sum_{k=0}^{d} \tau(\alpha_k)(\xi^{-1}X^p)^k e_1 = \\ &= \sum_{k=0}^{d-1} \tau(\alpha_k) e_{k+1} + \tau(\alpha_d) \sum_{k=0}^{d-1} \tau(-\alpha_k/\alpha_d) e_{k+1} = 0. \end{aligned} \tag{5.7}$$

From (5.5) and that $\{e_1, \dots, e_d\}$ generates $V_{\mathfrak{m}(\omega)}$ as an R-module, we see that the vector e_1 generates $V_{\mathfrak{m}(\omega)}$ as a P-module. By (5.7), we get an epimorphism of P-modules

$$\begin{aligned} \psi : P/P\tau(f) &\to V_{\mathfrak{m}(\omega)} \\ h + P\tau(f) &\mapsto h.e_1 \end{aligned}$$

Since $\dim_{\mathbb{K}_\omega} V_{\mathfrak{m}(\omega)} = d = \dim_{\mathbb{K}_\omega} P/P\tau(f)$, we deduce that ψ is an isomorphism. Since f is similar to $\tau(f)$, it follows that $V_{\mathfrak{m}(\omega)} \simeq P/Pf$. □

Now we come to the main result in this section.

Theorem 5.6. *Let $V = V(\omega, f)$, where ω is a finite and real orbit with $B_\omega = \emptyset$ and $f = \alpha_0 + \alpha_1 x + \cdots + \alpha_d x^d \in P = \mathbb{K}_\omega[x, x^{-1}; \tau]$, $\alpha_0 \neq 0 \neq \alpha_d$, is indecomposable. Then*

$$V(\omega, f)^\sharp \simeq V(\omega, f^\sharp)$$

with

$$f^\sharp = \sum_{k=0}^{d} \{k\}_\xi \cdot \tau^k(\overline{\alpha_{d-k}}) \cdot x^k, \tag{5.8}$$

where

$$\{k\}_\xi = \xi\tau(\xi)\cdots\tau^{k-1}(\xi) \quad \textit{for } k \ge 0, \tag{5.9}$$

and

$$\xi = \big(\sigma(t)\sigma^2(t)\cdots\sigma^p(t)\big)_{\mathfrak{m}(\omega)}. \tag{5.10}$$

In particular, $V \simeq V^\sharp$ iff f is similar to $f^\sharp$ in P.

Proof. By Lemma 5.1 and Proposition 3.14, $V^\sharp$ is indecomposable and the support $\mathrm{Supp}(V^\sharp) = \omega$. So by Theorem 4.1, we know that $V^\sharp \simeq V(\omega, h)$ for some $h \in P$. Then by Proposition 5.5, $(V^\sharp)_{\mathfrak{m}(\omega)} \simeq P/Ph$. Thus, it is enough to prove that $(V^\sharp)_{\mathfrak{m}(\omega)} \simeq P/Pf^\sharp$ as P-modules, because then h is similar to $f^\sharp$ which implies that $V^\sharp \simeq V(\omega, f^\sharp)$ by the isomorphism (4.5).

For this, let $e_i = (0, \dots, \overset{i}{1}, \dots, 0) \in V_{\mathfrak{m}(\omega)} = (\mathbb{K}_\omega)^d$, and define $e_i^\sharp \in V^\sharp$ by $e_i^\sharp(V_\mathfrak{n}) = 0$ for $\mathfrak{n} \in \omega$, $\mathfrak{n} \neq \mathfrak{m}(\omega)$ and $e_i^\sharp(e_k) = \delta_{ik} \cdot 1_{\mathfrak{m}(\omega)}$ for $i, k = 1, \dots d$. Since ω is real, $e_i^\sharp \in (V^\sharp)_{\mathfrak{m}(\omega)}$. By relation (4.2b),

$$Y^p e_k = \begin{cases} e_{k-1}, & k > 1, \\ F_f^{-1} e_1, & k = 1. \end{cases} \tag{5.11}$$

It is easy to check that

$$F_f^{-1} e_1 = -\alpha_0^{-1}(\alpha_1 e_1 + \alpha_2 e_2 + \cdots \alpha_d e_d). \tag{5.12}$$

Thus for any $i, k = 1, \dots, d$,

$$\begin{aligned}(X^p e_i^\sharp)(e_k) = \tau\big(e_i^\sharp(Y^p e_k)\big) &= \begin{cases} \delta_{i,k-1} \cdot 1_{\mathfrak{m}(\omega)}, & k > 1, \\ \tau(-\overline{\alpha_i}/\overline{\alpha_0}) \cdot 1_{\mathfrak{m}(\omega)}, & k = 1, \end{cases} \\ &= \big(e_{i+1}^\sharp - \tau(\overline{\alpha_i}/\overline{\alpha_0}) \cdot e_1^\sharp\big)(e_k) \end{aligned} \tag{5.13}$$

with the convention that $e_i^\sharp = 0$ for $i > d$. Let also $\alpha_i = 0$ for $i > d$. We claim that

$$\sum_{k=0}^{n} \tau^{k+1}\big(\overline{\alpha_{n-k}}/\overline{\alpha_0}\big) \cdot X^{pk} e_1^\sharp = e_{n+1}^\sharp, \quad \text{for all } n \geq 0. \tag{5.14}$$

We prove this by induction on n. For $n = 0$ it is trivial. Assume that

$$\sum_{k=0}^{n-1} \tau^{k+1}\big(\overline{\alpha_{n-1-k}}/\overline{\alpha_0}\big) \cdot X^{pk} e_1^\sharp = e_n^\sharp$$

Apply X^p to both sides to get

$$\sum_{k=0}^{n-1} \tau^{k+2}\big(\overline{\alpha_{n-1-k}}/\overline{\alpha_0}\big) \cdot X^{p(k+1)} e_1^\sharp = X^p e_n^\sharp$$

Use that, by (5.13), $X^p e_n^\sharp = e_{n+1}^\sharp - \tau(\overline{\alpha_n}/\overline{\alpha_0}) \cdot e_1^\sharp$ in the right hand side, add $\tau(\overline{\alpha_n}/\overline{\alpha_0}) \cdot e_1^\sharp$ to both sides, and replace k by $k - 1$ in the sum in the left hand side to obtain

$$\sum_{k=1}^{n} \tau^{k+1}\big(\overline{\alpha_{n-k}}/\overline{\alpha_0}\big) \cdot X^{pk} e_1^\sharp + \tau\big(\overline{\alpha_n}/\overline{\alpha_0}\big) \cdot e_1^\sharp = e_{n+1}^\sharp.$$

This proves (5.14). From (5.14) we see that $e_1^\sharp$ generates $(V^\sharp)_{\mathfrak{m}(\omega)}$ as a P-module and that $g.e_1^\sharp = 0$, where

$$g = \sum_{k=0}^{d} \tau^{k+1}(\overline{\alpha_{d-k}}/\overline{\alpha_0})(\xi x)^k = \sum_{k=0}^{d} \xi\tau(\xi)\cdots\tau^{k-1}(\xi)\cdot\tau^{k+1}(\overline{\alpha_{d-k}}/\overline{\alpha_0})x^k \in P.$$

Thus, as in the proof of Proposition 5.5, $(V^\sharp)_{\mathfrak{m}(\omega)} \simeq P/Pg$ as P-modules. Moreover, one verifies that $\tau^{-1}(\xi)\cdot\tau^{-1}(g)\cdot\tau^{-1}(\xi)\overline{\alpha_0} = f^\sharp$. Thus g is similar to $f^\sharp$ and we conclude that $(V^\sharp)_{\mathfrak{m}(\omega)} \simeq P/Pf^\sharp$. This finishes the proof of the theorem. $\square$

Remark 5.7. The example in Section 6.3, concerning $U_q(\mathfrak{sl}_2)$, shows that there exist non-simple indecomposable weight modules which are unitarizable with a non-degenerate admissable form. This is in contrast to the case of bounded $*$-representations of $*$-algebras on Hilbert spaces, that is, unitarizable modules with respect to a positive definite form, where any unitarizable module is semisimple. The example also shows that not all simple weight modules have a non-degenerate admissable form.

5.4 Finite orbit with breaks, first kind

Recall that we defined an automorphism of order two of the monoid $\mathbf{D}$ by $x^\sharp = y$ and $y^\sharp = x$. For example, $(xxy)^\sharp = yyx$.

Theorem 5.8. *Let ω be a finite real orbit with $m := |B_\omega| > 0$, let $j \in \mathbb{Z}_m$ and let $w \in \mathbf{D}$. Then $V(\omega, j, w)^\sharp \simeq V(\omega, j, w^\sharp)$. In particular $V(\omega, j, w)$ has a nondegenerate admissable form iff $w = \varepsilon$, the empty word (of length $n = 0$), which is equivalent to that $V(\omega, j, w)$ is simple.*

Proof. Define $\Phi : V(\omega, j, w) \to V(\omega, j, w^\sharp)^\sharp$ by $\Phi([\mathfrak{m}, e_k]) = c_{\mathfrak{m},k}[\mathfrak{m}, e_k^\sharp]$ where $[\mathfrak{m}, e_k^\sharp] \in V(\omega, j, w^\sharp)^\sharp$ are defined by $[\mathfrak{m}, e_k^\sharp]([\mathfrak{n}, e_l]) = \delta_{\mathfrak{n},\mathfrak{m}}\delta_{k,l}\cdot 1_\mathfrak{m}$ (where $1_\mathfrak{m} = 1 + \mathfrak{m} \in R/\mathfrak{m} \subseteq R_\omega$) and the coefficients $c_{\mathfrak{m},k} \in R/\mathfrak{m}$ are nonzero, to be determined later. Extend Φ to an R-module isomorphism.

Let $[\mathfrak{m}, e_k]$ be a basis vector of $V(\omega, j, w)$. Thus $j + k \equiv j(\mathfrak{m}) \pmod m$. Write

$w = z_1 \cdots z_n$. Consider a basis vector of the form $[\sigma(\mathfrak{m}), e_l] \in V(\omega, j, w^\sharp)$. We have

$$\begin{aligned}
&\Big(X\Phi([\mathfrak{m}, e_k])\Big)([\sigma(\mathfrak{m}), e_l]) = \sigma\Big(c_{\mathfrak{m},k}[\mathfrak{m}, e_k^\sharp](Y[\sigma(\mathfrak{m}), e_l])\Big) = \\
&= \begin{cases} \sigma\Big(c_{\mathfrak{m},k}[\mathfrak{m}, e_k^\sharp]([\mathfrak{m}, e_l])\Big), & \mathfrak{m} \notin B_\omega, \\ \sigma\Big(c_{\mathfrak{m},k}[\mathfrak{m}, e_k^\sharp]([\mathfrak{m}, e_{l-1}])\Big), & \mathfrak{m} \in B_\omega \text{ and } z_l^\sharp = y, \\ 0, & \text{otherwise} \end{cases} \\
&= \begin{cases} \sigma(c_{\mathfrak{m},k})\delta_{kl} \cdot 1_{\sigma(\mathfrak{m})}, & \mathfrak{m} \notin B_\omega, \\ \sigma(c_{\mathfrak{m},k})\delta_{k,l-1} \cdot 1_{\sigma(\mathfrak{m})}, & \mathfrak{m} \in B_\omega \text{ and } z_l = x, \\ 0, & \text{otherwise} \end{cases} \\
&= \begin{cases} \sigma(c_{\mathfrak{m},k})c_{\sigma(\mathfrak{m}),k}^{-1}\Big(\Phi([\sigma(\mathfrak{m}), e_k])\Big)([\sigma(\mathfrak{m}), e_l]), & \mathfrak{m} \notin B_\omega, \\ \sigma(c_{\mathfrak{m},k})c_{\sigma(\mathfrak{m}),k+1}^{-1}\Big(\Phi([\sigma(\mathfrak{m}), e_{k+1}])\Big)([\sigma(\mathfrak{m}), e_l]), & \mathfrak{m} \in B_\omega \text{ and } z_{k+1} = x, \\ 0, & \text{otherwise.} \end{cases} \\
&= \Big(\Phi(X[\mathfrak{m}, e_k])\Big)([\sigma(\mathfrak{m}), e_l])
\end{aligned}$$

if $c_{\mathfrak{m},k}$ are chosen in such a way that $\sigma(c_{\mathfrak{m},k})/c_{\sigma(\mathfrak{m}),k} = \sigma(t_\mathfrak{m})$ when $\mathfrak{m} \notin B_\omega$ and $\sigma(c_{\mathfrak{m},k})/c_{\sigma(\mathfrak{m}),k+1} = 1$ when $\mathfrak{m} \in B_\omega$ and $z_{k+1} = x$. On other basis vectors $[\mathfrak{n}, e_l]$, $\mathfrak{n} \neq \sigma(\mathfrak{m})$, both sides are zero:

$$\Big(X\Phi([\mathfrak{m}, e_k])\Big)([\mathfrak{n}, e_l]) = 0 = \Big(\Phi(X[\mathfrak{m}, e_k])\Big)([\mathfrak{n}, e_l]).$$

With this choice of coefficients, Φ commutes with the action of X. For the action of Y, suppose v is a basis vector of $V(\omega, j, w)$ which is equal to Xu for some u. Then

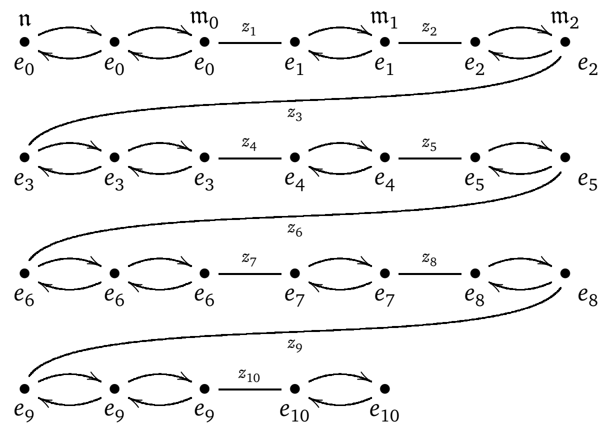
$$\Phi(Yv) = \Phi(YXu) = \Phi(tu) = t\Phi(u) = YX\Phi(u) = Y\Phi(Xu) = Y\Phi(v).$$

It remains to compare the results of applying ΦY and $Y\Phi$ on basis vectors which are not in the image of X. They have the form $[\sigma(\mathfrak{m}), e_k]$ where $\mathfrak{m} \in B_\omega$ and $z_k \neq x$, i.e. $z_k = y$ or $k = 0$.

$$\begin{aligned}
&\Big(Y\Phi([\sigma(\mathfrak{m}), e_k])\Big)([\mathfrak{m}, e_l]) = \sigma^{-1}\Big(c_{\sigma(\mathfrak{m}),k}[\sigma(\mathfrak{m}), e_k^\sharp](X[\mathfrak{m}, e_l])\Big) = \\
&= \begin{cases} \sigma^{-1}\Big(c_{\sigma(\mathfrak{m}),k}[\sigma(\mathfrak{m}), e_k^\sharp]([\sigma(\mathfrak{m}), e_{l+1}])\Big), & z_{l+1}^\sharp = x, \\ 0, & \text{otherwise} \end{cases} \\
&= \begin{cases} \sigma^{-1}(c_{\sigma(\mathfrak{m}),k})\delta_{k,l+1} \cdot 1_\mathfrak{m}, & z_{l+1} = y, \\ 0, & \text{otherwise} \end{cases} \\
&= \begin{cases} \sigma^{-1}(c_{\sigma(\mathfrak{m}),k})c_{\mathfrak{m},k-1}^{-1}\Big(\Phi([\mathfrak{m}, e_{k-1}])\Big)([\mathfrak{m}, e_l]), & z_k = y, \\ 0, & \text{otherwise} \end{cases} \\
&= \Big(\Phi(Y[\sigma(\mathfrak{m}), e_k])\Big)([\mathfrak{m}, e_l])
\end{aligned}$$

if the coefficients are chosen such that $\sigma^{-1}(c_{\sigma(\mathfrak{m}),k})/c_{\mathfrak{m},k-1} = 1$ when $\mathfrak{m} \in B_\omega$ and $z_k = y$. Choosing the coefficients in this way, which is always possible, Φ becomes an isomorphism of A-modules. $\square$

Example 5.9. Assume that $\omega \in \Omega$ is real and $p = |\omega| = 7$. Pick $\mathfrak{n} \in \omega$. Then $\omega = \{\sigma^j(\mathfrak{n}) \mid j = 0, \dots, 6\}$. Suppose that $B_\omega = \{\mathfrak{m}_0 := \sigma^2(\mathfrak{n}), \mathfrak{m}_1 := \sigma^4(\mathfrak{n}), \mathfrak{m}_2 := \sigma^6(\mathfrak{n})\}$, so and $m = |B_\omega| = 3$. The following is a weight diagram for $V(\omega, j, w)$ where $j = 0$ and $w = z_1 z_2 \cdots z_{10}$.



With ω as above, there are three modules of the form $V(\omega, j, \varepsilon)$ corresponding to $j = 0, 1, 2$. For example, $V(\omega, 1, \varepsilon)$ is two-dimensional with basis $\{[\sigma^{-1}(\mathfrak{m}_1), e_1], [\mathfrak{m}_1, e_1]\}$.

In general, let $j \in \mathbb{Z}_m$ and $V = V(\omega, j, \varepsilon)$. We determine all non-degenerate admissable forms on V. V has a basis

$$\{v_k := [\sigma^{-k}(\mathfrak{m}_j), e_j] \mid k = 0, 1, \dots, p_j - 1\},$$

where $p_j > 0$ is minimal such that $\sigma^{p_j}(\mathfrak{m}_{j-1}) = \mathfrak{m}_j$. Any A-module isomorphism $V \to V^\sharp$ has the form $\Phi_\lambda(v_0) = \lambda v_0^\sharp$ for some $\lambda \in \mathbb{K}_{\mathfrak{m}_j}$, where $v_0^\sharp = [\mathfrak{m}_j, e_j^\sharp]$. The corresponding admissable form satisfies

$$\begin{aligned}\widehat{\Phi_\lambda}(v_n, v_m) = \widehat{\Phi_\lambda}(Y^n v_0, Y^n v_0)\delta_{n,m} = \sigma^{-n}\big(\widehat{\Phi_\lambda}(X^n Y^n v_0, v_0)\big)\delta_{n,m} = \\ = \sigma^{-n}\big(\sigma(t)\sigma^2(t)\cdots\sigma^n(t)\lambda\big)\delta_{n,m} \quad (5.15)\end{aligned}$$

for $n, m = 0, 1, \dots, p_j - 1$. It is clearly non-degenerate iff $\lambda \neq 0$.

Suppose that ω is torsion trivial. Choose $\mathfrak{m}(\omega) = \mathfrak{m}_j$. Suppose that $\mathbb{K}_\omega \simeq \mathbb{C}$ and that conjugation is usual complex conjugation and assume that $\lambda \in \mathbb{R}$. Let Ψ_λ be the associated symmetric $\mathbb{C}$-form as described in Proposition 3.4. We have

$$\Psi_\lambda(v_n, v_m) = \big(\sigma(t)\sigma^2(t)\cdots\sigma^n(t)\big)_{\mathfrak{m}(\omega)}\lambda\delta_{n,m}$$

for $n, m = 0, 1, \dots, p_j - 1$. Let us calculate the index (n_+, n_-), (i.e. n_+ (n_-) is the number of positive (negative) eigenvalues) of the form Ψ_λ. Let $a_0 = \lambda$ and

$a_i = \sigma^i(t) + \mathfrak{m}(\omega) \in \mathbb{R}$, $i = 1, \ldots, p_j - 1$. Let $0 \le s_1 < s_2 < \cdots < s_r \le p_j - 1$ be the integers i for which $a_i < 0$ and put $s_i = 0$ for $i \le 0$ and put $s_i = p_j$ for $i > r$. Then one can check that Ψ_λ has index $\big(\sum_{i\in\mathbb{Z}}(s_{2i+1} - s_{2i}), \sum_{i\in\mathbb{Z}}(s_{2i} - s_{2i-1})\big)$. For example, if $p_j = 7$ and $\operatorname{sgn}(\lambda, a_1, a_2, \ldots, a_6) = (+,+,-,+,+,-,-)$, then the index of Ψ_λ is $(2+1, 3+1) = (3,4)$. All possible indices can occur. This can be seen as in Section 5.1.

5.5 Finite orbit with breaks, second kind

For $r \in R$ and $\mathfrak{m} \in \operatorname{Max}(R)$, we put $r_\mathfrak{m} = r + \mathfrak{m} \in R/\mathfrak{m}$ for brevity.

Theorem 5.10. *Let $\omega \in \Omega$ be a finite real orbit. Let $V = V(\omega, w, f)$ where $w = z_1 z_2 \cdots z_n$ is an m-word, and $f = a_1 + a_2 x + \cdots + a_d x^{d-1} + x^d \in \mathbb{K}_\omega[x;\tau^{n/m}]$ is any element with $a_1 \neq 0$. Then $V^\sharp \simeq V(\omega, w^\sharp, g)$ for some $g \in \mathbb{K}_\omega[x;\tau^{n/m}]$.*

Proof. For simplicity, we will assume that $z_1 = x$. The proof of the case $z_1 = y$ is similar.

Step 1. We find the action of X and Y on a dual basis in $V^\sharp$. Relations (4.3)-(4.4) for the module V can be written

$$X[\mathfrak{m}, e_{ks}] = \begin{cases} \sigma(t_\mathfrak{m}) \cdot [\sigma(\mathfrak{m}), e_{ks}], & \mathfrak{m} \notin B_\omega, \\ [\sigma(\mathfrak{m}), e_{k+1,s}], & \mathfrak{m} \in B_\omega, k < n, z_{k+1} = x, \\ 0, & \mathfrak{m} \in B_\omega, k < n, z_{k+1} = y, \\ [\sigma(\mathfrak{m}), e_{1,s+1}], & \mathfrak{m} \in B_\omega, k = n, s < d, \\ -\sum_{i=1}^d \sigma(a_i) \cdot [\sigma(\mathfrak{m}), e_{1i}], & \mathfrak{m} \in B_\omega, k = n, s = d, \end{cases} \tag{5.16}$$

$$Y[\mathfrak{m}, e_{ks}] = \begin{cases} [\sigma^{-1}(\mathfrak{m}), e_{ks}], & \sigma^{-1}(\mathfrak{m}) \notin B_\omega, \\ [\sigma^{-1}(\mathfrak{m}), e_{k-1,s}], & \sigma^{-1}(\mathfrak{m}) \in B_\omega, k > 1, z_k = y, \\ 0, & \sigma^{-1}(\mathfrak{m}) \in B_\omega, k > 1, z_k = x, \\ 0, & \sigma^{-1}(\mathfrak{m}) \in B_\omega, k = 1. \end{cases} \tag{5.17}$$

Let

$$\big\{[\mathfrak{m}, e_{ks}^\sharp] \mid s = 1, \ldots, d,\ k = 1, \ldots, n,\ k \equiv j(\mathfrak{m}) \pmod m\big\}$$

be the dual basis in $V^\sharp$, defined by requiring (recall that $1_\mathfrak{m}$ denotes $1 + \mathfrak{m} \in R/\mathfrak{m}$)

$$[\mathfrak{m}, e_{ks}^\sharp]\big([\mathfrak{n}, e_{lr}]\big) = \begin{cases} 1_\mathfrak{m}, & \text{if } \mathfrak{m} = \mathfrak{n}, k = l, s = r, \\ 0, & \text{otherwise}, \end{cases} \tag{5.18}$$

and $[\mathfrak{m}, e_{ks}^\sharp]$ to be additive and $[\mathfrak{m}, e_{ks}^\sharp](rv) = r^* \cdot [\mathfrak{m}, e_{ks}^\sharp](v)$ for any $r \in R$, $v \in V$. Then the following relations hold for the action of X and Y on this dual basis:

$$X[\mathfrak{m}, e_{ks}^\sharp] = \begin{cases} [\sigma(\mathfrak{m}), e_{ks}^\sharp], & \mathfrak{m} \notin B_\omega, \\ [\sigma(\mathfrak{m}), e_{k+1,s}^\sharp], & \mathfrak{m} \in B_\omega, k < n, z_{k+1} = y, \\ 0, & \text{otherwise}, \end{cases} \tag{5.19}$$

$$Y[\mathfrak{m},e^\sharp_{ks}]=$$
$$=\begin{cases} t_{\sigma^{-1}(\mathfrak{m})}\cdot[\sigma^{-1}(\mathfrak{m}),e^\sharp_{ks}], & \sigma^{-1}(\mathfrak{m})\notin B_\omega,\\ [\sigma^{-1}(\mathfrak{m}),e^\sharp_{k-1,s}], & \sigma^{-1}(\mathfrak{m})\in B_\omega, k>1, z_k=x,\\ 0, & \sigma^{-1}(\mathfrak{m})\in B_\omega, k>1, z_k=y,\\ [\sigma^{-1}(\mathfrak{m}),e^\sharp_{n,s-1}]-\overline{a_s}\cdot[\sigma^{-1}(\mathfrak{m}),e^\sharp_{nd}], & \sigma^{-1}(\mathfrak{m})\in B_\omega, k=1, s>1,\\ -\overline{a_1}\cdot[\sigma^{-1}(\mathfrak{m}),e^\sharp_{nd}], & \sigma^{-1}(\mathfrak{m})\in B_\omega, k=1, s=1.\end{cases} \tag{5.20}$$

Let us prove the first case in (5.20). If $\sigma^{-1}(\mathfrak{m})\notin B_\omega$, then

$$\begin{aligned}
&\big(Y[\mathfrak{m},e^\sharp_{ks}]\big)\big([\sigma^{-1}(\mathfrak{m}),e_{lr}]\big)=\\
&\quad=\sigma^{-1}\Big([\mathfrak{m},e^\sharp_{ks}]\big(X[\sigma^{-1}(\mathfrak{m}),e_{lr}]\big)\Big) && \text{by } A\text{-module str. of } V^\sharp,\\
&\quad=\sigma^{-1}\Big([\mathfrak{m},e^\sharp_{ks}]\big(\sigma(t)\cdot[\mathfrak{m},e_{lr}]\big)\Big) && \text{by (5.16)},\\
&\quad=\sigma^{-1}\Big(\sigma(t)^*\cdot[\mathfrak{m},e^\sharp_{ks}]\big([\mathfrak{m},e_{lr}]\big)\Big) && \text{by } R\text{-antilinearity},\\
&\quad=t\cdot\delta_{kl}\delta_{sr}\cdot\sigma^{-1}(1_\mathfrak{m}) && \text{by (5.18)},\\
&\quad=t\cdot[\sigma^{-1}(\mathfrak{m}),e^\sharp_{ks}]\big([\sigma^{-1}(\mathfrak{m}),e_{lr}]\big) && \text{by (5.18)}.
\end{aligned}$$

Furthermore, if $\mathfrak{n}\neq\sigma^{-1}(\mathfrak{m})$ then

$$\big(Y[\mathfrak{m},e^\sharp_{ks}]\big)\big([\mathfrak{n},e_{lr}]\big)=\sigma^{-1}\Big([\mathfrak{m},e^\sharp_{ks}]\big(X[\mathfrak{n},e_{lr}]\big)\Big)=0=t\cdot[\sigma^{-1}(\mathfrak{m}),e^\sharp_{ks}]\big([\mathfrak{n},e_{lr}]\big)$$

using that $X[\mathfrak{n},e_{lr}]\in V_{\sigma(\mathfrak{n})}$ and (5.18). This proves that $Y[\mathfrak{m},e^\sharp_{ks}]=t\cdot[\sigma^{-1}(\mathfrak{m}),e^\sharp_{ks}]=t_{\sigma^{-1}(\mathfrak{m})}\cdot[\sigma^{-1}(\mathfrak{m}),e^\sharp_{ks}]$ if $\sigma^{-1}(\mathfrak{m})\notin B_\omega$.

For the last two cases in (5.20), let us first note that if $\sigma^{-1}(\mathfrak{m})\in B_\omega$ and $j(\sigma^{-1}(\mathfrak{m}))\equiv n\equiv 0 \pmod m$ then in fact $\sigma^{-1}(\mathfrak{m})=\mathfrak{m}_0$. We have

$$\begin{aligned}
&\big(Y[\sigma(\mathfrak{m}_0),e^\sharp_{1s}]\big)\big([\mathfrak{m}_0,e_{lr}]\big)=\\
&=\sigma^{-1}\Big([\sigma(\mathfrak{m}_0),e^\sharp_{1s}]\big(X[\mathfrak{m}_0,e_{lr}]\big)\Big) \qquad \text{by } A\text{-module str. of } V^\sharp,\\
&=\sigma^{-1}\Big([\sigma(\mathfrak{m}_0),e^\sharp_{1s}]\big([\sigma(\mathfrak{m}_0),e_{1,r+1}]\delta_{ln}\delta_{r<d}-\sigma(a_s)[\sigma(\mathfrak{m}_0),e_{1s}]\delta_{ln}\delta_{rd}\big)\Big)\\
&=\delta_{s-1,r}\delta_{s>1}\delta_{ln}1_{\mathfrak{m}_0}-\overline{a_s}\delta_{ln}\delta_{rd}1_{\mathfrak{m}_0}\\
&=\big([\mathfrak{m}_0,e^\sharp_{n,s-1}]\delta_{s>1}-\overline{a_s}\cdot[\mathfrak{m}_0,e^\sharp_{nd}]\big)\big([\mathfrak{m}_0,e_{lr}]\big).
\end{aligned}$$

The other cases in (5.19),(5.20) are easily checked.

Step 2. We construct a basis $[\mathfrak{m},f_{ks}]$ for $V^\sharp$ such that $[\mathfrak{m},e_{ks}]\mapsto[\mathfrak{m},f_{ks}]$ is an isomorphism from $V(\omega,w^\sharp,g)$ to $V^\sharp$ for some g. We have a decomposition

$$(V^\sharp)_\mathfrak{m}=\bigoplus_{\substack{1\le k\le n,\\ k\equiv j(\mathfrak{m}) \pmod m}}(V^\sharp)^{(k)}_\mathfrak{m} \qquad \text{for any } \mathfrak{m}\in\omega, \tag{5.21}$$

$$(V^\sharp)^{(k)}_\mathfrak{m}=\oplus_{s=1}^d\mathbb{K}_\mathfrak{m}[\mathfrak{m},e^\sharp_{ks}]. \tag{5.22}$$

Note that, if $k > 1$ and $z_k^\sharp = y$ then $Y : (V^\sharp)^{(k)}_{\mathfrak{m}} \to (V^\sharp)^{(k-1)}_{\sigma^{-1}(\mathfrak{m})}$ is bijective, where $\sigma^{-1}(\mathfrak{m}) \in B_\omega$ is the unique break such that $j(\mathfrak{m}) \equiv k \pmod{m}$. Indeed this is trivial since $Y[\mathfrak{m}, e^\sharp_{ks}] = [\sigma^{-1}(\mathfrak{m}), e^\sharp_{k-1,s}]$ for $s = 1, \dots, d$ by the second case in (5.20). Also, $Y : (V^\sharp)^{(1)}_{\sigma(\mathfrak{m}_0)} \to (V^\sharp)^{(n)}_{\mathfrak{m}_0}$ is bijective by the fourth and fifth case in (5.20), using the assumption that $a_1 \neq 0$.

Put

$$[\sigma(\mathfrak{m}_0), f_{11}] = [\sigma(\mathfrak{m}_0), e^\sharp_{11}] \tag{5.23}$$

and recursively

$$[\mathfrak{m}, f_{ks}] = \begin{cases} \sigma(t)^{-1}_{\mathfrak{m}} X[\sigma^{-1}(\mathfrak{m}), f_{ks}], & \sigma^{-1}(\mathfrak{m}) \notin B_\omega, \\ X[\sigma^{-1}(\mathfrak{m}), f_{k-1,s}], & \sigma^{-1}(\mathfrak{m}) \in B_\omega, z^\sharp_k = x (\Rightarrow k > 1), \\ \big(Y|_{(V^\sharp)^{(k)}_{\mathfrak{m}}}\big)^{-1}[\sigma^{-1}(\mathfrak{m}), f_{k-1,s}], & \sigma^{-1}(\mathfrak{m}) \in B_\omega, k > 1, z^\sharp_k = y, \\ \big(Y|_{(V^\sharp)^{(1)}_{\mathfrak{m}}}\big)^{-1}[\sigma^{-1}(\mathfrak{m}), f_{n,s-1}], & \sigma^{-1}(\mathfrak{m}) \in B_\omega, k = 1. \end{cases} \tag{5.24}$$

Induction shows that each $[\mathfrak{m}, f_{ks}]$ is a linear combination of $[\mathfrak{m}, e^\sharp_{kr}]$ where $1 \le r \le s$ and the coefficient of $[\mathfrak{m}, e^\sharp_{ks}]$ is nonzero. Thus $\{[\mathfrak{m}, f_{ks}]\}_{s=1}^d$ is a basis for $(V^\sharp)^{(k)}_{\mathfrak{m}}$.

We prove that there exists a $g \in \mathbb{K}_\omega[x; \tau^{n/m}]$ such that the R-module isomorphism $\varphi : V(\omega, w^\sharp, g) \to V^\sharp$ defined by $\varphi([\mathfrak{m}, e_{ks}]) = [\mathfrak{m}, f_{ks}]$ is an A-module isomorphism. By (4.3),

$$\begin{aligned} \varphi(X[\mathfrak{m}, e_{ks}]) &= \begin{cases} \varphi\big(\sigma(t)_{\sigma(\mathfrak{m})} \cdot [\sigma(\mathfrak{m}), e_{ks}]\big), & \mathfrak{m} \notin B_\omega, \\ \varphi\big([\sigma(\mathfrak{m}), e_{k+1,s}]\big), & \mathfrak{m} \in B_\omega, k < n, z^\sharp_{k+1} = x, \\ 0, & \text{otherwise (since } z^\sharp_1 = y), \end{cases} \\ &= \begin{cases} \sigma(t)_{\sigma(\mathfrak{m})} \cdot [\sigma(\mathfrak{m}), f_{ks}], & \mathfrak{m} \notin B_\omega, \\ [\sigma(\mathfrak{m}), f_{k+1,s}], & \mathfrak{m} \in B_\omega, k < n, z^\sharp_{k+1} = x, \\ 0, & \text{otherwise,} \end{cases} \end{aligned} \tag{5.25}$$

while $X\varphi\big([\mathfrak{m}, e_{ks}]\big) = X[\mathfrak{m}, f_{ks}]$. By the recursive definition of $[\mathfrak{m}, f_{ks}]$, the vector $X[\mathfrak{m}, f_{ks}]$ equals the right hand side of (5.25). For example, $[\sigma(\mathfrak{m}), f_{ks}] = \sigma(t)^{-1}_{\sigma(\mathfrak{m})} \cdot X[\mathfrak{m}, f_{ks}]$ if $\mathfrak{m} \notin B_\omega$ by the first case in (5.24), which gives $X[\mathfrak{m}, f_{ks}] = \sigma(t)_{\sigma(\mathfrak{m})} \cdot [\sigma(\mathfrak{m}), f_{ks}]$. Similarly, by (4.4) and the construction of the basis $[\mathfrak{m}, f_{ks}]$, $\varphi\big(Y[\mathfrak{m}, e_{ks}]\big) = Y\varphi\big([\mathfrak{m}, e_{ks}]\big)$ when $k > 1$ or $s > 1$ or $\mathfrak{m} \neq \sigma(\mathfrak{m}_0)$. For the last case, $k = s = 1$ and $\mathfrak{m} = \sigma(\mathfrak{m}_0)$, we know that $Y : (V^\sharp)^{(1)}_{\sigma(\mathfrak{m}_0)} \to (V^\sharp)^{(n)}_{\mathfrak{m}_0}$ is bijective. Thus, since $\{[\mathfrak{m}_0, f_{ns}]\}_{s=1}^d$ is a basis for $(V^\sharp)^{(n)}_{\mathfrak{m}_0}$,

$$Y\varphi\big([\sigma(\mathfrak{m}_0), e_{11}]\big) = Y[\sigma(\mathfrak{m}_0), f_{11}] = -\sum_{r=1}^d c^\circ_r \cdot [\mathfrak{m}_0, f_{nr}]$$

for some constants $c_r \in \mathbb{K}_\omega$, where we denote $c^\circ_r = \tau^{d-r}(c_{d+1-r})$. Choose $g = c_1 + c_2 x + \cdots + c_d x^{d-1} + x_d$. Since $z^\sharp_1 = y$, relation (4.4) gives that, in $V(\omega, w^\sharp, g)$

we have $Y[\sigma(\mathfrak{m}_0), e_{11}] = -\sum_{r=1}^{d} c_r^\circ[\mathfrak{m}_0, e_{nr}]$ and thus

$$\varphi\big(Y[\sigma(\mathfrak{m}_0), e_{11}]\big) = \varphi\big(-\sum_{r=1}^{d} c_r^\circ[\mathfrak{m}_0, e_{nr}]\big) = -\sum_{r=1}^{d} c_r^\circ[\mathfrak{m}_0, f_{nr}].$$

This finishes the proof that $V^\sharp \simeq V(\omega, w^\sharp, g)$ for some g. □

Corollary 5.11. *Let ω be a finite real orbit. Let $V = V(\omega, w, f)$ where $w = z_1 z_2 \cdots z_n$ is a non-periodic m-word, and $f = a_1 + a_2 x + \cdots + a_d x^{d-1} + x^d \neq x^d$ is indecomposable in $\mathbb{K}_\omega[x; \tau^{n/m}]$. If $V \simeq V^\sharp$ then $w = w_0 w_0^\sharp$, where w_0 is an m-word.*

Proof. Since f is indecomposable and $f \neq x^d$ we have $a_1 \neq 0$. If $V \simeq V^\sharp$ then by Theorem 5.10, $V \simeq V(\omega, w^\sharp, g)$ for some $g \in \mathbb{K}_\omega[x; \tau^{n/m}]$. Thus by the classification in Theorem 4.1 we must have $w(lm) = w^\sharp$ for some integer $l \geq 0$, chosen minimal. Clearly, $lm < n$. Since the operation $\sharp$ on the monoid $\mathbf{D}$ commutes with the $\mathbb{Z}$-action, we have

$$w(lm + k) = w(k)^\sharp \qquad \forall\, k \in \mathbb{Z}. \tag{5.26}$$

We claim that $2lm \leq n$. Otherwise $lm < n < 2lm$ and thus $0 < n - lm < lm$. Also, $w(n - lm) = w(-lm) = w^\sharp$ since $w = w(-lm + lm) = w(-lm)^\sharp$ by (5.26) with $k = -lm$. Thus the properties of the number $\frac{n}{m} - l$ contradicts the minimality of l. Therefore $2lm \leq n$ as claimed.

Now let $k = \mathrm{GCD}(2lm, n)$. Trivially $w(n) = w$, and by (5.26), $w(2lm) = w(lm)^\sharp = w$. Hence $w(k) = w$ also. But $k|n$ and thus $w = (z_1 z_2 \cdots z_k)^{n/k}$. However w is non-periodic and thus $n = k$, forcing $n = 2lm$ so $w = w_0 w_0^\sharp$ where $w_0 = z_1 z_2 \cdots z_{lm}$ is an m-word. □

Theorem 5.12. *Let $\omega \in \Omega$ be a finite real orbit with $m := |B_\omega| > 0$. Let $w_0 \in \mathbf{D}\backslash\{\varepsilon\}$ be an m-word and put $l = |w_0|/m$ and $n = 2|w_0|$. Let $V = V(\omega, w_0 w_0^\sharp, f)$ where $f = \alpha_0 + \alpha_1 x + \cdots + \alpha_{d-1} x^{d-1} + \alpha_d x^d \in \mathbb{K}_\omega[x; \tau^{n/m}]$ is any element with $\alpha_0 \neq 0 \neq \alpha_d$. Then $V^\sharp \simeq V(\omega, w_0 w_0^\sharp, f^\sharp)$, where*

$$f^\sharp = \sum_{k=0}^{d} \{2lk\} \cdot \tau^{(2k+1)l}\big(\overline{\alpha_{d-k}}\big) \cdot x^k. \tag{5.27}$$

Here $\{k\}$ is a Pochhammer-type symbol:

$$\{k\} = \{k\}_{q,\tau} = q\tau(q) \cdots \tau^{k-1}(q) \in \mathbb{K}_\omega, \quad k \in \mathbb{Z}_{\geq 0}, \tag{5.28}$$

where $q \in \mathbb{K}_\omega \backslash \{0\}$ is given by

$$q = \sigma^{p_2 + p_3 + \cdots + p_m}(t_1)\sigma^{p_3 + p_4 + \cdots + p_m}(t_2) \cdots \sigma^{p_m}(t_{m-1}) t_m, \tag{5.29}$$

$$t_i = \big(\sigma(t)\sigma^2(t) \cdots \sigma^{p_i - 1}(t)\big)_{\mathfrak{m}_i} \qquad \text{for } i = 1, \ldots, m, \tag{5.30}$$

where $p_i \in \mathbb{Z}_{>0}$ are minimal such that $\sigma^{p_i}(\mathfrak{m}_{i-1}) = \mathfrak{m}_i$, $i = 1, \ldots, m$.

Combining Corollary 5.11 and Theorem 5.12 we obtain the following.

Theorem 5.13. *Let V be any indecomposable weight A-module of the type $V(\omega, w, f)$ with ω real. Thus $\omega \in \Omega$ is a finite real orbit with $m := |B_\omega| > 0$, $w \in \mathbf{D}\backslash\{\varepsilon\}$ is a non-periodic m-word, and $f = \alpha_0 + \alpha_1 x + \cdots + \alpha_d x^d \in \mathbb{K}_\omega[x; \tau^{n/m}]$, $\alpha_d \neq 0$, is an indecomposable element not equal to x^d. Then V has a non-degenerate admissable form iff $w = w_0 w_0^\sharp$ for some m-word $w_0 \in \mathbf{D}\backslash\{\varepsilon\}$ and f is similar to $f^\sharp$ in $\mathbb{K}_\omega[x; \tau^{n/m}]$, where $f^\sharp$ is given by* (5.27).

Remark 5.14. From Theorem 5.12 follows that $f^{\sharp\sharp}$ is similar to f. This is not apparent from (5.27) but by comparing the coefficients of f and $f^{\sharp\sharp}$ one can verify that

$$f^{\sharp\sharp} = \{(2d+1)l\} \cdot \tau^{\frac{n}{m}(m+1)}(f) \cdot \{l\}^{-1}.$$

Using that $\tau^{n/m}(f)$ is similar to f in $\mathbb{K}_\omega[x; \tau^{n/m}]$ we conclude that indeed $f^{\sharp\sharp} \sim f$.

Proof of Theorem 5.12. Let $z_1 z_2 \cdots z_n = w$. It will also be convenient to define $z_j = z_i$ when $j \equiv i \pmod{n}$. Assume for a moment that we have proved (5.27) for the case $z_1 = x$ and suppose that $z_1 = y$. By the shift isomorphism (4.6), which holds also for decomposable f, we have

$$V \simeq V(\omega, w(-lm), \tau^{-l}(f)) = V(\omega, w_0^\sharp w_0, \tau^{-l}(f)) \tag{5.31}$$

where $\tau^{-l}(f) = \tau^{-l}(\alpha_0) + \tau^{-l}(\alpha_1)x + \cdots + \tau^{-l}(\alpha_d)x^d$. By the assumption we then have

$$V(\omega, w_0^\sharp w_0, \tau^{-l}(f))^\sharp \simeq V(\omega, w_0^\sharp w_0, g), \tag{5.32}$$

where

$$g = \sum_{k=0}^{d} \{2lk\} \cdot \tau^{(2k+1)l}\Big(\overline{\tau^{-l}(\alpha_{d-k})}\Big) \cdot x^k = \sum_{k=0}^{d} \tau^{-l}\Big(\tau^{l}(\{2lk\}) \cdot \tau^{(2k+1)l}(\overline{\alpha_{d-k}})\Big) \cdot x^k.$$

Again by (4.6),

$$V(\omega, w_0^\sharp w_0, g) \simeq V(\omega, w_0 w_0^\sharp, \tau^{l}(g)). \tag{5.33}$$

From the formula

$$\tau^{l}(\{2lk\}) = \{l\}^{-1} \cdot \{2lk\} \cdot \tau^{2lk}(\{l\})$$

we see that $\tau^{l}(g) = \{l\}^{-1} \cdot f^\sharp \cdot \{l\}$ which is similar to $f^\sharp$. Combining this fact with the isomorphisms (5.31)-(5.33) we deduce that $V^\sharp \simeq V(\omega, w, f^\sharp)$. Therefore the case $z_1 = y$ follows from the case $z_1 = x$.

Thus we assume for the rest of the proof that $z_1 = x$.

Step 1. Put $a_k = \alpha_{k-1}/\alpha_d$ for $k = 1, 2, \ldots, d$. Let us replace f by $(1/\alpha_d)f = a_1 + a_2 x + \cdots + a_d x^{d-1} + x^d$. This does not change the isomorphism class of the module V. As in the proof of Theorem 5.10, we can construct a basis $[\mathfrak{m}, f_{ks}]$ for $V^\sharp$ such that

$$\begin{aligned} \varphi : V(\omega, w_0 w_0^\sharp, g) &\to V^\sharp \\ [\mathfrak{m}, e_{ks}] &\mapsto [\mathfrak{m}, f_{ks}] \end{aligned} \tag{5.34}$$

is an A-module isomorphism for some g. We use the decomposition (5.21). We put also $(V^\sharp)^{(l)}_{\mathfrak{m}} = (V^\sharp)^{(k)}_{\mathfrak{m}}$ whenever $l \in \mathbb{Z}, l \equiv k \pmod{n}$. By relation (5.20), which holds in $V^\sharp$ since $z_1 = x$, it follows that if $1 \le k \le n$ and $z_k = y$, so that $z_{lm+k} = z^\sharp_k = x$, then

$$Y : (V^\sharp)^{(lm+k)}_{\sigma(\mathfrak{m}_{k-1})} \to (V^\sharp)^{(lm+k-1)}_{\mathfrak{m}_{k-1}}$$

is bijective. For the case $k = lm+1$ it is essential that $a_1 \neq 0$. Put

$$[\sigma(\mathfrak{m}_0), f_{11}] = [\sigma(\mathfrak{m}_0), e^\sharp_{lm+1,1}] \tag{5.35}$$

and recursively

$$[\mathfrak{m}, f_{ks}] = \begin{cases} \sigma(t)^{-1}_{\mathfrak{m}} X[\sigma^{-1}(\mathfrak{m}), f_{ks}], & \sigma^{-1}(\mathfrak{m}) \notin B_\omega, \\ X[\sigma^{-1}(\mathfrak{m}), f_{k-1,s}], & \sigma^{-1}(\mathfrak{m}) \in B_\omega, k > 1, z_k = x, \\ \big(Y|_{(V^\sharp)^{(k+lm)}_{\mathfrak{m}}}\big)^{-1}[\sigma^{-1}(\mathfrak{m}), f_{k-1,s}], & \sigma^{-1}(\mathfrak{m}) \in B_\omega, z_k = y, (k>1), \\ X[\sigma^{-1}(\mathfrak{m}), f_{1,s-1}], & \sigma^{-1}(\mathfrak{m}) \in B_\omega, k = 1, (z_1 = x). \end{cases} \tag{5.36}$$

By induction, $[\mathfrak{m}, f_{ks}] \in (V^\sharp)^{(lm+k)}_{\mathfrak{m}}$ for each $\mathfrak{m} \in \omega$, $s = 1, \ldots, d$, $k = 1, \ldots, n$, $k \equiv j(\mathfrak{m}) \pmod{m}$.

Step 2. We will now show that the g such that $V(\omega, w_0 w_0^\sharp, g) \simeq V^\sharp$, is similar to $f^\sharp$, given by (5.27). Define an operator $Z : (V^\sharp)^{(lm)}_{\mathfrak{m}_0} \to (V^\sharp)^{(lm)}_{\mathfrak{m}_0}$ by

$$Z = Z_n \cdots Z_2 Z_1, \tag{5.37}$$

where $Z_i : (V^\sharp)^{(lm+i-1)}_{\mathfrak{m}_{i-1}} \to (V^\sharp)^{(lm+i)}_{\mathfrak{m}_i}$ are given by

$$Z_i = \begin{cases} (t_i)^{-1} X^{p_i}, & \text{if } z_i = x, \\ (t_i)^{-1} X^{p_i - 1}\big(Y|_{(V^\sharp)^{(lm+i)}_{\sigma(\mathfrak{m}_{i-1})}}\big)^{-1}, & \text{if } z_i = y. \end{cases} \tag{5.38}$$

Recall that $\mathfrak{m}_0, \mathfrak{m}_1, \ldots, \mathfrak{m}_{m-1}$ are the breaks in ω, ordered such that $\mathfrak{m}_{i-1} < \mathfrak{m}_i < \mathfrak{m}_{i+1}$ for $0 < i < m-1$. See also the weight diagram in Figure 1. For an interpretation of the operator Z, see Remark 5.15. It has the following properties:

$$Z[\mathfrak{m}_0, e^\sharp_{lm,1}] = [\mathfrak{m}_0, f_{n1}], \tag{5.39}$$

$$[\mathfrak{m}_0, f_{ns}] = Z^{s-1}[\mathfrak{m}_0, f_{n1}], \qquad \text{for } s = 1, 2, \ldots, d. \tag{5.40}$$

Let us prove (5.39). We have $Z = Z_n \cdots Z_2 Z_1$. First we prove that

$$Z_1[\mathfrak{m}_0, e^\sharp_{lm,1}] = [\mathfrak{m}_1, f_{11}]. \tag{5.41}$$

Since $z_1 = x$, and using relation (5.19) and that $z_{lm+1} = z^\sharp_1 = y$, we have

$$Z_1[\mathfrak{m}_0, e^\sharp_{lm,1}] = (t_1)^{-1} X^{p_1}[\mathfrak{m}_0, e^\sharp_{lm}] = (t_1)^{-1} X^{p_1 - 1}[\sigma(\mathfrak{m}_0), e^\sharp_{lm+1,1}].$$

By definition (5.30) of t_1 and of the vector $[\sigma(\mathfrak{m}_0), f_{11}]$, this is equal to

$$\big(\sigma(t)\sigma^2(t)\cdots\sigma^{p_1-1}(t)\big)_{\mathfrak{m}_1}^{-1}X^{p_1-1}[\sigma(\mathfrak{m}_0), f_{11}].$$

Using that $\sigma(r)_{\sigma(\mathfrak{m})}Xv = Xr_{\mathfrak{m}}v$ for any weight vector v of weight $\mathfrak{m}$ and any $r \in R$, where $r_{\mathfrak{m}}$ denotes $r + \mathfrak{m} \in R/\mathfrak{m}$ as usual, the expression can be rearranged into (recall that $\sigma^{p_1}(\mathfrak{m}_0) = \mathfrak{m}_1$)

$$\big(\sigma(t)^{-1}_{\sigma^{p_1}(\mathfrak{m}_0)}X\big)\big(\sigma(t)^{-1}_{\sigma^{p_1-1}(\mathfrak{m}_0)}X\big)\cdots\big(\sigma(t)^{-1}_{\sigma^{2}(\mathfrak{m}_0)}X\big)[\sigma(\mathfrak{m}_0), f_{11}].$$

By the recursive definition, (5.36), this is equal to $[\sigma^{p_1}(\mathfrak{m}_0), f_{11}] = [\mathfrak{m}_1, f_{11}]$, proving (5.41). Similarly one proves that

$$Z_k[\mathfrak{m}_{k-1}, f_{k-1,1}] = [\mathfrak{m}_k, f_{k1}] \quad \text{for } k = 2, 3, \ldots, n.$$

Combining this with (5.41), (5.39) is proved.
In the same way one shows that $[\mathfrak{m}_0, f_{ns}] = Z[\mathfrak{m}_0, f_{n,s-1}]$ for $s = 2, 3, \ldots, d$. Then (5.40) follows.

Step 3. We have

$$Z[\mathfrak{m}_0, e^\sharp_{lm,s}] = \begin{cases} \{2l\}^{-1}\cdot\big(-\tau^l(\overline{a_{s+1}}/\overline{a_1})[\mathfrak{m}_0, e^\sharp_{lm,1}] + [\mathfrak{m}_0, e^\sharp_{lm,s+1}]\big), & \text{if } s < d, \\ -\{2l\}^{-1}\tau^l(1/\overline{a_1})[\mathfrak{m}_0, e^\sharp_{lm,1}], & \text{if } s = d. \end{cases} \tag{5.42}$$

To prove this, we first prove that if $1 \le k \le lm$, so that $lm + k - 1 < n$, then

$$Z_k[\mathfrak{m}_{k-1}, e^\sharp_{lm+k-1,s}] = (t_k)^{-1}[\mathfrak{m}_k, e^\sharp_{lm+k,s}] \tag{5.43}$$

for any $1 \le s \le d$. Indeed, if $z_k = x$, then

$$\begin{aligned} Z_k[\mathfrak{m}_{k-1}, e^\sharp_{lm+k-1,s}] &= \\ &= (t_k)^{-1}X^{p_k}[\mathfrak{m}_{k-1}, e^\sharp_{lm+k-1,s}] && \text{by definition of } Z_k, \\ &= (t_k)^{-1}X^{p_k-1}[\sigma(\mathfrak{m}_{k-1}), e^\sharp_{lm+k,s}] && \text{by (5.19), since } z_{lm+k} = z_k^\sharp = y, \\ &= (t_k)^{-1}[\mathfrak{m}_k, e^\sharp_{lm+k,s}], && \text{by first case in (5.19).} \end{aligned}$$

We used that $\sigma^{p_k}(\mathfrak{m}_{k-1}) = \mathfrak{m}_k$ in the last step. Similarly, if $z_k = y$, then

$$Y[\sigma(\mathfrak{m}_{k-1}), e^\sharp_{lm+k,s}] = [\mathfrak{m}_{k-1}, e^\sharp_{lm+k-1,s}]$$

by (5.20) since $z_{lm+k} = z_k^\sharp = x$ and $1 < lm + k \le n$. Therefore

$$\big(Y|_{(V^\sharp)^{(lm+k)}_{\sigma(\mathfrak{m}_{k-1})}}\big)^{-1}[\mathfrak{m}_{k-1}, e^\sharp_{lm+k-1,s}] = [\sigma(\mathfrak{m}_{k-1}), e^\sharp_{lm+k,s}]$$

and

$$\begin{aligned} Z_k[\mathfrak{m}_{k-1}, e^\sharp_{lm+k-1,s}] &= (t_k)^{-1}X^{p_k-1}\big(Y_{(V^\sharp)^{(lm+k)}_{\sigma(\mathfrak{m}_{k-1})}}\big)^{-1}[\mathfrak{m}_{k-1}, e^\sharp_{lm+k-1,s}] = \\ &= (t_k)^{-1}X^{p_k-1}[\sigma(\mathfrak{m}_{k-1}), e^\sharp_{lm+k,s}] = \\ &= (t_k)^{-1}[\mathfrak{m}_k, e^\sharp_{lm+k,s}]. \end{aligned}$$

This proves (5.43).
Using (5.43) repeatedly for $k = 1, 2, \dots, lm$ while moving the t_i's to the left, we have

$$\begin{aligned}
&Z_m Z_{m-1} \cdots Z_2 Z_1 [\mathfrak{m}_0, e^\sharp_{lm,s}] = \\
&= Z_m Z_{m-1} \cdots Z_2 \cdot (t_1)^{-1} [\mathfrak{m}_1, e^\sharp_{lm+1,s}] = \\
&= \sigma^{p_2+p_3+\cdots+p_m}(t_1)^{-1} Z_m Z_{m-1} \cdots Z_2 [\mathfrak{m}_1, e^\sharp_{lm+1,s}] = \cdots = \\
&= \sigma^{p_2+p_3+\cdots+p_m}(t_1)^{-1} \sigma^{p_3+p_4+\cdots+p_m}(t_2)^{-1} \cdots \sigma^{p_m}(t_{m-1})^{-1} \cdot (t_m)^{-1} \cdot \\
&\quad \cdot [\mathfrak{m}_m, e^\sharp_{lm+m,s}] = \\
&= q^{-1} \cdot [\mathfrak{m}_0, e^\sharp_{(l+1)m,s}].
\end{aligned}$$

Here we use that, from the definition of Z_k, $Z_k \lambda v = \sigma^{p_k}(\lambda) Z_k v$ for $\lambda \in R/\mathfrak{m}$ and v a weight vector of weight $\mathfrak{m}$, and σ denotes the map $R/\mathfrak{m} \to R/\sigma(\mathfrak{m})$ induced by σ. In particular, $Z_m Z_{m-1} \cdots Z_1 \lambda v = \tau(\lambda) Z_m Z_{m-1} \cdots Z_1 v$ since $\tau = \sigma^p$ and $p = p_1 + p_2 + \cdots p_m$. Therefore, using (5.43) as in the above calculation we get

$$\begin{aligned}
Z_{lm} Z_{lm-1} \cdots Z_1 [\mathfrak{m}_0, e^\sharp_{lm,s}] &= Z_{lm} Z_{lm-1} \cdots Z_{m+1} \cdot q^{-1} [\mathfrak{m}_0, e^\sharp_{(l+1)m,s}] = \\
&= \tau^{l-1}(q^{-1}) Z_{lm} Z_{lm-1} \cdots Z_{m+1} [\mathfrak{m}_0, e^\sharp_{(l+1)m,s}] = \\
&\dots \\
&= \tau^{l-1}(q^{-1}) \tau^{l-2}(q^{-1}) \cdots \tau(q^{-1}) q^{-1} \cdot [\mathfrak{m}_0, e^\sharp_{2lm,s}] = \\
&= \{l\}^{-1} \cdot [\mathfrak{m}_0, e^\sharp_{n,s}]. \qquad (5.44)
\end{aligned}$$

It remains to calculate $Z_{2lm} Z_{2lm-1} \cdots Z_{lm+1} [\mathfrak{m}_0, e^\sharp_{n,s}]$. To calculate $Z_{lm+1} [\mathfrak{m}_0, e^\sharp_{n,s}]$ we need to find, by definition of Z_{lm+1},

$$\big(Y|_{(V^\sharp)^{(1)}_{\sigma(\mathfrak{m}_0)}}\big)^{-1} [\mathfrak{m}_0, e^\sharp_{ns}]$$

because $z_{lm+1} = z^\sharp_1 = y$. By (5.20),

$$Y[\sigma(\mathfrak{m}_0), e^\sharp_{1,s+1}] = [\mathfrak{m}_0, e^\sharp_{n,s}] - \overline{a_{s+1}} \cdot [\mathfrak{m}_0, e^\sharp_{n,d}], \quad \text{if } s < d, \qquad (5.45)$$

$$Y[\sigma(\mathfrak{m}_0), e^\sharp_{1,1}] = -\overline{a_1} \cdot [\mathfrak{m}_0, e^\sharp_{n,d}]. \qquad (5.46)$$

Therefore

$$\begin{aligned}
&\big(Y|_{(V^\sharp)^{(1)}_{\sigma(\mathfrak{m}_0)}}\big)^{-1} [\mathfrak{m}_0, e^\sharp_{n,s}] = \\
&\qquad = \begin{cases} [\sigma(\mathfrak{m}_0), e^\sharp_{1,s+1}] - \sigma(\overline{a_{s+1}}/\overline{a_1}) \cdot [\sigma(\mathfrak{m}_0), e^\sharp_{1,1}], & s < d, \\ -\sigma(1/\overline{a_1}) \cdot [\sigma(\mathfrak{m}_0), e^\sharp_{1,1}], & s = d. \end{cases} \qquad (5.47)
\end{aligned}$$

Applying $(t_1)^{-1} X^{p_1 - 1}$ to both sides of (5.47) we deduce that

$$Z_{lm+1}[\mathfrak{m}_0, e^\sharp_{n,s}] = (t_1)^{-1} \cdot \begin{cases} [\mathfrak{m}_1, e^\sharp_{1,s+1}] - \sigma^{p_1}(\overline{a_{s+1}}/\overline{a_1}) \cdot [\mathfrak{m}_1, e^\sharp_{1,1}], & s < d, \\ -\sigma^{p_1}(1/\overline{a_1}) \cdot [\mathfrak{m}_1, e^\sharp_{1,1}], & s = d. \end{cases} \qquad (5.48)$$

Similar to relation (5.43) we have the formula

$$Z_{lm+k}[\mathfrak{m}_{k-1}, e^\sharp_{k-1,s}] = (t_k)^{-1}[\mathfrak{m}_k, e^\sharp_{k,s}] \quad \text{for } 1 < k \le lm \text{ and } 1 \le s \le d, \tag{5.49}$$

which can be proved using (5.19), (5.20). Note that $t_{lm+k} = t_k$ by the notational assumptions on $\mathfrak{m}_k$ and t_k. Using (5.49) repeatedly we get

$$\begin{aligned} &Z_{(l+1)m}Z_{(l+1)m-1}\cdots Z_{lm+1}[\mathfrak{m}_0, e^\sharp_{n,s}] = \\ &\qquad = q^{-1}\cdot \begin{cases} [\mathfrak{m}_0, e^\sharp_{m,s+1}] - \tau(\overline{a_{s+1}}/\overline{a_1})\cdot[\mathfrak{m}_0, e^\sharp_{m,1}], & s < d, \\ -\tau(1/\overline{a_1})\cdot[\mathfrak{m}_0, e^\sharp_{m,1}], & s = d. \end{cases} \end{aligned} \tag{5.50}$$

Repeating we get

$$\begin{aligned} &Z_{2lm}Z_{2lm-1}\cdots Z_{lm+1}[\mathfrak{m}_0, e^\sharp_{n,s}] = \\ &\qquad = \{l\}^{-1}\cdot \begin{cases} [\mathfrak{m}_0, e^\sharp_{lm,s+1}] - \tau^l(\overline{a_{s+1}}/\overline{a_1})\cdot[\mathfrak{m}_0, e^\sharp_{lm,1}], & s < d, \\ -\tau^l(1/\overline{a_1})\cdot[\mathfrak{m}_0, e^\sharp_{lm,1}], & s = d. \end{cases} \end{aligned} \tag{5.51}$$

Thus, combining (5.44) and (5.51) we obtain (5.42) as desired.

Step 4. Set $b_s = -\overline{a_s}/\overline{a_1}$ for $2 \le s \le d$ and $b_1 = -1/\overline{a_1}$. We claim that for $1 \le s < d$, there are constants $C_{s1}, C_{s2}, \dots, C_{ss} \in \mathbb{K}_\omega$ such that

$$\begin{aligned} [\mathfrak{m}_0, f_{ns}] = C_{s1}\tau^{3l}(b_s)[\mathfrak{m}_0, f_{n1}] + \cdots + C_{s,s-1}\tau^{l+2l(s-1)}(b_2)[\mathfrak{m}_0, f_{n,s-1}] + \\ + C_{s,s}\big(\tau^l(b_{s+1})[\mathfrak{m}_0, e^\sharp_{lm,1}] + [\mathfrak{m}_0, e^\sharp_{lm,s+1}]\big) \end{aligned} \tag{5.52}$$

We prove this by induction on s. If $s = 1$ we can take

$$C_{11} = \{2l\}^{-1} \tag{5.53}$$

by (5.39) and (5.42). Assume (5.52) holds for some $s < d - 1$. Then, using (5.40) and that $Z\lambda = \tau^{2l}(\lambda)Z$ for any $\lambda \in \mathbb{K}_{\mathfrak{m}_0}$, we have

$$\begin{aligned} &[\mathfrak{m}_0, f_{n,s+1}] = Z[\mathfrak{m}_0, f_{ns}] = \\ &= \tau^{2l}(C_{s1})\tau^{5l}(b_s)Z[\mathfrak{m}_0, f_{n1}] + \cdots + \tau^{2l}(C_{s,s-1})\tau^{l+2ls}(b_2)Z[\mathfrak{m}_0, f_{n,s-1}] + \\ &\quad + \tau^{2l}(C_{s,s})\big(\tau^{3l}(b_{s+1})Z[\mathfrak{m}_0, e^\sharp_{lm,1}] + Z[\mathfrak{m}_0, e^\sharp_{lm,s+1}]\big) \end{aligned}$$

By (5.39),(5.40) and (5.42) this equals

$$\begin{aligned} &\tau^{2l}(C_{s,s})\tau^{3l}(b_{s+1})[\mathfrak{m}_0, f_{n1}] + \\ &\quad + \tau^{2l}(C_{s1})\tau^{5l}(b_s)[\mathfrak{m}_0, f_{n2}] + \cdots + \tau^{2l}(C_{s,s-1})\tau^{l+2ls}(b_2)[\mathfrak{m}_0, f_{n,s}] + \\ &\quad + \tau^{2l}(C_{s,s})\{2l\}^{-1}\cdot\big(\tau^l(b_{s+2})[\mathfrak{m}_0, e^\sharp_{lm,1}] + [\mathfrak{m}_0, e^\sharp_{lm,s+2}]\big). \end{aligned}$$

Thus we seek the solution to the following system of equations

$$C_{s+1,1} = \tau^{2l}(C_{s,s}), \tag{5.54}$$

$$C_{s+1,r} = \tau^{2l}(C_{s,r-1}),\ 2 \le r \le s, \tag{5.55}$$

$$C_{s+1,s+1} = \tau^{2l}(C_{s,s})\{2l\}^{-1}. \tag{5.56}$$

From (5.56),(5.53) we deduce

$$C_{s,s} = \{2ls\}^{-1} \qquad 1 \le s < d. \tag{5.57}$$

Repeated use of (5.55) gives For $1 \le r < s < d$ we have

$$\begin{aligned} C_{s,r} &= \tau^{2l}(C_{s-1,r-1}) = \cdots = \tau^{2l(r-1)}(C_{s-r+1,1}) && \text{by (5.55)} \\ &= \tau^{2lr}(C_{s-r,s-r}) && \text{by (5.54)} \\ &= \{2lr\}\{2ls\}^{-1} && \text{by (5.57).} \end{aligned}$$

Substituting this and (5.57) into (5.52) we obtain that, for $1 \le s < d$,

$$\begin{aligned} [\mathfrak{m}_0, f_{ns}] = \{2l\}\{2ls\}^{-1} \cdot \tau^{3l}(b_s) \cdot [\mathfrak{m}_0, f_{n1}] + \\ + \{4l\}\{2ls\}^{-1} \cdot \tau^{5l}(b_{s-1}) \cdot [\mathfrak{m}_0, f_{n2}] + \\ \cdots \\ + \{2l(s-1)\}\{2ls\}^{-1} \cdot \tau^{l+2l(s-1)}(b_2) \cdot [\mathfrak{m}_0, f_{n,s-1}] + \\ + \{2ls\}^{-1}\big(\tau^{l}(b_{s+1})[\mathfrak{m}_0, e^{\sharp}_{lm,1}] + [\mathfrak{m}_0, e^{\sharp}_{lm,s+1}]\big) \end{aligned} \tag{5.58}$$

In particular, taking $s = d-1$ and applying Z we have

$$\begin{aligned} [\mathfrak{m}_0, f_{nd}] = Z[\mathfrak{m}_0, f_{n,d-1}] = \\ = \{4l\}\{2ld\}^{-1} \cdot \tau^{5l}(b_{d-1}) \cdot [\mathfrak{m}_0, f_{n2}] + \\ + \{6l\}\{2ld\}^{-1} \cdot \tau^{7l}(b_{d-2}) \cdot [\mathfrak{m}_0, f_{n3}] + \\ \cdots \\ + \{2l(d-1)\}\{2ld\}^{-1} \cdot \tau^{l+2l(d-1)}(b_2) \cdot [\mathfrak{m}_0, f_{n,d-1}] + \\ + \{2l\}\{2ld\}^{-1} \cdot \big(\tau^{3l}(b_d)[\mathfrak{m}_0, f_{n1}] + \{2l\}^{-1}\tau^{l}(b_1)[\mathfrak{m}_0, e^{\sharp}_{lm,1}]\big) \end{aligned}$$

where we applied (5.42) in the last term. Hence, using that

$$X[\mathfrak{m}_0, e^{\sharp}_{lm,1}] = [\sigma(\mathfrak{m}_0), f_{11}] = [\sigma(\mathfrak{m}_0), e^{\sharp}_{lm+1,1}]$$

by (5.19) and that $z_{lm+1} = z^{\sharp}_1 = y$, together with the relation (recall φ from (5.34))

$$\begin{aligned} X[\mathfrak{m}_0, f_{ns}] = X\varphi\big([\mathfrak{m}_0, e_{ns}]\big) = \varphi\big(X[\mathfrak{m}_0, e_{ns}]\big) = \\ = \varphi\big([\sigma(\mathfrak{m}_0), e_{1,s+1}]\big) = [\sigma(\mathfrak{m}_0), f_{1,s+1}] \end{aligned}$$

holding for $s < d$, we obtain that

$$\begin{aligned} X[\mathfrak{m}_0, f_{nd}] = \sigma\big(\{2ld\}^{-1}\tau^{l}(b_1)\big) \cdot [\sigma(\mathfrak{m}_0), f_{11}] + \\ + \sigma\big(\{2l\}\{2ld\}^{-1}\tau^{3l}(b_d)\big) \cdot [\sigma(\mathfrak{m}_0), f_{12}] + \\ + \sigma\big(\{4l\}\{2ld\}^{-1}\tau^{5l}(b_{d-1})\big) \cdot [\sigma(\mathfrak{m}_0), f_{13}] + \\ \cdots \\ + \sigma\big(\{2l(d-1)\}\{2ld\}^{-1}\tau^{l+2l(d-1)}(b_2)\big) \cdot [\sigma(\mathfrak{m}_0), f_{1d}]. \end{aligned}$$

Resubstituting $b_1 = -1/\overline{\alpha_1} = -\overline{\alpha_d}/\overline{\alpha_0}$ and $b_s = -\overline{\alpha_s}/\overline{\alpha_1} = -\overline{\alpha_{s-1}}/\overline{\alpha_0}$ (for $s > 1$), we conclude that, in view of the final case in relation (5.16), that the map $V(\omega, w_0 w_0^\sharp, g) \to V^\sharp$, $[\mathfrak{m}, e_{ks}] \mapsto [\mathfrak{m}, f_{ks}]$ will be an A-module isomorphism if g is given by

$$\begin{aligned}
\{2ld\} \cdot g = {} & \tau^l(\overline{\alpha_d}/\overline{\alpha_0}) + \\
& + \{2l\} \cdot \tau^{3l}(\overline{\alpha_{d-1}}/\overline{\alpha_0}) \cdot x + \\
& + \{4l\} \cdot \tau^{5l}(\overline{\alpha_{d-2}}/\overline{\alpha_0}) \cdot x^2 + \\
& \quad \cdots \\
& + \{2l(d-1)\} \cdot \tau^{l+2l(d-1)}(\overline{\alpha_1}/\overline{\alpha_0}) \cdot x^{d-1} + \\
& + \{2ld\} \cdot x^d.
\end{aligned}$$

Thus $\{2ld\} \cdot g \cdot \tau^l(\overline{\alpha_0}) = f^\sharp$ so g is similar to $f^\sharp$. This finishes the proof that $V^\sharp \simeq V(\omega, w_0 w_0^\sharp, f^\sharp)$. □

Remark 5.15. The indecomposable weight module $V = V(\omega, w, f)$, $w = z_1 \cdots z_n$, has the the following characterizing properties:

1) the operator $Z = Z(w) : V_{\mathfrak{m}_0} \to V_{\mathfrak{m}_0}$ given by $Z = Z_n \cdots Z_2 Z_1$ where

$$Z_i = \begin{cases} (t_i)^{-1} X^{p_i}, & z_i = x, \\ (t_i)^{-1} X^{p_i - 1} Y^{-1}, & z_i = y, \end{cases}$$

is well-defined and single-valued (since w is non-periodic), and

2) giving $V_{\mathfrak{m}_0}$ the structure of a module over $\mathbb{K}_\omega[x; \tau^{n/m}]$ by

$$x.v = Zv, \quad v \in V_{\mathfrak{m}_0},$$

there exists a nonzero vector in $V_{\mathfrak{m}_0}$ which is annihilated by f.

What we prove in Theorem 5.10 is that $Z(w^\sharp)$ is well-defined on the $\mathfrak{m}_0$-weight space of $V(\omega, w, f)^\sharp$, while in Theorem 5.12 we prove that when $V = V(\omega, w_0 w_0^\sharp, f)$, the space $(V^\sharp)_{\mathfrak{m}_0}$ contains a nonzero vector annihilated by a skew polynomial similar to $f^\sharp$. Therefore $V^\sharp \simeq V(\omega, w_0 w_0^\sharp, f^\sharp)$.

6 Examples

6.1 Noncommutative type-A Kleinian singularities

Let $R = \mathbb{C}[H]$ and $\sigma \in \mathrm{Aut}_{\mathbb{C}}(H)$ be given by $\sigma(H) = H - 1$ and $t \in R$ be arbitrary. The generalized Weyl algebra $A = R(\sigma, t)$ was studied in [B] and [H]. For example, all simple modules (not only weight modules) were classified in [B]. Let $*$ be the $\mathbb{R}$-algebra automorphism of R given by $i^* = -i$, $H^* = H$. Suppose that $t^* = t$ i.e. that $t = f(H)$, where the polynomial f has real coefficients. Since any orbit is infinite, Theorem 5.2 and Theorem 5.3 implies that an indecomposable weight module with real support has a non-degenerate admissable form iff it is simple.

6.2 The enveloping algebra of $\mathfrak{sl}_2$

Let $R = \mathbb{C}[h,t]$ and let $\sigma \in \mathrm{Aut}_{\mathbb{C}}(R)$ be given by $\sigma(h) = h-2$, $\sigma(t) = t+h$. Then $A = R(\sigma,t) \simeq U(\mathfrak{sl}_2)$. Define $* \in \mathrm{Aut}_{\mathbb{R}}(R)$ by $h^* = h, t^* = t, i^* = -i$. Here, as in the previous example, all orbits are infinite so indecomposable weight modules with real support are non-degenerately unitarizable iff they are simple.

By induction one checks that $\sigma^n(t) = -n^2 + (h+1)n + t$, $\forall n \in \mathbb{Z}$. Thus, for any $\mu, \alpha \in \mathbb{R}$,

$$\lim_{n\to\pm\infty} \{\sigma^n(t) \bmod (h-\mu, t-\alpha)\} = \lim_{n\to\pm\infty} -n^2 + (\mu+1)n + \alpha = -\infty.$$

In view of formulas (5.1),(5.2),(5.3), this shows that any non-degenerate symmetric admissable form on an infinite-dimensional simple weight module with real support is necessarily indefinite.

On the other hand, on a finite-dimensional simple weight module $V(N)$ (with highest weight $N \in \mathbb{Z}_{\geq 0}$ and of dimension $N+1$), the form Ψ_λ given by (5.2) with $\lambda > 0$ is positive definite because

$$\sigma^n(t) \bmod (t, h-N) = n(N-n+1) > 0$$

for $n = 1,2,\dots,N$ so that $\Psi_\lambda(Y^n e_0, Y^n e_0) > 0$ for $n = 0,1,\dots,N$.

6.3 The quantum enveloping algebra of $\mathfrak{sl}_2$

Let $R = \mathbb{C}[K,K^{-1},t]$ and $q \in \mathbb{C}\backslash\{-1,0,1\}$. Define $\sigma \in \mathrm{Aut}_{\mathbb{C}}(R)$ by $\sigma(K) = q^{-2}K, \sigma(t) = t + \frac{K-K^{-1}}{q-q^{-1}}$. Then $R(\sigma,t) \simeq U_q(\mathfrak{sl}_2)$. We assume here that q^2 is a root of unity of order $p > 1$. Let $* \in \mathrm{Aut}_{\mathbb{R}}(R)$ be given by $K^* = K^{-1}$, $i^* = -i$, $t^* = t$. One verifies that σ commutes with $*$ and that σ has order p. All orbits have p elements and are torsion trivial. Let $\omega \in \Omega$ and $\mathfrak{m} = (K-\mu, t-\alpha) \in \omega$. Then ω is real iff $\mathfrak{m}^* = \mathfrak{m}$ which holds iff $|\mu| = 1$ and $\alpha \in \mathbb{R}$. Assume ω is real and put $\mathfrak{m}(\omega) = \mathfrak{m}$. We identify $\mathbb{K}_\omega = R/\mathfrak{m}$ with $\mathbb{C}$. The real number

$$\xi = \big(\sigma(t)\sigma^2(t)\cdots\sigma^p(t)\big)_{\mathfrak{m}} = \prod_{k=0}^{p-1}\Big(\alpha + \sum_{i=0}^{k} \frac{q^{-2i}\mu - q^{2i}\mu^{-1}}{q-q^{-1}}\Big) \tag{6.1}$$

is nonzero iff there are no breaks in ω.

Assume that $\xi \neq 0$ and consider the modules $V(\omega, f)$. Since $\sigma^p = \mathrm{Id}$, the skew Laurent polynomial ring $\mathbb{K}_\omega[x,x^{-1};\tau]$, to which f belongs, is just the ordinary commutative Laurent polynomial ring $P = \mathbb{C}[x,x^{-1}]$. Similarity in P just means equality up to multiplication by nonzero homogenous term. Any indecomposable element in P is similar to $f = (x-a)^d$ for some $a \in \mathbb{C}\backslash\{0\}$, $d \geq 1$. By Theorem 5.6, $V(\omega,f)^\sharp \simeq V(\omega, f^\sharp)$ where $f^\sharp = (\xi x)^d((\xi x)^{-1} - \overline{a})^d = (1-\overline{a}\xi x)^d \sim (x - (\overline{a}\xi)^{-1})^d$. Thus we conclude that $V(\omega,f)$, where ω is a real orbit without breaks containing $(K-\mu, t-\alpha)$ and $f = (x-a)^d$, has a non-degenerate admissable form iff $a = (\overline{a}\xi)^{-1}$, that is, iff $|a|^2 = \xi^{-1}$, where ξ is given by (6.1). It would be interesting to determine the values of α and μ for which ξ is positive so that $|a|^2 = \xi^{-1}$ can hold.

We only note here that for any fixed μ, the quantity ξ is a polynomial of degree p in α with positive leading coefficient and thus $\xi > 0$ if α is sufficiently big.

Assume now that $\xi = 0$. Then ω has breaks and we can assume $\alpha = 0$. Recall that the break $\mathfrak{m}_0 = \mathfrak{m}(\omega) = \mathfrak{m}$. For $k \geq 0$ we have

$$\sigma^{k+1}(t) = t + \sum_{i=0}^{k} \frac{q^{-2i}K - q^{2i}K^{-1}}{q - q^{-1}}.$$

Thus the reduction modulo $\mathfrak{m}_0$ is

$$\big(\sigma^{k+1}(t)\big)_{\mathfrak{m}_0} = \sum_{i=0}^{k} \frac{q^{-2i}\mu - q^{2i}\mu^{-1}}{q - q^{-1}} = \frac{(1 - q^{2(k+1)})(1 - \mu^2 q^{-2k})}{\mu q (q - q^{-1})^2} \tag{6.2}$$

This shows that, for $0 \leq k \leq p - 2$,

$$\sigma^{-(k+1)}(\mathfrak{m}_0) \in B_\omega \iff \mu^2 = q^{2k}. \tag{6.3}$$

By (6.3) we have

$$B_\omega = \begin{cases} \{\mathfrak{m}_0, \mathfrak{m}_1 = \sigma^{-(k+1)}(\mathfrak{m}_0)\}, & \text{if } \mu^2 = q^{2k} \text{ where } 0 \leq k \leq p-2, \\ \{\mathfrak{m}_0\}, & \text{if } \mu \notin \{\pm 1, \pm q, \dots, \pm q^{p-2}\}, \end{cases}$$

Call μ *generic* if $\mu \notin \{\pm 1, \pm q, \dots, \pm q^{p-2}\}$ and *specific* otherwise. If μ is specific, we let r $(0 \leq r \leq p-2)$ denote the unique integer such that $\mu^2 = q^{2r}$. Let $m = |B_\omega|$. By (6.3), $m = 1$ if μ is generic and $m = 2$ if μ is specific. Recall the definition of p_i from Section 4.1. For specific μ we have $p_1 = p - (r+1)$ and $p_2 = r + 1$.

By Theorem 5.8, a module of the form $V(\omega, j, w)$ has a non-degenerate admissable form iff it is simple, which holds iff $w = \varepsilon$, the empty word. If μ is generic then there is only one such module, $V(\omega, 0, \varepsilon)$. If μ is specific then there are two such modules, $V(\omega, 0, \varepsilon)$ and $V(\omega, 1, \varepsilon)$.

If $V = V(\omega, w = z_1 \cdots z_n, f = (x-a)^d)$, then by Theorem 5.13, V has a non-degenerate admissable form iff $w = w_0 w_0^\sharp$ where w_0 is a non-empty m-word (so for generic μ the word w_0 is arbitrary, while for specific μ, it has to be of even length) and f is similar to $f^\sharp$ in $\mathbb{C}[x]$. Let $(a;s)_i$ denote the shifted factorial

$$(a;s)_i = (1-a)(1-as)\cdots(1-as^{i-1})$$

and for $j < i$ let $(a;s)_i^{(j)}$ denote $(a;s)_i$ but with the factor $(1 - as^j)$ omitted. By (5.27) the polynomial $f^\sharp$ is given by

$$f^\sharp = \sum_{k=0}^{d} Q^{nk}\overline{\alpha_{d-k}} \cdot x^k = (Q^n x)^d \cdot \overline{f\big((Q^n x)^{-1}\big)} = (1 - Q^n \overline{a} x)^d \sim \big(x - (Q^n \overline{a})^{-1}\big)^d,$$

where Q is the nonzero real number given by

$$Q = t_1 = \frac{(q^2;q^2)_{p-1} \cdot (\mu^2;q^{-2})_{p-1}}{(\mu q (q - q^{-1})^2)^{p-1}}, \quad \text{if } \mu \text{ is generic,} \tag{6.4}$$

and

$$Q=\sigma^{p_2}(t_1)t_2=\frac{(q^2;q^2)^{(r)}_{p-1}\cdot(\mu^2;q^{-2})^{(r)}_{p-1}}{(\mu q(q-q^{-1})^2)^{p-2}},\quad \text{if } \mu \text{ is specific, } \mu^2=q^{2r}. \tag{6.5}$$

We conclude that $V=V(\omega,w=z_1\cdots z_n,f=(x-a)^d)$, ($\omega$ a real orbit containing a break $\mathfrak{m}=(t,K-\mu)$) has a non-degenerate admissable form iff $w=w_0w_0^\sharp$, where $w_0\in\mathbf{D}\backslash\{\varepsilon\}$ has even length if μ is specific, and $|a|^2=Q^{-n}$. Since n is even, solutions $a\in\mathbb{C}$ to this equation always exist.

Irreducible representations of $U_q(\mathfrak{sl}_2)$ which are unitarizable with respect to a positive definite form were described in [V]. This corresponds to the case when all the factors in (6.1) are nonnegative.

6.4 When R is a field

We note that in the special case when $R=\mathbb{K}$ is a field, there is only one orbit ω_0 consiting of the zero ideal alone. The orbit ω_0 is real, and contains a break iff $t=0$. Furthermore, ω_0 is torsion trivial iff σ is trivial. An indecomposable weight module over $A=R(\sigma,t)$ is then of the form $V(\omega,f)$ if $t\neq 0$, where $f\in\mathbb{K}[x,x^{-1};\sigma]$ and $V(\omega,j,w)$ or $V(\omega,w,f)$ if $t=0$, where $f\in\mathbb{K}[x;\sigma^n]$ ($n=|w|$). This shows that any skew polynomial ring can occur.

6.5 An example of a module of the second kind

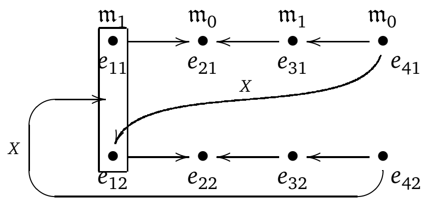
Let $R=\mathbb{C}[u,t]$, $\sigma\in\mathrm{Aut}_{\mathbb{C}}(R)$ defined by $\sigma(u)=1-u,\sigma(t)=t$. Then the orbits have the form $\omega_{\mu,\alpha}=\{(u-\mu,t-\alpha),(u-(1-\mu),t-\alpha)\}$, where $\mu,\alpha\in\mathbb{C}$. All orbits are torsion trivial and have two elements, except for $\omega_{1/2,\alpha}$ which has only one element. The orbit $\omega_{\mu,\alpha}$ contains no breaks if $\alpha\neq 0$, and all elements of $\omega_{\mu,0}$ are breaks. Define $*\in\mathrm{Aut}_{\mathbb{R}}(R)$ by $u^*=u$, $t^*=t$, $i^*=-i$. Then $\omega_{\mu,\alpha}$ is real iff $\mu,\alpha\in\mathbb{R}$.

Let $\omega=\omega_{0,0}$. Let $\mathfrak{m}(\omega)=\mathfrak{m}_0=(u,t)$ and $\sigma(\mathfrak{m}_0)=\mathfrak{m}_1=(u-1,t)$. Then $B_\omega=\omega$, $p=|\omega|=2$, $m=|B_\omega|=2$. We identify $\mathbb{K}_\omega=R/\mathfrak{m}(\omega)$ with $\mathbb{C}$. The map τ is the identity since ω is torsion trivial. Let $f=a_1+a_2x+x^2\in\mathbb{C}[x]$, $a_1\neq 0$, let $w=xxyy$ and let $V=V(\omega,w,f)$. The weight module V is decomposable iff f has distinct roots.

Since $\sigma(\mathfrak{m}_0)=\mathfrak{m}_1$ and $\sigma(\mathfrak{m}_1)=\mathfrak{m}_0$, the integers p_1 and p_2 (defined in Section 4.1) both equal one. Thus, recalling definitions (5.29), (5.30) of q, t_1,t_2, we have $t_1=t_2=1$ and $q=1$. By Theorem 5.12, $V^\sharp\simeq V(\omega,w,f^\sharp)$ where $f^\sharp=1+\overline{a_2}x+\overline{a_1}x^2\sim 1/\overline{a_1}+\overline{a_2}/\overline{a_1}\cdot x+x^2$. Thus $V\simeq V^\sharp$ iff $a_1=1/\overline{a_1}$, $a_2=\overline{a_2}/\overline{a_1}$.

The module V has the following structure. We have $V=V_{\mathfrak{m}_0}\oplus V_{\mathfrak{m}_1}$. Since $j(\mathfrak{m}_0)=0$ and $j(\mathfrak{m}_1)=1$, $V_{\mathfrak{m}_0}$ has a basis $\{e_{21},e_{22},e_{41},e_{42}\}$ and $V_{\mathfrak{m}_1}$ has a basis

$\{e_{11}, e_{12}, e_{31}, e_{32}\}$.



The module structure on V is given by the following, where $s = 1, 2$:

$$\begin{cases} Xe_{1s} = e_{2s}, \\ Xe_{2s} = Xe_{3s} = 0, \\ Xe_{41} = e_{12}, \\ Xe_{42} = -a_1 e_{11} - a_2 e_{12}, \end{cases} \qquad \begin{cases} Ye_{1s} = 0, \\ Ye_{2s} = 0, \\ Ye_{3s} = e_{2s}, \\ Ye_{4s} = e_{3s}. \end{cases}$$

Let us show explicitly that $V^\sharp \simeq V(\omega, w, f^\sharp)$. Let $\{e^\sharp_{ks} : 1 \le k \le 4, s = 1, 2\}$ be the dual basis in $V^\sharp$, i.e. $e^\sharp_{ks}(e_{ij}) = \delta_{ki}\delta_{sj}$. Then $\{e^\sharp_{2s}, e^\sharp_{4s} : s = 1, 2\}$ is a basis for $(V^\sharp)_{\mathfrak{m}_0}$ and $\{e^\sharp_{1s}, e^\sharp_{3s} : s = 1, 2\}$ is a basis for $(V^\sharp)_{\mathfrak{m}_1}$. For $s = 1, 2$ we have

$$\begin{cases} Xe^\sharp_{1s} = 0, \\ Xe^\sharp_{2s} = e^\sharp_{3s}, \\ Xe^\sharp_{3s} = e^\sharp_{4s}, \\ Xe^\sharp_{4s} = 0, \end{cases} \qquad \begin{cases} Ye^\sharp_{11} = -\overline{a_1} e^\sharp_{42}, \\ Ye^\sharp_{12} = e^\sharp_{41} - \overline{a_2} e^\sharp_{42}, \\ Ye^\sharp_{2s} = e^\sharp_{1s}, \\ Ye^\sharp_{3s} = Ye^\sharp_{4s} = 0. \end{cases}$$

Set $b_1 = -1/\overline{a_1}$ and $b_2 = -\overline{a_2}/\overline{a_1}$ and

$$\begin{cases} f_{11} = e^\sharp_{31}, \\ f_{21} = e^\sharp_{41}, \\ f_{31} = b_2 e^\sharp_{11} + e^\sharp_{12}, \\ f_{41} = b_2 e^\sharp_{21} + e^\sharp_{22}, \end{cases} \qquad \begin{cases} f_{12} = b_2 e^\sharp_{31} + e^\sharp_{32}, \\ f_{22} = b_2 e^\sharp_{41} + e^\sharp_{42}, \\ f_{32} = (b_1 + b_2^2) e^\sharp_{11} + b_2 e^\sharp_{12}, \\ f_{42} = (b_1 + b_2^2) e^\sharp_{21} + b_2 e^\sharp_{22}. \end{cases} \tag{6.6}$$

We have $Xf_{42} = b_1 f_{11} + b_2 f_{12}$. Set $g(x) = -b_1 - b_2 x + x^2$. Then one verifies that $V^\sharp \simeq V(\omega, w, g)$ via the map $f_{ks} \mapsto e_{ks}$. Since $g \sim f^\sharp$ we deduce that $V^\sharp \simeq V(\omega, w, f^\sharp)$. Thus, since polynomials in $\mathbb{C}[x]$ are similar iff they differ by a multiplicative scalar, $V \simeq V^\sharp$ iff $f = g$, i.e. iff $a_1 = 1/\overline{a_1}$ and $a_2 = \overline{a_2}/\overline{a_1}$. It is easy to check that

$$E := \{(a_1, a_2) \in \mathbb{C}^2 \;:\; a_1 = 1/\overline{a_1}, a_2 = \overline{a_2}/\overline{a_1}\} = \{(\zeta^2, x\zeta) \;:\; x \in \mathbb{R}, \zeta \in \mathbb{C}, |\zeta| = 1\}$$

and $(\zeta_1^2, x_1\zeta_1) = (\zeta_2^2, x\zeta_2)$ iff $(\zeta_1, x_1) = \pm(\zeta_2, x_2)$.

If $(a_1, a_2) \in E$, the non-degenerate admissable $\mathbb{C}$-form $\widehat{\Phi}$ corresponding to the isomorphism $\Phi : V \to V^\sharp$, $\Phi(e_{ks}) = f_{ks}$ is

$$\widehat{\Phi}(e_{ks}, e_{lr}) = \big(\Phi(e_{ks})\big)(e_{lr}) = f_{ks}(e_{lr}).$$

Using (6.6) and that $(e^\sharp_{ks})(e_{lr}) = \delta_{kl}\delta_{sr}$, and explicit matrix for $\widehat{\Phi}$ in the basis $\{e_{ks}\}$ can be written down. As a curious aside we mention that the zero-set of the determinant of the symmetrized form $\widehat{\Phi} + \widehat{\Phi}^\sharp$ as a function of $z \in \mathbb{C}\backslash\{1\}$ via $a_2 = 1 - z$, $a_1 = (1 - z)/(1 - \overline{z})$ is the curve known as the *limaçon trisectrix*. It has certain special geometric properties and is parametrized in polar coordinates by $r = 1 + 2\cos\theta$. Thus, for points outside of this curve, $\widehat{\Phi} + \widehat{\Phi}^\sharp$ is the unique symmetric non-degenerate admissable form, by Remark 3.22.

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References

[B] V. V. Bavula, *Generalized Weyl algebras and their representations*, Algebra i Analiz **4** No. 1 (1992) 75–97; English transl. in St Petersburg Math. J. **4** (1993), 71–92.

[BB] V. V. Bavula, V. Bekkert, *Indecomposable representations of generalized Weyl algebras*, Comm. Algebra, **28** (2000) 5067–5100.

[BL] V. V. Bavula, T. H. Lenagan, *Generalized Weyl algebras are tensor Krull minimal*, J. Algebra, **239** No. 1 (2001) 93–111.

[BO] V. V. Bavula, F. van Oystaeyen, *Simple modules of the Witten-Woronowicz algebra*, J. Algebra, **271** (2004) 827–845.

[CS] T. Cassidy, B. Shelton, *Basic properties of generalized down-up algebras*, J. Algebra, **279** (2004) 402–421.

[DGO] Yu. A. Drozd, B. L. Guzner, S. A. Ovsienko, *Weight modules over generalized Weyl algebras*, J. Algebra, **184** (1996) 491–504.

[G] J. Germoni, *Contravariant forms for weight modules*, preprint.

[H] T. J. Hodges, *Noncommutative deformation of type-A Kleinian singularities*, J. Algebra, **161** No. 2 (1993) 271–290.

[J] D. A. Jordan, *Down-up algebras and ambiskew polynomial rings*, J. Algebra, **228** (2000) 311–346.

[KS] E. Kissin, V. Shulman, *Representations on Krein spaces and derivations of C^*-algebras*, Pitman Monographs and Surveys in Pure and Applied Mathematics 89, Longman, Harlow, (1997).

[MS] G. Morchio, F. Strocchi, *Representations of $*$-algebras in indefinite inner product spaces*, CMS Conf. Proc. **29** (2000) 491–503.

[MT1] V. Mazorchuk, L. Turowska, *Existence and unicity of σ-forms on finite-dimensional modules* Meth. Funct. Anal. Top. **7** (2001) 53–62.

[MT2] V. Mazorchuk, L. Turowska, *σ-unitarizability of simple weight modules*, Proc. Int. Conf. "Information Processes of formation of Open Systems", Vologda, 26–28 June 2001, 253–257.

[MT3] V. Mazorchuk, L. Turowska, *$*$-Representations of twisted generalized Weyl constructions*, Algebr. Represent. Theory **5** No. 2 (2002) 163–186.

[V] E. Vaysleb, *Infinite-dimensional $*$-representations of the Sklyanin algebra and of the quantum algebra $U_q(sl(2))$*, Sel. Math. **12** No. 1 (1993) 57–73.

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Paper IV

The elliptic $GL(n)$ dynamical quantum group as an $\mathfrak{h}$-Hopf algebroid

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Abstract

Using the language of $\mathfrak{h}$-Hopf algebroids which was introduced by Etingof and Varchenko, we construct a dynamical quantum group, $\mathscr{F}_{\mathrm{ell}}(GL(n))$, from Felder's elliptic solution of the quantum dynamical Yang-Baxter equation with spectral parameter associated to the Lie algebra $\mathfrak{sl}_n$. First we apply the generalized FRST construction and obtain a bialgebroid $\mathscr{F}_{\mathrm{ell}}(M(n))$. Then, analogues of the exterior algebra and their matrix elements, elliptic minors, are defined and studied. In particular we define the elliptic determinant and prove that it is grouplike and almost central. Localizing at this determinant and constructing an antipode we obtain the $\mathfrak{h}$-Hopf algebroid $\mathscr{F}_{\mathrm{ell}}(GL(n))$.

1 Introduction

The quantum dynamical Yang-Baxter (QDYB) equation was introduced by Gervais and Neveu [GN84]. It was realized by Felder [F95] that this equation is equivalent to the Star-Triangle relation in statistical mechanics. It is a generalization of the quantum Yang-Baxter equation, involving an extra, so called dynamical, parameter. In [F95] an interesting elliptic solution to the QDYB equation with spectral parameter was given, adapted from the $A_n^{(1)}$ solution to the Star-Triangle relation constructed in [JKMO88]. Felder also defined the concept of a representation of the corresponding quantum group $E_{\tau,\eta}(\mathfrak{g})$. These representations were further studied in [FV96] in the case $\mathfrak{g} = \mathfrak{sl}_2$.

In [FV97], the authors consider exterior and symmetric powers of the vector representation of the elliptic quantum group $E_{\tau,\eta}(\mathfrak{gl}_n)$. In particular they obtain the quantum determinant and mention how to prove that it is central in the so called operator algebra. This is also proved in [TV01] (appendix B) in more detail and in [ZSY03] using a different approach.

An algebraic framework for studying dynamical R-matrices was introduced in [EV98]. There the authors defined the notion of $\mathfrak{h}$-bialgebroids and $\mathfrak{h}$-Hopf algebroids, a special case of the Hopf algebroids defined by Lu [L96]. They also show, using a generalized version of the FRST construction, how to associate to every solution R of the quantum dynamical Yang-Baxter equation (without spectral parameter) an $\mathfrak{h}$-bialgebroid. Under some extra condition they get an $\mathfrak{h}$-Hopf algebroid by adjoining formally the matrix elements of the inverse L-matrix. This correspondence gives a tensor equivalence between the category of representations of the R-matrix and the category of so called dynamical representations of the $\mathfrak{h}$-bialgebroid.

In this paper we define an $\mathfrak{h}$-Hopf algebroid associated to Felder's elliptic R-matrix with both dynamical and spectral parameter for $\mathfrak{g} = \mathfrak{sl}_n$.

This generalizes the spectral elliptic dynamical $GL(2)$ quantum group from [KNR04] and the non-spectral trigonometric dynamical $GL(n)$ quantum group from [KN06]. As in [KNR04], this is done by first using the the generalized FRST construction, modified to also include spectral parameters. In addition to the usual RLL-relation, residual relations must be added "by hand" to be able to prove that different expressions for the determinant are equal.

Instead of, as in [EV98], adjoining formally all the matrix elements of the inverse L-matrix, we want to adjoin only the inverse of the determinant, as in [KNR04]. Then we express the antipode using this inverse. The main problem is to find the correct formula for the determinant, to prove that it is central, and to provide row and column expansion formulas for the determinant in the setting of $\mathfrak{h}$-bialgebroids.

The plan of this paper is as follows. After introducing some notation in Section 2 and the R-matrix in Section 3, we recall the definition of $\mathfrak{h}$-bialgebroids and the generalized FRST construction in Section 4. The relations for the resulting algebra $\mathscr{F}_{\mathrm{ell}}(M(n))$ are written more explicitly in Section 5.

Left and right analogs of the exterior algebra over $\mathbb{C}^n$ is defined in Section 6 in a similar way as in [KN06]. They are certain comodule algebras over $\mathscr{F}_{\mathrm{ell}}(M(n))$.

The matrix elements of these corepresentations are generalized minors depending on a spectral parameter. Their properties are studied in Section 7. In particular we show that the left and right versions of the minors in fact coincide.

In Section 8 we show that one can define a cobraiding on $\mathscr{F}_{\mathrm{ell}}(M(n))$, in the sense of [R04]. We use this and the the ideas as in [FV97] and [TV01] to prove that the determinant commutes with the generators for almost all values of the spectral parameters. This implies that the determinant is central in the operator algebra as shown in [FV97].

Finally, in Section 9.1 we prove Laplace expansion formulas for the quantum elliptic minors and define the antipode in Section 9.2.

2 Notation

Let $p, q \in \mathbb{R}$, $0 < p, q < 1$. We assume p, q are generic in the sense that if $p^a q^b = 1$ for some $a, b \in \mathbb{Z}$, then $a = b = 0$.

Denote by θ the normalized Jacobi theta function:

$$\theta(z) = \theta(z; p) = \prod_{j=0}^{\infty}(1 - zp^j)(1 - p^{j+1}/z). \tag{2.1}$$

It is holomorphic on $\mathbb{C}^\times := \mathbb{C}\backslash\{0\}$ with zero-set $\{p^k : k \in \mathbb{Z}\}$ and satisfies

$$\theta(z^{-1}) = \theta(pz) = -z^{-1}\theta(z) \tag{2.2}$$

and the addition formula

$$\theta(xy, x/y, zw, z/w) = \theta(xw, x/w, zy, z/y) + (z/y)\theta(xz, x/z, yw, y/w), \tag{2.3}$$

where we use the notation

$$\theta(z_1, \ldots, z_n) = \theta(z_1)\cdots\theta(z_n).$$

Recall also the Jacobi triple product identity, which can be written

$$\sum_{k\in\mathbb{Z}}(-z)^k p^{\frac{k(k-1)}{2}} = \theta(z)\prod_{j=1}^{\infty}(1-p^j). \tag{2.4}$$

It will sometimes be convenient to use the auxiliary function E given by

$$E : \mathbb{C} \to \mathbb{C}, \qquad E(s) = q^s\theta(q^{-2s}). \tag{2.5}$$

Relation (2.2) implies that $E(-s) = -E(s)$.

The set $\{1, 2, \ldots, n\}$ will be denoted by $[1, n]$.

3 The R-matrix

Let $\mathfrak{h}$ be a complex vector space, viewed as an abelian Lie algebra, $\mathfrak{h}^*$ its dual space and $V = \bigoplus_{\lambda\in\mathfrak{h}^*} V_\lambda$ a diagonalizable $\mathfrak{h}$-module. A dynamical R-matrix is by definition a meromorphic function

$$R : \mathfrak{h}^* \times \mathbb{C}^\times \to \mathrm{End}_{\mathfrak{h}}(V \otimes V)$$

satisfying the quantum dynamical Yang-Baxter equation with spectral parameter (QDYBE):

$$\begin{aligned} R(\lambda, \frac{z_2}{z_3})^{(23)} R(\lambda - h^2, \frac{z_1}{z_3})^{(13)} R(\lambda, \frac{z_1}{z_2})^{(12)} = \\ = R(\lambda - h^3, \frac{z_1}{z_2})^{(12)} R(\lambda, \frac{z_1}{z_3})^{(13)} R(\lambda - h^1, \frac{z_2}{z_3})^{(23)}. \end{aligned} \tag{3.1}$$

Equation (3.1) is an equality in the algebra of meromorphic functions $\mathfrak{h}^* \times \mathbb{C}^\times \to \mathrm{End}(V^{\otimes 3})$. Upper indices are leg-numbering notation and h indicates the action of $\mathfrak{h}$. For example,

$$R(\lambda - h^3, \frac{z_1}{z_2})^{(12)}(u \otimes v \otimes w) = R(\lambda - \alpha, \frac{z_1}{z_2})(u \otimes v) \otimes w, \qquad \text{if } w \in V_\alpha.$$

An R-matrix R is called *unitary* if

$$R(\lambda, z)R(\lambda, z^{-1})^{(21)} = \mathrm{Id}_{V\otimes V} \tag{3.2}$$

as meromorphic functions on $\mathfrak{h}^* \times \mathbb{C}^\times$ with values in $\mathrm{End}_{\mathfrak{h}}(V \otimes V)$.

In the example we study, $\mathfrak{h}$ is the Cartan subalgebra of $\mathfrak{sl}(n)$. Thus $\mathfrak{h}$ is the abelian Lie algebra of all traceless diagonal complex $n \times n$ matrices. Let V be the $\mathfrak{h}$-module $\mathbb{C}^n$ with standard basis $e_1, \ldots, e_n$. Define $\omega(i) \in \mathfrak{h}^*$ $(i = 1, \ldots, n)$ by

$$\omega(i)(h) = h_i, \quad \text{if } h = \mathrm{diag}(h_1, \ldots, h_n) \in \mathfrak{h}.$$

We have $V = \bigoplus_{i=1}^n V_{\omega(i)}$ and $V_{\omega(i)} = \mathbb{C}e_i$. Define

$$R : \mathfrak{h}^* \times \mathbb{C}^\times \to \mathrm{End}(V \otimes V)$$

by

$$R(\lambda, z) = \sum_{i=1}^n E_{ii} \otimes E_{ii} + \sum_{i \neq j} \alpha(\lambda_{ij}, z) E_{ii} \otimes E_{jj} + \sum_{i \neq j} \beta(\lambda_{ij}, z) E_{ij} \otimes E_{ji} \tag{3.3}$$

where $E_{ij} \in \mathrm{End}(V)$ are the matrix units, λ_{ij} $(\lambda \in \mathfrak{h}^*)$ is an abbrevation for $\lambda(E_{ii} - E_{jj})$, and

$$\alpha, \beta : \mathbb{C} \times \mathbb{C}^\times \to \mathbb{C}$$

are given by

$$\alpha(\lambda, z) = \alpha(\lambda, z; p, q) = \frac{\theta(z)\theta(q^{2(\lambda+1)})}{\theta(q^2 z)\theta(q^{2\lambda})}, \tag{3.4}$$

$$\beta(\lambda, z) = \beta(\lambda, z; p, q) = \frac{\theta(q^2)\theta(q^{-2\lambda} z)}{\theta(q^2 z)\theta(q^{-2\lambda})}. \tag{3.5}$$

Proposition 3.1. *The map R is a unitary R-matrix.*

Proof. Since θ is holomorphic on $\mathbb{C}^\times$, R is meromorphic. From (3.3) we see that

$$R(\lambda, z)(e_a \otimes e_b) \in \mathbb{C}e_a \otimes e_b + \mathbb{C}e_b \otimes e_a$$

proving that $R(\lambda, z)$ commutes with the $\mathfrak{h}$-action on $V \otimes V$.

The definition of R we use is a slight modification of the one from [FV97] as shown in Lemma 3.2 below. From this it will follow that R satisfies (3.1), (3.2). This can also be proved directly by extracting the matrix elements of both sides and using the addition formula (2.3).

Let us recall Felders elliptic R-matrix from [FV97], denoted here by R_1. Let $\mathfrak{h}_1$ be the Cartan subalgebra of $\mathfrak{gl}(n)$. Define $R_1 : \mathfrak{h}_1^* \times \mathbb{C} \to \mathrm{End}(V \otimes V)$ as in (3.3) with α, β replaced by $\alpha_1, \beta_1 : \mathbb{C}^2 \to \mathbb{C}$ defined as

$$\alpha_1(\lambda, x) = \alpha_1(\lambda, x; \tau, \gamma) = \frac{\theta_1(x; \tau)\theta_1(\lambda + \gamma; \tau)}{\theta_1(x - \gamma; \tau)\theta_1(\lambda; \tau)}, \tag{3.6}$$

$$\beta_1(\lambda, x) = \beta_1(\lambda, x; \tau, \gamma) = -\frac{\theta_1(x + \lambda; \tau)\theta_1(\gamma; \tau)}{\theta_1(x - \gamma; \tau)\theta_1(\lambda; \tau)}. \tag{3.7}$$

Here $\tau, \gamma \in \mathbb{C}$ with $\operatorname{Im}\tau > 0$ and θ_1 is the first Jacobi theta function

$$\theta_1(x;\tau) = -\sum_{j\in\mathbb{Z}+\frac{1}{2}} e^{\pi i j^2\tau + 2\pi i j(x+1/2)}.$$

As proved in [F95], R_1 satisfies the following version of the QDYBE:

$$\begin{aligned} R_1(\lambda-\gamma h^3, x_1-x_2)^{(12)} R_1(\lambda, x_1-x_3)^{(13)} R_1(\lambda-\gamma h^1, x_2-x_3)^{(23)} = \\ = R_1(\lambda, x_2-x_3)^{(23)} R_1(\lambda-\gamma h^2, x_1-x_3)^{(13)} R_1(\lambda, x_1-x_2)^{(12)} \end{aligned} \tag{3.8}$$

and the unitarity condition

$$R_1(\lambda, x) R_1^{21}(\lambda, -x) = \mathrm{Id}_{V\otimes V}. \tag{3.9}$$

We can identify $\mathfrak{h}^* \simeq \mathfrak{h}_1^*/\mathbb{C}\,\mathrm{tr}$ where $\mathrm{tr} \in \mathfrak{h}_1^*$ is the trace. Since R_1 has the form (3.3), it is constant, as a function of $\lambda \in \mathfrak{h}_1^*$, on the cosets modulo $\mathbb{C}\,\mathrm{tr}$. So R_1 induces a map $\mathfrak{h}^* \times \mathbb{C} \to \mathrm{End}(V\otimes V)$, which we also denote by R_1, still satisfying (3.8),(3.9).

Lemma 3.2. *Let $\tau, \gamma \in \mathbb{C}$ with $\operatorname{Im}\tau > 0$ be such that $p = e^{\pi i\tau}$, $q = e^{\pi i\gamma}$. Then, as meromorphic functions of $(\lambda, x) \in \mathfrak{h}^* \times \mathbb{C}$,*

$$R_1(\gamma\lambda, -x; \tau/2, \gamma) = R(\lambda, z; p, q) \tag{3.10}$$

where $z = e^{2\pi i x}$.

Proof. Using the Jacobi triple product identity (2.4) we have

$$\theta_1(x;\tau/2) = i e^{\pi i(\tau/2-x)} \theta(z) \prod_{j=1}^{\infty} (1-p^j)$$

Substituting this into (3.6) and (3.7) gives $\alpha_1(\gamma\lambda, -x; \tau/2, \gamma) = \alpha(\lambda, z; p, q)$ and $\beta_1(\gamma\lambda, -x; \tau/2, \gamma) = \beta(\lambda, z; p, q)$ which proves (3.10). □

To finish the proof of Proposition 3.1, replace λ, x_i by $\gamma\lambda$, $-x_i$ in (3.8) and use (3.10) to obtain (3.1) with $z_i = e^{2\pi i x_i}$, and similarly for the unitarity condition. □

We end this section by recording some useful identities. Recall the definitions of α, β in (3.4),(3.5). It is immediate that

$$\alpha(\lambda, q^2) = \beta(-\lambda, q^2). \tag{3.11}$$

By induction, one generalizes (2.2) to

$$\theta(p^s z) = (-1)^s (p^{s(s-1)/2} z^s)^{-1} \theta(z), \quad \text{for } s \in \mathbb{Z}. \tag{3.12}$$

Applying (3.12) to the definitions of α,β we get

$$\alpha(\lambda, p^k z) = q^{2k} \alpha(\lambda, z), \quad \beta(\lambda, p^k z) = q^{2k(\lambda+1)} \beta(\lambda, z), \tag{3.13}$$

and, using also $\theta(z^{-1})=-z^{-1}\theta(z)$,

$$\begin{aligned}\lim_{z\to p^{-k}q^{-2}}\frac{q^{-1}\theta(q^2z)}{q\theta(q^{-2}z)}\alpha(\lambda,z)&=\alpha(\lambda,p^kq^2),\\ \lim_{z\to p^{-k}q^{-2}}\frac{q^{-1}\theta(q^2z)}{q\theta(q^{-2}z)}\beta(\lambda,z)&=-\beta(-\lambda,p^kq^2),\end{aligned}\tag{3.14}$$

for $\lambda\in\mathbb{C}$, $z\in\mathbb{C}^\times$, and $k\in\mathbb{Z}$. By the addition formula (2.3) with $(x,y,z,w)=(z^{1/2}q^{-\lambda+1},z^{1/2}q^{\lambda-1},z^{1/2}q^{\lambda+1},z^{1/2}q^{-\lambda-1})$ we have

$$\alpha(\lambda,z)\alpha(-\lambda,z)-\beta(\lambda,z)\beta(-\lambda,z)=q^2\frac{\theta(q^{-2}z)}{\theta(q^2z)}.\tag{3.15}$$

4 $\mathfrak{h}$-Bialgebroids

4.1 Definitions

We recall the some definitions from [EV98]. Let $\mathfrak{h}^*$ be a finite-dimensional complex vector space (for example the dual space of an abelian Lie algebra) and $M_{\mathfrak{h}^*}$ be the field of meromorphic functions on $\mathfrak{h}^*$.

Definition 4.1. An $\mathfrak{h}$-*algebra* is a complex associative algebra A with 1 which is bigraded over $\mathfrak{h}^*$, $A=\bigoplus_{\alpha,\beta\in\mathfrak{h}^*}A_{\alpha\beta}$, and equipped with two algebra embeddings $\mu_l,\mu_r:M_{\mathfrak{h}^*}\to A$, called the left and right moment maps, such that

$$\mu_l(f)a=a\mu_l(T_\alpha f),\qquad \mu_r(f)a=a\mu_r(T_\beta f),\quad \text{for } a\in A_{\alpha\beta},\ f\in M_{\mathfrak{h}^*},\tag{4.1}$$

where T_α denotes the automorphism $(T_\alpha f)(\zeta)=f(\zeta+\alpha)$ of $M_{\mathfrak{h}^*}$. A morphism of $\mathfrak{h}$-algebras is an algebra homomorphism preserving the bigrading and the moment maps.

The *matrix tensor product* $A\widetilde{\otimes}B$ of two $\mathfrak{h}$-algebras A, B is the $\mathfrak{h}^*$-bigraded vector space with $(A\widetilde{\otimes}B)_{\alpha\beta}=\bigoplus_{\gamma\in\mathfrak{h}^*}(A_{\alpha\gamma}\otimes_{M_{\mathfrak{h}^*}}B_{\gamma\beta})$, where $\otimes_{M_{\mathfrak{h}^*}}$ denotes tensor product over $\mathbb{C}$ modulo the relations

$$\mu_r^A(f)a\otimes b=a\otimes\mu_l^B(f)b,\quad \text{for all } a\in A,\ b\in B,\ f\in M_{\mathfrak{h}^*}.\tag{4.2}$$

The multiplication $(a\otimes b)(c\otimes d)=ac\otimes bd$ for $a,c\in A$ and $b,d\in B$ and the moment maps $\mu_l(f)=\mu_l^A(f)\otimes 1$ and $\mu_r(f)=1\otimes\mu_r^B(f)$ make $A\widetilde{\otimes}B$ into an $\mathfrak{h}$-algebra.

Example 4.2. Let $D_{\mathfrak{h}}$ be the algebra of operators on $M_{\mathfrak{h}^*}$ of the form $\sum_i f_iT_{\alpha_i}$ with $f_i\in M_{\mathfrak{h}^*}$ and $\alpha_i\in\mathfrak{h}^*$. It is an $\mathfrak{h}$-algebra with bigrading $fT_{-\alpha}\in(D_{\mathfrak{h}})_{\alpha\alpha}$ and both moment maps equal to the natural embedding.

For any $\mathfrak{h}$-algebra A, there are canonical isomorphisms $A\simeq A\widetilde{\otimes}D_{\mathfrak{h}}\simeq D_{\mathfrak{h}}\widetilde{\otimes}A$ defined by

$$x\simeq x\otimes T_{-\beta}\simeq T_{-\alpha}\otimes x,\quad \text{for } x\in A_{\alpha\beta}.\tag{4.3}$$

Definition 4.3. An $\mathfrak{h}$-*bialgebroid* is an $\mathfrak{h}$-algebra A equipped with two $\mathfrak{h}$-algebra morphisms, the comultiplication $\Delta:A\to A\widetilde{\otimes}A$ and the counit $\varepsilon:A\to D_{\mathfrak{h}}$ such that $(\Delta\otimes\mathrm{Id})\circ\Delta=(\mathrm{Id}\otimes\Delta)\circ\Delta$ and $(\varepsilon\otimes\mathrm{Id})\circ\Delta=\mathrm{Id}=(\mathrm{Id}\otimes\varepsilon)\circ\Delta$, under the identifications (4.3).

4.2 The generalized FRST-construction

Let $\mathfrak{h}$ be a finite-dimensional abelian Lie algebra, $V = \bigoplus_{\alpha\in\mathfrak{h}^*} V_\alpha$ a finite-dimensional diagonalizable $\mathfrak{h}$-module and $R : \mathfrak{h}^* \times \mathbb{C}^\times \to \mathrm{End}_{\mathfrak{h}}(V \otimes V)$ a meromorphic function. The generalized FRST-construction attaches to this data an $\mathfrak{h}$-bialgebroid A_R as follows. Let $\{e_x\}_{x\in X}$ be a homogeneous basis of V, where X is an index set. The matrix elements $R^{ab}_{xy} : \mathfrak{h}^* \times \mathbb{C}^\times \to \mathbb{C}$ of R are given by

$$R(\zeta,z)(e_a \otimes e_b) = \sum_{x,y\in X} R^{ab}_{xy}(\zeta,z)e_x \otimes e_y. \tag{4.4}$$

They are meromorphic on $\mathfrak{h}^* \times \mathbb{C}^\times$. Define $\omega : X \to \mathfrak{h}^*$ by $e_x \in V_{\omega(x)}$. Let $\tilde{A}_R$ be the complex associative algebra with 1 generated by $\{L_{xy}(z) : x,y \in X, z \in \mathbb{C}^\times\}$ and two copies of $M_{\mathfrak{h}^*}$, whose elements are denoted by $f(\lambda)$ and $f(\rho)$, respectively, with defining relations $f(\lambda)g(\rho) = g(\rho)f(\lambda)$ for $f,g \in M_{\mathfrak{h}^*}$ and

$$f(\lambda)L_{xy}(z) = L_{xy}(z)f(\lambda+\omega(x)), \quad f(\rho)L_{xy}(z) = L_{xy}(z)f(\rho+\omega(y)), \tag{4.5}$$

for all $x,y \in X$, $z \in \mathbb{C}^\times$ and $f \in M_{\mathfrak{h}^*}$. The bigrading on $\tilde{A}_R$ is given by $L_{xy}(z) \in (\tilde{A}_R)_{\omega(x),\omega(y)}$ for $x,y \in X$, $z \in \mathbb{C}^\times$ and $f(\lambda), f(\rho) \in (\tilde{A}_R)_{00}$ for $f \in M_{\mathfrak{h}^*}$. The moment maps are defined by $\mu_l(f) = f(\lambda)$, $\mu_r(f) = f(\rho)$. The counit and comultiplication are defined by

$$\varepsilon(L_{ab}(z)) = \delta_{ab}T_{-\omega(a)}, \quad \varepsilon(f(\lambda)) = \varepsilon(f(\rho)) = fT_0, \tag{4.6}$$

$$\Delta(L_{ab}(z)) = \sum_{x\in X} L_{ax}(z) \otimes L_{xb}(z), \tag{4.7}$$

$$\Delta(f(\lambda)) = f(\lambda)\otimes 1, \quad \Delta(f(\rho)) = 1 \otimes f(\rho). \tag{4.8}$$

This makes $\tilde{A}_R$ into an $\mathfrak{h}$-bialgebroid.

Consider the ideal in $\tilde{A}_R$ generated by the RLL-relations

$$\sum_{x,y\in X} R^{xy}_{ac}(\lambda, \frac{z_1}{z_2})L_{xb}(z_1)L_{yd}(z_2) = \sum_{x,y\in X} R^{bd}_{xy}(\rho, \frac{z_1}{z_2})L_{cy}(z_2)L_{ax}(z_1), \tag{4.9}$$

where $a,b,c,d \in X$, and $z_1, z_2 \in \mathbb{C}^\times$. More precisely, to account for possible singularities of R, we let I_R be the ideal in $\tilde{A}_R$ generated by all relations of the form

$$\sum_{x,y\in X} \lim_{w\to z_1/z_2} \left(\varphi(w)R^{xy}_{ac}(\lambda,w)\right) L_{xb}(z_1)L_{yd}(z_2) = \\ = \sum_{x,y\in X} \lim_{w\to z_1/z_2} \left(\varphi(w)R^{bd}_{xy}(\rho,w)\right) L_{cy}(z_2)L_{ax}(z_1), \tag{4.10}$$

where $a,b,c,d \in X$, $z_1,z_2 \in \mathbb{C}^\times$ and $\varphi : \mathbb{C}^\times \to \mathbb{C}$ is a meromorphic function such that the limits exist.

We define A_R to be $\tilde{A}_R/I_R$. The bigrading descends to A_R because (4.10) is homogeneous, of bidegree $\omega(a)+\omega(c), \omega(b)+\omega(d)$, by the $\mathfrak{h}$-invariance of R. One checks that $\Delta(I_R) \subseteq \tilde{A}_R \widetilde{\otimes} I_R + I_R \widetilde{\otimes} \tilde{A}_R$ and $\varepsilon(I_R) = 0$. Thus A_R is an $\mathfrak{h}$-bialgebroid with the induced maps.

4.3 Operator form of the RLL relations

The RLL-relations (4.9) can be understood as follows. Assume $R^{ab}_{xy}(\zeta,z)$ are defined, as meromorphic functions of $\zeta\in\mathfrak{h}^*$, for any $z\in\mathbb{C}^\times$. Define $\mathsf{R}(\lambda,z),\mathsf{R}(\rho,z)\in\mathrm{End}(V\otimes V\otimes A)$ by

$$\mathsf{R}(\lambda,z)(e_a\otimes e_b\otimes u)=\sum_{x,y\in X}e_x\otimes e_y\otimes R^{ab}_{xy}(\lambda,z)u,$$
$$\mathsf{R}(\rho,z)(e_a\otimes e_b\otimes u)=\sum_{x,y\in X}e_x\otimes e_y\otimes R^{ab}_{xy}(\rho,z)u,$$

for $a,b\in X$, $u\in A$. Note that the λ and ρ in the left hand side are not variables but merely indicates which moment map is to be used. For $z\in\mathbb{C}^\times$ we also define $\mathsf{L}(z)\in\mathrm{End}(V\otimes A)$ by

$$\mathsf{L}(z)=\sum_{x,y\in X}E_{xy}\otimes L_{xy}(z).$$

Here E_{xy} are the matrix units in $\mathrm{End}(V)$ and A acts on itself by left multiplication. The RLL relation (4.9) is equivalent to

$$\mathsf{R}(\lambda,z_1/z_2)\mathsf{L}^1(z_1)\mathsf{L}^2(z_2)=\mathsf{L}^2(z_2)\mathsf{L}^1(z_1)\mathsf{R}(\rho+h^1+h^2,z_1/z_2) \tag{4.11}$$

in $\mathrm{End}(V\otimes V\otimes A)$, where $\mathsf{L}^i(z)=\mathsf{L}(z)^{(i,3)}\in\mathrm{End}(V\otimes V\otimes A)$ for $i=1,2$. This can be seen by acting on $e_b\otimes e_d\otimes 1$ in both sides of (4.11), and collecting and equating terms of the form $e_a\otimes e_c\otimes u$. The matrix elements of the R-matrix in the right hand side can then be moved to the left using that R is $\mathfrak{h}$-invariant, and relation (4.5).

5 The algebra $\mathscr{F}_{\mathrm{ell}}(M(n))$

Let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{sl}(n)$, $V=\mathbb{C}^n$ and R be given by (3.3)-(3.5). Applying the generalized FRST-construction to these data we obtain an $\mathfrak{h}$-bialgebroid which we denote by $\mathscr{F}_{\mathrm{ell}}(M(n))$. The generators $L_{ij}(z)$ will be denoted by $e_{ij}(z)$. Thus $\mathscr{F}_{\mathrm{ell}}(M(n))$ is the unital associative $\mathbb{C}$-algebra generated by $e_{ij}(z)$, $i,j\in[1,n]$, $z\in\mathbb{C}^\times$, and two copies of $M_{\mathfrak{h}^*}$, whose elements are denoted by $f(\lambda)$ and $f(\rho)$ for $f\in M_{\mathfrak{h}^*}$, subject to the following relations

$$f(\lambda)e_{ij}(z)=e_{ij}(z)f(\lambda+\omega(i)),\qquad f(\rho)e_{ij}(z)=e_{ij}(z)f(\rho+\omega(j)), \tag{5.1}$$

for all $f\in M_{\mathfrak{h}^*}$, $i,j\in[1,n]$ and $z\in\mathbb{C}^\times$, and

$$\sum_{x,y=1}^n R^{xy}_{ac}(\lambda,\frac{z_1}{z_2})e_{xb}(z_1)e_{yd}(z_2)=\sum_{x,y=1}^n R^{bd}_{xy}(\rho,\frac{z_1}{z_2})e_{cy}(z_2)e_{ax}(z_1), \tag{5.2}$$

for all $a,b,c,d\in[1,n]$. More explicitly, from (3.3) we have

$$R^{ab}_{xy}(\zeta,z)=\begin{cases}1, & a=b=x=y,\\ \alpha(\zeta_{xy},z), & a\neq b, x=a, y=b,\\ \beta(\zeta_{xy},z), & a\neq b, x=b, y=a,\\ 0, & \text{otherwise,}\end{cases} \tag{5.3}$$

which substituted into (5.2) yields four families of relations:

$$e_{ab}(z_1)e_{ab}(z_2) = e_{ab}(z_2)e_{ab}(z_1), \tag{5.4a}$$

$$e_{ab}(z_1)e_{ad}(z_2) = \alpha(\rho_{bd}, \frac{z_1}{z_2})e_{ad}(z_2)e_{ab}(z_1) + \beta(\rho_{db}, \frac{z_1}{z_2})e_{ab}(z_2)e_{ad}(z_1), \tag{5.4b}$$

$$\alpha(\lambda_{ac}, \frac{z_1}{z_2})e_{ab}(z_1)e_{cb}(z_2) + \beta(\lambda_{ac}, \frac{z_1}{z_2})e_{cb}(z_1)e_{ab}(z_2) = e_{cb}(z_2)e_{ab}(z_1), \tag{5.4c}$$

$$\begin{aligned}\alpha(\lambda_{ac}, \frac{z_1}{z_2})e_{ab}(z_1)e_{cd}(z_2) + \beta(\lambda_{ac}, \frac{z_1}{z_2})e_{cb}(z_1)e_{ad}(z_2) = \\ = \alpha(\rho_{bd}, \frac{z_1}{z_2})e_{cd}(z_2)e_{ab}(z_1) + \beta(\rho_{db}, \frac{z_1}{z_2})e_{cb}(z_2)e_{ad}(z_1),\end{aligned} \tag{5.4d}$$

where $a, b, c, d \in [1, n]$, $a \neq c$ and $b \neq d$. Since θ has zeros precisely at $p^k, k \in \mathbb{Z}$, α and β have poles at $z = q^{-2}p^k, k \in \mathbb{Z}$. Thus (5.4b)-(5.4d) are to hold for $z_1, z_2 \in \mathbb{C}^\times$ with $z_1/z_2 \notin \{p^k q^{-2} : k \in \mathbb{Z}\}$.

In (4.10), assuming $a \neq c$, $b \neq d$, and taking $z_1 = z$, $z_2 = p^k q^2 z$, $\varphi(w) = \frac{q^{-1}\theta(q^2 w)}{q\theta(q^{-2}w)}$, and using the limit formulas (3.14), we obtain the relation

$$\begin{aligned}\alpha(\lambda_{ac}, q^2)\big(e_{ab}(z)e_{cd}(p^k q^2 z) - q^{2k\lambda_{ca}} e_{cb}(z)e_{ad}(p^k q^2 z)\big) = \\ = \alpha(\rho_{bd}, q^2)e_{cd}(p^k q^2 z)e_{ab}(z) - q^{2k\rho_{bd}}\beta(\rho_{bd}, q^2)e_{cb}(p^k q^2 z)e_{ad}(z).\end{aligned} \tag{5.5}$$

This identity does not follow from (5.4a)-(5.4d) in an obvious way. It will be called the *residual RLL relation*.

Proposition 5.1. *Relations* (5.4), (5.5) *generate the ideal I_R. Hence* (5.1), (5.4), (5.5) *consitute the defining relations of the algebra $\mathscr{F}_{ell}(M(n))$.*

Proof. Assume we have a relation of the form (4.10) and that a limit in one of the terms, $\lim_{w\to z}\varphi(w)R^{ab}_{xy}(\lambda, w)$, say, exists and is nonzero. Then one of the following cases occurs.

1. At $w = z$, $\varphi(w)$ and $R^{ab}_{xy}(\lambda, w)$ are both regular. If this holds for all terms, then the relation is just a multiple of one of (5.4a)-(5.4d).

2. At $w = z$, $\varphi(w)$ has a pole while $R^{ab}_{xy}(\lambda, w)$ is regular. Then $R^{ab}_{xy}(\lambda, w)$ must vanish identically at $w = z$. The only case where this is possible is when $x \neq y$ and $R^{ab}_{xy}(\lambda, w) = \alpha(\lambda_{xy}, w)$ and $z = p^k$. But then there is another term containing β which is never identically zero for any z, and hence the limit in that term does not exist.

3. At $w = z$, $\varphi(w)$ is regular while $R^{ab}_{xy}(\lambda, w)$ has a pole. Since these poles are simple and occur only when $z \in q^{-2}p^{\mathbb{Z}}$, the function φ must have a zero of multiplicity one there. We can assume without loss of generality that φ has the specific form
$$\varphi(w) = \frac{q^{-1}\theta(q^2 w)}{q\theta(q^{-2}w)}.$$

Then, if $a \neq c$ and $b \neq d$, (4.10) becomes the residual RLL relation (5.5).

If instead $c = a$, $b \neq d$, and we take $z_1 = z$, $z_2 = p^k q^2 z$ in (4.10) we get, using (3.14),

$$0 = \alpha(\rho_{bd}, p^k q^2) e_{ad}(p^k q^2 z) e_{ab}(z) - \beta(\rho_{bd}, p^k q^2) e_{ab}(p^k q^2 z) e_{ad}(z),$$

or, rewritten,

$$e_{ad}(p^k q^2 z) e_{ab}(z) = q^{2k\rho_{bd}} \frac{E(\rho_{bd} - 1)}{E(\rho_{bd} + 1)} e_{ab}(p^k q^2 z) e_{ad}(z).$$

However this relation is already derivable from (5.4b) as follows. Take $z_1 = p^k q^2 z$ and $z_2 = z$ in (5.4b) and multiply both sides by $q^{2k\rho_{bd}} \frac{E(\rho_{bd}-1)}{E(\rho_{bd}+1)}$ and then use (5.4b) on the right hand side.

Similarly, if $a \neq c$, $d = b$, $z_1 = z$, $z_2 = p^k q^2 z$, $\varphi(w) = \frac{q^{-1}\theta(q^2 w)}{q\theta(q^{-2} w)}$ in (4.10) and using (3.14) we get

$$\alpha(\lambda_{ac}, p^k q^2) e_{ab}(z) e_{cb}(p^k q^2 z) - \beta(\lambda_{ca}, p^k q^2) e_{cb}(z) e_{ab}(p^k q^2 z) = 0,$$

or,

$$e_{ab}(z) e_{cb}(p^k q^2 z) = q^{2k\lambda_{ca}} e_{cb}(z) e_{ab}(p^k q^2 z).$$

Similarly to the previous case, this identity follows already from (5.4c).

□

6 Left and right elliptic exterior algebras

6.1 Corepresentations of $\mathfrak{h}$-bialgebroids

We recall the definition of corepresentations of an $\mathfrak{h}$-bialgebroid given in [KR01].

Definition 6.1. An $\mathfrak{h}$-*space* V is an $\mathfrak{h}^*$-graded vector space over $M_{\mathfrak{h}^*}$, $V = \bigoplus_{\alpha \in \mathfrak{h}^*} V_\alpha$, where each V_α is $M_{\mathfrak{h}^*}$-invariant. A morphism of $\mathfrak{h}$-spaces is a degree-preserving $M_{\mathfrak{h}^*}$-linear map.

Given an $\mathfrak{h}$-space V and an $\mathfrak{h}$-bialgebroid A, we define $A \widetilde{\otimes} V$ to be the $\mathfrak{h}^*$-graded space with $(A \widetilde{\otimes} V)_\alpha = \bigoplus_{\beta \in \mathfrak{h}^*} (A_{\alpha\beta} \otimes_{M_{\mathfrak{h}^*}} V_\beta)$ where $\otimes_{M_{\mathfrak{h}^*}}$ denotes $\otimes_{\mathbb{C}}$ modulo the relations

$$\mu_r(f) a \otimes v = a \otimes f v,$$

for $f \in M_{\mathfrak{h}^*}$, $a \in A$, $v \in V$. $A \widetilde{\otimes} V$ becomes an $\mathfrak{h}$-space with the $M_{\mathfrak{h}^*}$-action $f(a \otimes v) = \mu_l(f) a \otimes v$. Similarly we define $V \widetilde{\otimes} A$ as an $\mathfrak{h}$-space by $(V \widetilde{\otimes} A)_\beta = \bigoplus_\alpha V_\alpha \otimes_{M_{\mathfrak{h}^*}} A_{\alpha\beta}$ where $\otimes_{M_{\mathfrak{h}^*}}$ here means $\otimes_{\mathbb{C}}$ modulo the relation $v \otimes \mu_l(f) a = f v \otimes a$, and $M_{\mathfrak{h}^*}$-action given by $f(v \otimes a) = v \otimes \mu_r(f) a$.

For any $\mathfrak{h}$-space V we have isomorphisms $D_{\mathfrak{h}} \widetilde{\otimes} V \simeq V \simeq V \widetilde{\otimes} D_{\mathfrak{h}}$ given by

$$T_{-\alpha} \otimes v \simeq v \simeq v \otimes T_\alpha, \quad \text{for } v \in V_\alpha, \tag{6.1}$$

extended to $\mathfrak{h}$-space morphisms.

Definition 6.2. A left corepresentation V of an $\mathfrak{h}$-bialgebroid A is an $\mathfrak{h}$-space equipped with an $\mathfrak{h}$-space morphism $\Delta_V : V \to A\widetilde{\otimes}V$ such that $(\Delta_V \otimes 1)\circ\Delta_V = (1\otimes\Delta)\circ\Delta_V$ and $(\varepsilon\otimes 1)\circ\Delta_V = \mathrm{Id}_V$ (under the identification (6.1)).

Definition 6.3. A left $\mathfrak{h}$-comodule algebra V over an $\mathfrak{h}$-bialgebroid A is a left corepresentation $\Delta_V : V \to A\widetilde{\otimes}V$ and in addition a $\mathbb{C}$-algebra such that $V_\alpha V_\beta \subseteq V_{\alpha+\beta}$ and such that Δ_V is an algebra morphism, when $A\widetilde{\otimes}V$ is given the natural algebra structure.

Right corepresentations and comodule algebras are defined analogously.

6.2 The comodule algebras Λ and Λ'.

Let Λ be the unital associative $\mathbb{C}$-algebra generated by $v_i(z)$, $1\le i\le n$, $z\in\mathbb{C}^\times$ and a copy of $M_{\mathfrak{h}^*}$ embedded as a subalgebra subject to the relations

$$f(\zeta)v_i(z) = v_i(z)f(\zeta+\omega(i)), \tag{6.2a}$$

$$v_i(z)v_i(w) = 0, \tag{6.2b}$$

$$\alpha(\zeta_{kj}, z/w)v_k(z)v_j(w) + \beta(\zeta_{kj}, z/w)v_j(z)v_k(w) = 0, \tag{6.2c}$$

for $i,j,k\in[1,n]$, $j\neq k$, $z,w\in\mathbb{C}^\times$, $z/w\notin\{p^kq^{-2} : k\in\mathbb{Z}\}$ and $f\in M_{\mathfrak{h}^*}$. We require also the residual relation of (6.2c) obtained by multiplying by $\varphi(z/w) = \frac{q^{-1}\theta(q^2z/w)}{q\theta(q^{-2}z/w)}$ and letting $z/w\to p^{-k}q^{-2}$. After simplification using (3.14), we get

$$v_k(z)v_j(p^kq^2z) = q^{2k\zeta_{jk}}v_j(z)v_k(p^kq^2z). \tag{6.2d}$$

Λ becomes an $\mathfrak{h}$-space by

$$\mu_\Lambda(f)v = f(\zeta)v$$

and requiring $v_i(z)\in\Lambda_{\omega(i)}$ for each i,z.

Proposition 6.4. *Λ is a left comodule algebra over $\mathscr{F}_{ell}(M(n))$ with left coaction $\Delta_\Lambda : \Lambda \to \mathscr{F}_{ell}(M(n))\widetilde{\otimes}\Lambda$ satisfying*

$$\Delta_\Lambda(v_i(z)) = \sum_j e_{ij}(z)\otimes v_j(z),$$

$$\Delta_\Lambda(f(\zeta)) = f(\lambda)\otimes 1.$$

Proof. We have

$$\Delta_\Lambda(v_i(z))\Delta_\Lambda(v_i(w)) = \sum_{jk} e_{ij}(z)e_{ik}(w)\otimes v_j(z)v_k(w) =$$

$$= \sum_{j\neq k}\big(\alpha(\mu_{jk},\frac{z}{w})e_{ik}(w)e_{ij}(z) + \beta(\mu_{kj},\frac{z}{w})e_{ij}(w)e_{ik}(z)\big)\otimes v_j(z)v_k(w) =$$

$$= \sum_{j\neq k} e_{ij}(w)e_{ik}(z)\otimes\big(\alpha(\zeta_{kj},\frac{z}{w})v_k(z)v_j(w) + \beta(\zeta_{kj},\frac{z}{w})v_j(z)v_k(w)\big) = 0.$$

Similarly one proves that (6.2c),(6.2d) are preserved. ◻

We will use the function E, defined in (2.5).

Proposition 6.5. *(i) The following is a complete set of relations for* Λ

$$f(\zeta)v_i(z) = v_i(z)f(\zeta + \omega(i)), \tag{6.3a}$$

$$v_k(p^s q^2 z)v_j(z) = -q^{2s\zeta_{kj}}\frac{E(\zeta_{kj}-1)}{E(\zeta_{kj}+1)}v_j(p^s q^2 z)v_k(z), \quad \forall s \in \mathbb{Z}, k \neq j, \tag{6.3b}$$

$$v_k(z)v_j(p^s q^2 z) = q^{2s\zeta_{jk}}v_j(z)v_k(p^s q^2 z), \tag{6.3c}$$

$$v_k(z)v_j(w) = 0 \quad \text{if } z/w \notin \{p^s q^{\pm 2} | s \in \mathbb{Z}\} \text{ or if } k = j. \tag{6.3d}$$

(ii) The set

$$\{v_{i_d}(z_d)\cdots v_{i_1}(z_1) : 1 \le i_1 < \cdots < i_d \le n, \frac{z_{i+1}}{z_i} \in p^{\mathbb{Z}}q^{\pm 2}\} \tag{6.4}$$

is a basis for Λ *over* $M_{\mathfrak{h}^*}$.

Proof. (i) Elimination of the $v_j(z)v_k(w)$-term in (6.2c) yields

$$\big(\alpha(\zeta_{jk},\frac{z}{w})\alpha(\zeta_{kj},\frac{z}{w}) - \beta(\zeta_{kj},\frac{z}{w})\beta(\zeta_{jk},\frac{z}{w})\big)v_k(z)v_j(w) = 0. \tag{6.5}$$

Combining (6.5), (3.15) and the fact that the $\theta(z)$ is zero precisely for $z \in \{p^k | k \in \mathbb{Z}\}$ we deduce that in Λ,

$$v_k(z)v_j(w) \neq 0 \Longrightarrow \frac{z}{w} = p^s q^2 \text{ for some } s \in \mathbb{Z}. \tag{6.6}$$

Using (3.13) we obtain from (6.6) and (6.2b),(6.2c) that relations (6.3b),(6.3d) hold in the left elliptic exterior algebra Λ. Relations (6.3a),(6.3c) are just repetitions of (6.2a),(6.2d).

(ii) It follows from the relations that each monomial in Λ can be uniquely written as $f(\zeta)v_{i_d}(z_d)\cdots v_{i_1}(z_1)$ where $1 \le i_1 < \cdots < i_d \le n$ and $f \in M_{\mathfrak{h}^*}$. It remains to show that the set (6.4) is linearly independent over $M_{\mathfrak{h}^*}$. Assume that a linear combination of basis elements is zero, and that the sum has minimal number of terms. By multiplying from the right or left by $v_j(w)$ for appropriate j, w we can assume the sum is of the form

$$f_1(\zeta)v_{i_d}(z_d^1)\cdots v_{i_1}(z_1^1) + \cdots + f_r(\zeta)v_{i_d}(z_d^r)\cdots v_{i_1}(z_1^r) = 0$$

for some fixed set $\{i_1, \ldots, i_d\}$. By the relations, a monomial $v_{i_d}(z_d)\cdots v_{i_1}(z_1)$ can be given the "degree" $\sum_{i=1}^d z_i t^{i-1} \in \mathbb{C}[t]$, where t is an indeterminate. Formally, consider $\mathbb{C}(t)\otimes\Lambda$, the tensor product (over $\mathbb{C}$) of Λ by the field of rational functions in t. We identify Λ with its image under $\Lambda \ni v \mapsto 1\otimes v \in \mathbb{C}(t)\otimes\Lambda$, and view $\mathbb{C}(t)\otimes\Lambda$ naturally as a vector space over $\mathbb{C}(t)$. By relations (6.3a)-(6.3d), there is a $\mathbb{C}$-algebra automorphism T of $\mathbb{C}(t)\otimes\Lambda$ satisfying $T(v_j(z)) = tv_j(z)$, $T(f(\zeta)) = f(\zeta)$ and $T(p \otimes 1) = p \otimes 1$. Define

$$D(v_i(z)) = zv_i(z), \quad D(f(\zeta)) = 0, \quad D(p \otimes 1) = 0,$$

for $f \in M_{\mathfrak{h}^*}$, $p \in \mathbb{C}(t)$ and $i \in [1,n]$, $z \in \mathbb{C}^\times$ and extend D to a $\mathbb{C}$-linear map $D : \mathbb{C}(t) \otimes \Lambda \to \mathbb{C}(t) \otimes \Lambda$ by requiring

$$D(ab) = D(a)T(b) + aD(b) \tag{6.7}$$

for $a, b \in \mathbb{C}(t) \otimes \Lambda$. The point is that the requirement (6.7) respects relations (6.3a)-(6.3d), making D well defined. Write $u_j = f_j(\zeta)v_{i_d}(z_d^j)\cdots v_{i_1}(z_1^j)$. Then one checks that $D(u_j) = p_j(t)u_j$, where $p_j(t) = \sum_{i=1}^d z_i^j t^{i-1}$. By applying D repeatedly we get

$$\begin{gathered} u_1(z^1)+\cdots+u_r(z^r) = 0, \\ p_1(t)u_1(z^1)+\cdots+p_r(t)u_r(z^r) = 0, \\ \vdots \\ p_1(t)^{r-1}u_1(z^1)+\cdots+p_r(t)^{r-1}u_r(z^r) = 0. \end{gathered}$$

Inverting the Vandermonde matrix $(p_j(t)^{i-1})_{ij}$ we obtain $u_j(z^j) = 0$ for each j, i.e. $f_j(\zeta) = 0$ for each j. This proves linear independence of (6.4). $\square$

Analogously one defines a right comodule algebra Λ' with generators $w^i(z)$ and $f(\zeta) \in M_{\mathfrak{h}^*}$. The following relations will be used:

$$w^k(z)w^j(p^s q^2 z) = -q^{2s\zeta_{kj}} w^j(z)w^k(p^s q^2 z), \quad \forall s \in \mathbb{Z}, k \neq j, \tag{6.8a}$$

$$w^k(z_1)w^j(z_2) = 0 \quad \text{if } z_2/z_1 \notin \{p^s q^{\pm 2} | s \in \mathbb{Z}\} \text{ or if } k = j. \tag{6.8b}$$

Λ' has also $M_{\mathfrak{h}^*}$-basis of the form (6.4). In fact Λ and Λ' are isomorphic as algebras.

6.3 Action of S_n

From (5.4),(5.5) we see that $S_n \times S_n$ acts by $\mathbb{C}$-algebra automorphisms on $\mathscr{F}_{\mathrm{ell}}(M(n))$ as follows

$$(\sigma,\tau)(f(\lambda)) = f(\lambda \circ L_\sigma), \qquad (\sigma,\tau)(f(\mu)) = f(\mu \circ L_\tau),$$
$$(\sigma,\tau)(e_{ij}(z)) = e_{\sigma(i)\tau(j)}(z),$$

where $L_\sigma : \mathfrak{h} \to \mathfrak{h}$ ($\sigma \in S_n$) is given by permutation of coordinates:

$$L_\sigma(\mathrm{diag}(h_1,\ldots,h_n)) = \mathrm{diag}(h_{\sigma(1)},\ldots,h_{\sigma(n)}).$$

Also, S_n acts on Λ by $\mathbb{C}$-algebra automorphisms via

$$\sigma(f(\zeta)) = f(\zeta \circ L_\sigma), \qquad \sigma(v_i(z)) = v_{\sigma(i)}(z). \tag{6.9}$$

Similarly we define an S_n action on Λ'.

Lemma 6.6. *For each $v \in \Lambda$, $w \in \Lambda'$ and any $\sigma, \tau \in S_n$ we have*

$$\Delta_\Lambda(\sigma(v)) = ((\sigma,\tau)\otimes\tau)(\Delta_\Lambda(v)), \tag{6.10}$$

$$\Delta_{\Lambda'}(\tau(w)) = (\sigma\otimes(\sigma,\tau))(\Delta_{\Lambda'}(w)). \tag{6.11}$$

Proof. By multiplicativity, it is enough to prove these claims on the generators, which is easy. $\square$

7 Elliptic quantum minors

7.1 Definition

For $I \subseteq [1,n]$ we set

$$F_I(\zeta) = \prod_{i,j\in I, i<j} E(\zeta_{ij}+1), \qquad F^I(\zeta) = \prod_{i,j\in I, i<j} E(\zeta_{ij}), \tag{7.1}$$

and define the left and right elliptic sign functions

$$\operatorname{sgn}_I(\sigma;\zeta) = \frac{\sigma(F_I(\zeta))}{F_{\sigma(I)}(\zeta)} = \prod_{i,j\in I, i<j, \sigma(i)>\sigma(j)} \frac{E(\zeta_{\sigma(i)\sigma(j)}+1)}{E(\zeta_{\sigma(j)\sigma(i)}+1)}, \tag{7.2}$$

$$\operatorname{sgn}^I(\sigma;\zeta) = \frac{F^{\sigma(I)}(\zeta)}{\sigma(F^I(\zeta))} = \prod_{i,j\in I, i<j, \sigma(i)>\sigma(j)} \frac{E(\zeta_{\sigma(j)\sigma(i)})}{E(\zeta_{\sigma(i)\sigma(j)})}, \tag{7.3}$$

for $\sigma \in S_n$. In fact, $E(\zeta_{ij})/E(\zeta_{ji}) = -1$ so $\operatorname{sgn}^{[1,n]}(\sigma;\zeta)$ is just $\operatorname{sgn}(\sigma)$.

We will denote the elements of a subset $I \subseteq [1,n]$ by $i_1 < i_2 < \cdots$.

Proposition 7.1. *Let $I \subseteq [1,n]$, $d=\#I$, $\sigma \in S_n$ and $J = \sigma(I)$. Then for $z \in \mathbb{C}^\times$,*

$$v_{\sigma(i_d)}(q^{2(d-1)}z)\cdots v_{\sigma(i_1)}(z) = \operatorname{sgn}_I(\sigma;\zeta) v_{j_d}(q^{2(d-1)}z)\cdots v_{j_1}(z) \tag{7.4}$$

and

$$w^{\sigma(i_1)}(z)\cdots w^{\sigma(i_d)}(q^{2(d-1)}z) = \operatorname{sgn}^I(\sigma;\zeta) w^{j_1}(z)\cdots w^{j_d}(q^{2(d-1)}z). \tag{7.5}$$

Proof. We prove (7.4). The proof of (7.5) is analogous. We proceed by induction on $\#I = d$, the case $d=1$ being clear. If $d>1$, set $I' = \{i_1,\dots,i_{d-1}\}, J' = \sigma(I')$. Let $1 \le j_1' < \cdots < j_{d-1}' \le n$ be the elements of J'. By the induction hypothesis, the left hand side of (7.4) equals

$$v_{\sigma(i_d)}(q^{2(d-1)}z)\operatorname{sgn}_{I'}(\sigma,\zeta) v_{j_{d-1}'}(q^{2(d-2)}z)\cdots v_{j_1'}(z). \tag{7.6}$$

Now $v_{\sigma(i_d)}(q^{2(d-1)}z)$ commutes with $\operatorname{sgn}_{I'}(\sigma,\zeta)$ since the latter only involves ζ_{ij} with $i,j \ne \sigma(i_d)$. Using the commutation relations (6.3b) we obtain

$$\operatorname{sgn}_{I'}(\sigma,\zeta)\cdot \prod_{j\in J', j>\sigma(i_d)} \frac{E(\zeta_{j\sigma(i_d)}+1)}{E(\zeta_{\sigma(i_d)j}+1)} \cdot v_{j_d}(q^{2(d-1)}z)\cdots v_{j_1}(z). \tag{7.7}$$

Replace $j \in J'$ such that $j > \sigma(i_d)$ by $\sigma(i)$ where $i \in I, i < i_d, \sigma(i) > \sigma(i_d)$. $\square$

Introduce the normalized monomials

$$v_I(z) = F_I(\zeta)^{-1} v_{i_r}(q^{2(d-1)}z) v_{i_{r-1}}(q^{2(d-2)}z)\cdots v_{i_1}(z) \in \Lambda, \tag{7.8}$$

$$w^I(z) = F^I(\zeta) w^{i_1}(z) w^{i_2}(q^2 z)\cdots w^{i_d}(q^{2(d-1)}z) \in \Lambda'. \tag{7.9}$$

Corollary 7.2. *Let $I \subseteq [1,n]$. For any permutation $\sigma \in S_n$,*

$$\sigma(v_I(z)) = v_{\sigma(I)}(z), \qquad \sigma(w^I(z)) = w^{\sigma(I)}(z), \tag{7.10}$$

for any $z \in \mathbb{C}^\times$. In particular $v_I(z)$ and $w^I(z)$ are fixed by any permutation which preserves the subset I.

Proof. Let $J = \sigma(I)$. Then

$$\begin{aligned}\sigma(v_I(z)) &= \sigma(F_I(\zeta)^{-1})v_{\sigma(i_d)}(q^{2(d-1)}z)\cdots v_{\sigma(i_1)}(z) = \\ &= \sigma(F_I(\zeta))^{-1}\operatorname{sgn}_I(\sigma;\zeta)v_{j_d}(q^{2(d-1)}z)\cdots v_{j_1}(z) = v_{\sigma(I)}(z).\end{aligned}$$

The proof for $w^I(z)$ is analogous. □

Proposition 7.3. *For $I,J \subseteq [1,n]$ and $z \in \mathbb{C}^\times$, the left and right elliptic minors, $\overleftarrow{\xi}{}^J_I(z)$ and $\overrightarrow{\xi}{}^J_I(z)$ respectively, can be defined by*

$$\Delta_\Lambda(v_I(z)) = \sum_J \overleftarrow{\xi}{}^J_I(z) \otimes v_J(z), \tag{7.11}$$

$$\Delta_{\Lambda'}(w^J(z)) = \sum_I w^I(z) \otimes \overrightarrow{\xi}{}^J_I(z), \tag{7.12}$$

where the sums are taken over all subsets of $[1,n]$.

If $\#I \neq \#J$, then $\overleftarrow{\xi}{}^J_I(z) = 0 = \overrightarrow{\xi}{}^J_I(z)$ for all z. If $\#I = \#J = d$, they are explicitly given by

$$\begin{aligned}&\overleftarrow{\xi}{}^J_I(z) = \\ &= \frac{F_J(\rho)}{F_I(\lambda)}\sum_{\tau \in S_J}\frac{\operatorname{sgn}_J(\tau;\rho)}{\operatorname{sgn}_I(\sigma;\lambda)}e_{\sigma(i_d)\tau(j_d)}(q^{2(d-1)}z)e_{\sigma(i_{d-1})\tau(j_{d-1})}(q^{2(d-2)}z)\cdots e_{\sigma(i_1)\tau(j_1)}(z)\end{aligned} \tag{7.13}$$

for any $\sigma \in S_I$, and

$$\overrightarrow{\xi}{}^J_I(z) = \frac{F^J(\rho)}{F^I(\lambda)}\sum_{\sigma \in S_I}\frac{\operatorname{sgn}^J(\tau;\rho)}{\operatorname{sgn}^I(\sigma;\lambda)}e_{\sigma(i_1)\tau(j_1)}(z)e_{\sigma(i_2)\tau(j_2)}(q^2z)\cdots e_{\sigma(i_d)\tau(j_d)}(q^{2(d-1)}z) \tag{7.14}$$

for any $\tau \in S_J$. Moreover,

$$(\sigma,\tau)(\overleftarrow{\xi}{}^J_I(z)) = \overleftarrow{\xi}{}^{\tau(J)}_{\sigma(I)}(z) \quad \text{and} \quad (\sigma,\tau)(\overrightarrow{\xi}{}^J_I(z)) = \overrightarrow{\xi}{}^{\tau(J)}_{\sigma(I)}(z) \tag{7.15}$$

for any $(\sigma,\tau) \in S_n \times S_n$ and $z \in \mathbb{C}^\times$.

Remark 7.4. In Theorem 7.10 we will prove that, in fact, $\overleftarrow{\xi}{}^J_I(z) = \overrightarrow{\xi}{}^J_I(z)$.

Proof. We prove the statements concerning the left elliptic minor $\overleftarrow{\xi}{}^J_I(z)$. We have

$$\Delta_\Lambda(v_I(z)) = \sum_{1\le k_1,\dots,k_d\le n} F_I(\lambda)^{-1} e_{i_1k_1}(q^{2(d-1)}z)\cdots e_{i_dk_d}(z)\otimes v_{k_1}(q^{2(d-1)}z)\cdots v_{k_d}(z) =$$
$$= \sum_{J,\#J=d}\sum_{\tau\in S_J} F_I(\lambda)^{-1} e_{i_1\tau(j_1)}(q^{2(d-1)}z)\cdots e_{i_d\tau(j_d)}(z)\otimes v_{\tau(j_1)}(q^{2(d-1)}z)\cdots v_{\tau(j_d)}(z) =$$
$$= \sum_{J,\#J=d}\left(\sum_{\tau\in S_J}\frac{\tau(F_J(\rho))}{F_I(\lambda)} e_{i_1\tau(j_1)}(q^{2(d-1)}z)\cdots e_{i_d\tau(j_d)}(z)\right)\otimes v_J(z).$$

Thus (7.11) holds when $\overleftarrow{\xi}{}^J_I(z)$ is defined by (7.13) with $\sigma = \mathrm{Id}$. Then the right hand side of (7.13) equals $(\sigma,\mathrm{Id})(\overleftarrow{\xi}{}^J_I(z))$. Thus only (7.15) remains. Using (6.10) and Corollary 7.2 we have

$$\Delta_\Lambda\big(\sigma(v_I(z))\big) = ((\sigma,\tau)\otimes\tau)\big(\Delta_\Lambda(v_I(z))\big) = \sum_J (\sigma,\tau)(\overleftarrow{\xi}{}^J_I(z))\otimes v_{\tau(J)}(z)).$$

On the other hand, again by Corollary 7.2,

$$\Delta_\Lambda\big(\sigma(v_I(z))\big) = \Delta_\Lambda(v_{\sigma(I)}(z)) = \sum_J \overleftarrow{\xi}{}^{\tau(J)}_{\sigma(I)}(z)\otimes v_{\tau(J)}(z)),$$

where we made the substitution $J\mapsto\tau(J)$. This proves the first equality in (7.15). The statements concerning the right elliptic minors are proved analogously. □

7.2 The Cherednik operator

The goal of this section is to prove Theorem 7.10 stating that the left and right elliptic minors coincide. We use ideas from Section 3 of [FV97], where the authors study representations of the elliptic quantum group $E_{r,\gamma/2}(\mathfrak{gl}_N)$ and associate a linear operator (product of R-matrices) on $V^{\otimes n}$ to each diagram of a certain form, a kind of braid group representation. Then they consider the operator associated to the longest permutation, in [ZSY03] called the Cherednik operator. Instead of working with representations, we proceed inside the $\mathfrak{h}$-bialgebroid $\mathscr{F}_{\mathrm{ell}}(M(n))$ and consider certain operators on $V^{\otimes n}\otimes\mathscr{F}_{\mathrm{ell}}(M(n))$ depending on n spectral parameters. Using the analog of the Cherednik operator we prove an extended RLL-relation (7.25). Theorem 7.10 then follows by extracting matrix elements from both sides of this matrix equation.

In this section, we set $\mathscr{F}=\mathscr{F}_{\mathrm{ell}}(M(n))$. Recall the operators $\mathsf{R}(\lambda,z)$, $\mathsf{R}(\rho,z)$ from Section 4.3. For $z\in\mathbb{C}^\times$, define the following linear operators on $V^{\otimes n}\otimes\mathscr{F}$:

$$\mathsf{R}^{ij}(\lambda,z) := \lim_{w\to z}\theta(q^2w)\mathsf{R}(\lambda,w)^{(i,j,n+1)},\qquad \mathsf{R}^{ij}(\rho,z) := \lim_{w\to z}\theta(q^2w)\mathsf{R}(\rho,w)^{(i,j,n+1)}.$$

The limits are taken in the sense of taking limits of each matrix element. These operators are well-defined for any z, since we multiply away the singularities in z of α and β (3.4),(3.5). Furthermore, due to the RLL relations (4.10) we have

$$\mathsf{R}^{12}(\lambda,\frac{z_1}{z_2})L^1(z_1)L^2(z_2) = L^2(z_2)L^1(z_1)\mathsf{R}^{12}(\rho+h^1+h^2,\frac{z_1}{z_2}) \tag{7.16}$$

for any $z_1, z_2 \in \mathbb{C}^\times$.

Let $\mathscr{E}_n$ denote the algebra of all functions

$$F : (\mathbb{C}^\times)^n \to \mathrm{End}(V^{\otimes n} \otimes \mathscr{F}).$$

The symmetric group S_n acts on $\mathscr{E}_n$ by

$$\sigma(F(z)) = (\sigma \otimes \mathrm{Id}_{\mathscr{F}}) \circ F(\sigma(z)) \circ (\sigma^{-1} \otimes \mathrm{Id}_{\mathscr{F}}) \tag{7.17}$$

for $F(z) \in \mathscr{E}_n$ and $\sigma \in S_n$. In the right hand side of (7.17), σ acts on $(\mathbb{C}^\times)^n$ by permuting coordinates, and on $V^{\otimes n}$ by permuting the tensor factors. For example we have

$$(23)\left(\mathsf{R}^{12}(\lambda, z_1/z_2)\right) = \mathsf{R}^{13}(\lambda, z_1/z_3).$$

Consider the skew group algebra $\mathscr{E}_n * S_n$, defined as the algebra with underlying space $\mathscr{E}_n \otimes \mathbb{C}S_n$, where $\mathbb{C}S_n$ is the group algebra, with the multiplication

$$(F(z) \otimes \sigma)(G(z) \otimes \tau) = F(z)\sigma(G(z)) \otimes \sigma\tau \tag{7.18}$$

for $\sigma, \tau \in S_n$, $F(z), G(z) \in \mathscr{E}_n$. Since σ acts on $\mathscr{E}_n$ by automorphisms, $\mathscr{E}_n * S_n$ is an associative algebra. The constant function $z \mapsto \mathrm{Id}_{V^{\otimes n} \otimes \mathscr{F}} \otimes (1)$ is the unit element. Let B_n be the monoid (set with unital associative multiplication) generated by $\{s_1, \ldots, s_{n-1}\}$ and relations

$$\begin{aligned} s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1} \quad \text{for } 1 \le i \le n-2, \\ s_i s_j &= s_j s_i \quad \text{if } |i-j| > 1. \end{aligned}$$

Let $\sigma_i = (i \;\; i+1) \in S_n$. We have an epimorphism $\pi : B_n \to S_n$ given by $\pi(s_i) = \sigma_i$, $\pi(1) = (1)$. Define

$$\begin{aligned} W(1) &= \mathrm{Id}_{V^{\otimes n} \otimes \mathscr{F}} \otimes (1), \\ W(s_i) &= \mathsf{R}^{i,i+1}(\lambda - h^{\ge i+2}, z_i/z_{i+1}) \otimes \sigma_i. \end{aligned}$$

Here and below we use $h^{\ge k}$ to denote the expression $\sum_{j=k}^n h^j$.

Proposition 7.5. *W extends to a well-defined map*

$$W : B_n \to \mathscr{E}_n * S_n$$

satisfying $W(b_1 b_2) = W(b_1)W(b_2)$ for any $b_1, b_2 \in B_n$.

Proof. We have to show the relations

$$W(s_i)W(s_{i+1})W(s_i) = W(s_{i+1})W(s_i)W(s_{i+1}), \tag{7.19}$$

$$W(s_i)W(s_j) = W(s_j)W(s_i) \quad \text{if } |i-j| > 1. \tag{7.20}$$

Relation (7.19) follows from the QDYBE (3.1). For example, $W(s_i)W(s_{i+1})W(s_i)$ equals

$$\mathsf{R}^{i,i+1}(\lambda - h^{\ge i+2}, \frac{z_i}{z_{i+1}})\mathsf{R}^{i,i+2}(\lambda - h^{\ge i+3}, \frac{z_i}{z_{i+2}})\mathsf{R}^{i+1,i+2}(\lambda - h^i - h^{\ge i+3}, \frac{z_{i+1}}{z_{i+2}}) \otimes \sigma_i\sigma_{i+1}\sigma_i$$

Relation (7.20) is easy to check, using the $\mathfrak{h}$-invariance of R. ∎

For $b \in B_n$ we define $W_b(\lambda,z) \in \mathscr{E}_n$ by

$$W(b) = W_b(\lambda,z) \otimes \pi(b). \tag{7.21}$$

From this and the product rule (7.18) follows that

$$W_{b_1 b_2}(\lambda,z) = W_{b_1}(\lambda,z) \cdot \pi(b_1)\left(W_{b_2}(\lambda,z)\right) \tag{7.22}$$

for $b_1, b_2 \in B_n$. By replacing λ by ρ we get similarly operators $W_b(\rho,z)$.

Recall the operator $\mathsf{L}(z)$ from Section 4.3. Define for $z \in \mathbb{C}^\times$, $i \in [1,n]$,

$$\mathsf{L}^i(z) = \mathsf{L}(z)^{(i,n+1)} \in \mathrm{End}(V^{\otimes n} \otimes \mathscr{F}).$$

If i,j,k are distinct, then one can check that

$$\mathsf{R}^{ij}(\lambda - h^k, z)\mathsf{L}^k(w) = \mathsf{L}^k(w)\mathsf{R}^{ij}(\lambda,z), \tag{7.23}$$

$$\mathsf{R}^{ij}(\rho,z)\mathsf{L}^k(w) = \mathsf{L}^k(w)\mathsf{R}^{ij}(\rho + h^k, z). \tag{7.24}$$

Define $t_d \in B_n$, $d \in [1,n]$, recursively by

$$t_d = \begin{cases} t_{d-1}s_{d-1}s_{d-2}\cdots s_1, & d > 1 \\ 1, & d = 1. \end{cases}$$

Let τ_d be the image of t_d in S_n:

$$\tau_d := \pi(t_d) = \begin{pmatrix} 1 & 2 & \cdots & d & d+1 & \cdots & n \\ d & d-1 & \cdots & 1 & d+1 & \cdots & n \end{pmatrix} \in S_n.$$

Proposition 7.6. *Let $1 \le d \le n$. For any $z = (z_1,\dots,z_d) \in (\mathbb{C}^\times)^d$ we have*

$$W_{t_d}(\lambda,z)\mathsf{L}^1(z_1)\cdots\mathsf{L}^d(z_d) = \mathsf{L}^d(z_d)\cdots\mathsf{L}^1(z_1)W_{t_d}(\rho + h^{\le d}, z). \tag{7.25}$$

Proof. We use induction on d. The case $d=1$ is trivial, while $d=2$ is the RLL relation (7.16). If $d>2$, write $t_d = t_{d-1}u_d$, where $u_d = s_{d-1}s_{d-2}\cdots s_1$. Thus, by (7.22),

$$W_{t_d}(\lambda,z) = W_{t_{d-1}}(\lambda,z)\cdot\tau_{d-1}\left(W_{u_d}(\lambda,z)\right). \tag{7.26}$$

We claim that

$$\begin{aligned}\tau_{d-1}\left(W_{u_d}(\lambda,z)\right) L^1(z_1)\cdots L^d(z_d) &= \\ &= L^d(z_d)L^1(z_1)\cdots L^{d-1}(z_{d-1})\tau_{d-1}\left(W_{u_d}(\rho+h^{\le d},z)\right).\end{aligned} \tag{7.27}$$

For notational simplicity, set $\lambda' = \lambda - h^{>d}$.

$$W_{u_d}(\lambda,z) = \mathsf{R}^{d-1,d}(\lambda', \frac{z_{d-1}}{z_d})\mathsf{R}^{d-2,d}(\lambda' - h^{d-1}, \frac{z_{d-2}}{z_d})\cdots\mathsf{R}^{1,d}(\lambda' - h^{[2,d-1]}, \frac{z_1}{z_d}),$$

where $h^{[a,b]}$ means $\sum_{a\le j\le b} h^j$. Thus

$$\tau_{d-1}\left(W_{u_d}(\lambda,z)\right) = \mathsf{R}^{1,d}(\lambda', \frac{z_1}{z_d})\mathsf{R}^{2,d}(\lambda' - h^1, \frac{z_2}{z_d})\cdots\mathsf{R}^{d-1,d}(\lambda' - h^{\le d-2}, \frac{z_{d-1}}{z_d}). \tag{7.28}$$

Using (7.23) and the RLL relation (7.16) repeatedly, we obtain (7.27). Now the proposition follows by induction on d, using that

$$W_{t_{d-1}}(\lambda,z)L^d(z_d) = L^d(z_d)W_{t_{d-1}}(\lambda+h^d,z)$$

which follows from (7.23).

□

The operator $\mathsf{C}(\lambda,z) := W_{t_n}(\lambda,z)$ is called the *Cherednik operator*. For an operator $F(z) \in \mathscr{E}_n$ we define its matrix elements $F(z)^{a_1,\dots,a_n}_{x_1,\dots,x_n} \in \mathscr{F}$ by

$$F(z)(e_{a_1}\otimes\cdots\otimes e_{a_n}\otimes 1) = \sum_{x_1,\dots,x_n} e_{x_1}\otimes\cdots\otimes e_{x_n}\otimes F(z)^{a_1,\dots,a_n}_{x_1,\dots,x_n}.$$

Proposition 7.7. *Let*

$$\tilde\alpha(\lambda,z) = \lim_{w\to z}\theta(q^2w)\alpha(\lambda,w) = \theta(z)\theta(q^{2(\lambda+1)})/\theta(q^{2\lambda}). \tag{7.29}$$

Then

$$\mathsf{C}(\lambda,z)^{1,\dots,n}_{1,\dots,n} = \prod_{i<j}\tilde\alpha(\lambda_{ij},z_i/z_j) = \prod_{i<j} q\theta(z_i/z_j)\cdot\frac{F_{[1,n]}(\lambda)}{F^{[1,n]}(\lambda)}.$$

Proof. The second equality follows from the definition, (7.1), of F_I and F^I. We prove by induction on d that $W_{t_d}(\lambda,z)^{1,\dots,n}_{1,\dots,n} = \prod_{i<j\le d}\tilde\alpha(\lambda_{ij},z_i/z_j)$. For $d=2$ we have $t_d = s_1$ and $W_{s_1}(\lambda,z)^{1,\dots,n}_{1,\dots,n} = \mathsf{R}^{12}(\lambda-h^{>2},z_1/z_2)^{1,\dots,n}_{1,\dots,n} = \tilde\alpha(\lambda_{12},z_1/z_2)$ as claimed. For $d>2$, using the factorization (7.26) we have

$$W_{t_d}(\lambda,z)^{1,\dots,n}_{1,\dots,n} = \sum_{x_1,\dots,x_n} W_{t_{d-1}}(\lambda,z)^{x_1,\dots,x_n}_{1,\dots,n}\tau_{d-1}(W_{u_d}(\lambda,z))^{1,\dots,n}_{x_1,\dots,x_n}. \tag{7.30}$$

Since $W_{t_{d-1}}(\lambda,z)$ is a product of operators of the form $\sigma(\mathsf{R}^{ii+1}(\lambda,z_i/z_{i+1}))$ where $1\le i\le d-2$ and $\sigma\in S_n$, $\sigma(j)=j, j>d-1$, and each of these operators preserve the subspace spanned by $e_{\tau(1)}\otimes\cdots\otimes e_{\tau(d-1)}\otimes e_d\otimes\cdots\otimes e_n\otimes a$ where $\tau\in S_{d-1}$ and $a\in\mathscr{F}$, the operator $W_{t_{d-1}}(\lambda,z)$ also preserves this subspace. This means that $W_{t_{d-1}}(\lambda,z)^{x_1,\dots,x_n}_{1,\dots,n} = 0$ unless $x_j = j$ for $j\ge d$ and $\{x_1,\dots,x_{d-1}\} = \{1,\dots,d-1\}$. Furthermore, by (7.28),

$$\begin{aligned}\tau_{d-1}(W_{u_d}(\lambda,z))^{1,\dots,n}_{x_1,\dots,x_{d-1},d,\dots,n} &= \\ = \sum_{y_2,\dots,y_{d-1}} \tilde R^{1y_2}_{x_1d}(\lambda,\frac{z_1}{z_d})\tilde R^{2y_3}_{x_2y_2}(\lambda-\omega(1),\frac{z_2}{z_d})&\cdots\tilde R^{d-1,d}_{x_{d-1}y_{d-1}}(\lambda-\sum_{k\le d-2}\omega(k),\frac{z_{d-1}}{z_d}).\end{aligned} \tag{7.31}$$

Here $\tilde R^{ab}_{xy}(\lambda,z) = \lim_{w\to z}\theta(q^2w)R^{ab}_{xy}(\lambda,w)$. Since $\tilde R^{ab}_{xy}(\lambda,z) = 0$ unless $\{x,y\} = \{a,b\}$, we deduce that, when $\{x_1,\dots,x_{d-1}\} = \{1,\dots,d-1\}$, the terms in the sum (7.31) are zero unless $x_i = i$ for all i and $y_j = d$ for all j. Substituting into (7.30) the claim follows by induction. □

Lemma 7.8. *Fix $2 \le d \le n$ and $i < d$. Then there are elements $b, c \in B_n$ such that $t_d = s_i b$ and $t_d = c s_i$.*

Proof. Since $t_2 = s_1$ and $t_3 = s_1 s_2 s_1 = s_2 s_1 s_2$, the statement clearly holds for $d = 2, 3$. Assuming $d > 3$, we first prove the existence of b. If $i < d-1$ then by induction there is a $b' \in B_n$ such that $t_{d-1} = s_i b'$. Hence $t_d = t_{d-1} s_{d-1} \cdots s_1 = s_i b' s_{d-1} \cdots s_1$. Thus we can take $b = b' s_{d-1} \cdots s_1$. If $i = d-1$, write $t_d = t_{d-2} s_{d-2} \cdots s_1 s_{d-1} \cdots s_1$. Then move each of the $d-1$ rightmost factors $s_{d-1}, \ldots, s_1$ as far to the left as possible, using that $s_j s_k = s_k s_j$ when $|j-k| > 1$. This gives

$$t_d = t_{d-2} s_{d-2} s_{d-1} s_{d-3} s_{d-2} s_{d-4} \cdots s_2 s_3 s_1 s_2 s_1.$$

Then use $s_j s_{j+1} s_j = s_{j+1} s_j s_{j+1}$ repeatedly, working from right to left, to obtain

$$t_d = t_{d-2} s_{d-1} s_{d-2} s_{d-1} s_{d-3} s_{d-2} \cdots s_4 s_2 s_3 s_1 s_2.$$

Finally, s_{d-1} can be moved to the left of t_{d-2} since the latter is a product of s_j's with $j \le d-3$.

To prove the existence of c we note that B_n carries an involution $* : B_n \to B_n$ satisfying $(a_1 a_2)^* = a_2^* a_1^*$ for any $a_1, a_2 \in B_n$, defined by $s_j^* = s_j$ for $j \in [1, n]$ and $1^* = 1$. Thus it suffices to show that $t_d^* = t_d$ for any d. This is trivial for $d = 2, 3$. When $d > 3$ we have, by induction on d,

$$\begin{aligned} t_d^* &= (t_{d-1} s_{d-1} \cdots s_1)^* = s_1 \cdots s_{d-1} t_{d-1} = \\ &= s_1 \cdots s_{d-1} t_{d-2} s_{d-2} \cdots s_1 = \\ &= s_1 \cdots s_{d-2} t_{d-2} s_{d-1} s_{d-2} \cdots s_1 = \qquad \text{(since } s_{d-1} \text{ commutes with } t_{d-2}) \\ &= t_{d-1}^* s_{d-1} \cdots s_1 = t_d. \end{aligned}$$

□

Proposition 7.9. *Let $w = (z_0, q^2 z_0, \ldots, q^{2(n-1)} z_0)$, where $z_0 \neq 0$ is arbitrary, and let $\sigma, \tau \in S_n$. Then*

$$\mathsf{C}(\lambda, w)^{\tau(1),\ldots,\tau(n)}_{\sigma(1),\ldots,\sigma(n)} = \frac{\mathrm{sgn}_{[1,n]}(\sigma; \lambda)}{\mathrm{sgn}^{[1,n]}(\tau; \lambda)} \mathsf{C}(\lambda, w)^{1,\ldots,n}_{1,\ldots,n}. \tag{7.32}$$

Proof. First we claim that for all $\sigma, \tau \in S_n$ and each $i \in [1, n]$,

$$W_{s_i}(\lambda, w)^{\tau(1),\ldots,\tau(n)}_{\sigma\sigma_i(1),\ldots,\sigma\sigma_i(n)} = \sigma(\mathrm{sgn}_{[1,n]}(\sigma_i; \lambda)) W_{s_i}(\lambda, w)^{\tau(1),\ldots,\tau(n)}_{\sigma(1),\ldots,\sigma(n)}, \tag{7.33}$$

and

$$\begin{aligned} W_{s_i}(\lambda, w)^{\tau\sigma_i(1),\ldots,\tau\sigma_i(n)}_{\sigma(1),\ldots,\sigma(n)} &= \tau(\mathrm{sgn}^{[1,n]}(\sigma_i; \lambda)) W_{s_i}(\lambda, w)^{\tau(1),\ldots,\tau(n)}_{\sigma(1),\ldots,\sigma(n)} = \\ &= -W_{s_i}(\lambda, w)^{\tau(1),\ldots,\tau(n)}_{\sigma(1),\ldots,\sigma(n)}. \end{aligned} \tag{7.34}$$

Indeed, assume that $z_i / z_{i+1} = q^{-2}$ and that $\{a_1, \ldots, a_n\} = \{b_1, \ldots, b_n\} = [1, n]$. Then $W_{s_i}(\lambda, z)^{b_1,\ldots,b_n}_{a_1,\ldots,a_n} \neq 0$ iff $\{a_i, a_{i+1}\} = \{b_i, b_{i+1}\}$ in which case

$$W_{s_i}(\lambda, z)^{b_1,\ldots,b_n}_{a_1,\ldots,a_n} = \frac{E(1) E(\lambda_{a_i a_{i+1}} + 1)}{E(\lambda_{b_{i+1} b_i})}. \tag{7.35}$$

From this and the definitions of the sign functions, (7.2)-(7.3), the claims follow. Next, we prove (7.32) by induction on the sum ℓ of the lengths of σ and τ. If $\ell = 0$ it is trivial. Assuming (7.32) holds for (σ, τ) we prove it holds for $(\sigma\sigma_i, \tau)$ and $(\sigma, \tau\sigma_i)$ where i is arbitrary.

Let $i \in [1, n]$. By Lemma 7.8 we have $t_n = s_i b$ for some $b \in B_n$. We have

$$W_{t_n}(\lambda, w)^{\tau(1),\dots,\tau(n)}_{\sigma\sigma_i(1),\dots,\sigma\sigma_i(n)} = \left(W_{s_i}(\lambda, w)\sigma_i\left(W_b(\lambda, w)\right)\right)^{\tau(1),\dots,\tau(n)}_{\sigma\sigma_i(1),\dots,\sigma\sigma_i(n)} = \\ = \sum_{x_1,\dots,x_n} W_{s_i}(\lambda, w)^{x_1,\dots,x_n}_{\sigma\sigma_i(1),\dots,\sigma\sigma_i(n)}\sigma_i\left(W_b(\lambda, w)\right)^{\tau(1),\dots,\tau(n)}_{x_1,\dots,x_n}.$$

As in the proof of Proposition 7.7, $W_{s_i}(\lambda, w)^{x_1,\dots,x_n}_{\sigma\sigma_i(1),\dots,\sigma\sigma_i(n)}$ is zero if $x_1, \dots, x_n$ is not a permutation of $1, \dots, n$. Using (7.33) we obtain

$$\sigma(\mathrm{sgn}_{[1,n]}(\sigma_i; \lambda)) \sum_{x_1,\dots,x_n} W_{s_i}(\lambda, w)^{x_1,\dots,x_n}_{\sigma(1),\dots,\sigma(n)}\sigma_i\left(W_b(\lambda, w)\right)^{\tau(1),\dots,\tau(n)}_{x_1,\dots,x_n} = \\ = \sigma(\mathrm{sgn}_{[1,n]}(\sigma_i; \lambda))W_{t_n}(\lambda, w)^{\tau(1),\dots,\tau(n)}_{\sigma(1),\dots,\sigma(n)}.$$

Using the induction hypothesis and the relation $\mathrm{sgn}_{[1,n]}(\sigma; \lambda)\sigma(\mathrm{sgn}_{[1,n]}(\sigma_i; \lambda)) = \mathrm{sgn}_{[1,n]}(\sigma\sigma_i; \lambda)$ we obtain (7.32) for $(\sigma\sigma_i, \tau)$.

For the other case, let i be arbitrary and set $j = \tau_n(i)$. By Lemma 7.8 there is a $c \in B_n$ such that $t_n = cs_j$. Recall the surjective morphism $\pi : B_n \to S_n$ sending s_i to $\sigma_i = (i\; i+1)$. Then $\sigma_j\pi(c) = \pi(c)\sigma_i$. We have

$$W_{t_n}(\lambda, w)^{\tau\sigma_i(1),\dots,\tau\sigma_i(n)}_{\sigma(1),\dots,\sigma(n)} = \left(W_c(\lambda, w)\cdot\pi(c)(W_{s_j}(\lambda, w))\right)^{\tau\sigma_i(1),\dots\tau\sigma_i(n)}_{\sigma(1),\dots,\sigma(n)} = \\ = \sum_{x_1,\dots,x_n} W_c(\lambda, w)^{x_1,\dots,x_n}_{\sigma(1),\dots,\sigma(n)}\pi(c)\left(W_{s_j}(\lambda, w)\right)^{\tau\sigma_i(1),\dots,\tau\sigma_i(n)}_{x_1,\dots,x_n}.$$

It is easy to check that $\sigma(F(z))^{b_1,\dots,b_n}_{a_1,\dots,a_n} = F(\sigma(z))^{b_{\sigma(1)},\dots,b_{\sigma(n)}}_{a_{\sigma(1)},\dots,a_{\sigma(n)}}$ for any $F(z) \in \mathscr{E}_n$ and $\sigma \in S_n$. Define w_i by $(w_1, \dots, w_n) = w = (z_0, q^2z_0, \dots, q^{2(n-1)}z_0)$. Then $w_i/w_{i+1} = q^{-2}$ for each i. Set $w' = (w_{\pi(c)(1)}, \dots, w_{\pi(c)(n)})$. For each i, $w_{\pi(c)(i)}/w_{\pi(c)(i+1)} = w_{\tau_n(i+1)}/w_{\tau_n(i)} = q^{-2}$ also. Therefore

$$W_{t_n}(\lambda, w)^{\tau\sigma_i(1),\dots,\tau\sigma_i(n)}_{\sigma(1),\dots,\sigma(n)} = \sum_{x_1,\dots,x_n} W_c(\lambda, w)^{x_1,\dots,x_n}_{\sigma(1),\dots,\sigma(n)}W_{s_j}(\lambda, w')^{\tau\sigma_i\pi(c)(1),\dots,\tau\sigma_i\pi(c)(n)}_{x_{\pi(c)(1)},\dots,x_{\pi(c)(n)}} = \\ = \sum_{x_1,\dots,x_n} W_c(\lambda, w)^{x_1,\dots,x_n}_{\sigma(1),\dots,\sigma(n)}W_{s_j}(\lambda, w')^{\tau\pi(c)\sigma_j(1),\dots,\tau\pi(c)\sigma_j(n)}_{x_{\pi(c)(1)},\dots,x_{\pi(c)(n)}} = \\ = \sum_{x_1,\dots,x_n} W_c(\lambda, w)^{x_1,\dots,x_n}_{\sigma(1),\dots,\sigma(n)}(\mathrm{sgn}\,\sigma_j)W_{s_j}(\lambda, w')^{\tau\pi(c)(1),\dots,\tau\pi(c)(n)}_{x_{\pi(c)(1)},\dots,x_{\pi(c)(n)}} = \\ = \sum_{x_1,\dots,x_n} W_c(\lambda, w)^{x_1,\dots,x_n}_{\sigma(1),\dots,\sigma(n)}(-1)\pi(c)(W_{s_j}(\lambda, w)^{\tau(1),\dots,\tau(n)}_{x_1,\dots,x_n} = \\ = -W_{t_n}(\lambda, w)^{\tau(1),\dots,\tau(n)}_{\sigma(1),\dots,\sigma(n)}.$$

By the induction hypothesis it follows that (7.32) holds for $(\sigma, \tau\sigma_i)$. This proves the formula (7.32). ☐

Theorem 7.10. *For any subsets $I, J \subseteq [1,n]$ and $z \in \mathbb{C}^\times$, the left and right elliptic minors coincide:*

$$\overleftarrow{\xi}^{J}_{I}(z) = \overrightarrow{\xi}^{J}_{I}(z).$$

We denote this common element by $\xi^J_I(z)$.

Proof. If $\#I \neq \#J$ both sides are zero. Suppose $\#I = \#J = d$. By relation (7.15) we can, after applying a suitable automorphism, assume that $I = J = [1,d]$. Since the subalgebra of $\mathscr{F}$ generated by $e_{ij}(z)$, $i,j \in [1,d]$, $z \in \mathbb{C}^\times$ and $f(\lambda), f(\rho)$ with $f \in M_{\mathfrak{h}^*_d} \subseteq M_{\mathfrak{h}^*}$, $\mathfrak{h}_d$ being the Cartan subalgebra of $\mathfrak{sl}(d)$, is isomorphic to $\mathscr{F}_{\mathrm{ell}}(M(d))$, we can also assume $d = n$. Identifying the matrix element ${}^{1,\dots,n}_{1,\dots,n}$ on both sides of (7.25) we get

$$\sum_{x_1,\dots,x_n} \mathsf{C}(\lambda,z)^{x_1,\dots,x_n}_{1,\dots,n} e_{x_1,1}(z_1)\cdots e_{x_n,n}(z_n) = \\ = \sum_{x_1,\dots,x_n} e_{n,x_n}(z_n)\cdots e_{1,x_1}(z_1)\mathsf{C}(\rho+h^{\le n},z)^{1,\dots,n}_{x_1,\dots,x_n}.$$

As in the proof of Proposition 7.7, $\mathsf{C}(\lambda,z)^{x_1,\dots,x_n}_{1,\dots,n}$ is zero if $x_1,\dots,x_n$ is not a permutation of $1,\dots,n$. Taking $z = w = (z_0, q^2 z_0, \dots, q^{2(n-1)}z_0)$ and dividing both sides by $\prod_{i<j} q\theta(w_i/w_j) = \prod_{i<j} q\theta(q^{2(i-j)})$ we get

$$\frac{F_{[1,n]}(\lambda)}{F^{[1,n]}(\lambda)} \sum_{\sigma\in S_n} \mathrm{sgn}^{[1,n]}(\sigma;\lambda)^{-1} e_{\sigma(1)1}(z_0)\cdots e_{\sigma(n)n}(q^{2(n-1)}z_0) = \\ = \frac{F_{[1,n]}(\rho)}{F^{[1,n]}(\rho)} \sum_{\tau\in S_n} \mathrm{sgn}_{[1,n]}(\tau;\rho) e_{n\tau(n)}(q^{2(n-1)}z_0)\cdots e_{1\sigma(1)}(z_0)$$

Multiplying by $\frac{F^{[1,n]}(\rho)}{F_{[1,n]}(\lambda)}$ and comparing with (7.13) and (7.14), we deduce that $\overrightarrow{\xi}^{[1,n]}_{[1,n]}(z_0) = \overleftarrow{\xi}^{[1,n]}_{[1,n]}(z_0)$, as desired. □

8 The cobraiding and the quantum determinant

8.1 A cobraiding for $\mathscr{F}_{\mathrm{ell}}(M(n))$

The following definition was given in [R04].

Definition 8.1. A *cobraiding* on an $\mathfrak{h}$-bialgebroid A is a $\mathbb{C}$-bilinear map $\langle\cdot,\cdot\rangle : A\times A \to$

$D_{\mathfrak{h}}$ such that, for any $a, b, c \in A$ and $f \in M_{\mathfrak{h}^*}$,

$$\langle A_{\alpha\beta}, A_{\gamma\delta}\rangle \subseteq (D_{\mathfrak{h}})_{\alpha+\gamma,\beta+\delta}, \tag{8.1a}$$

$$\langle \mu_r(f)a, b\rangle = \langle a, \mu_l(f)b\rangle = f\,T_0 \circ \langle a, b\rangle, \tag{8.1b}$$

$$\langle a\mu_l(f), b\rangle = \langle a, b\mu_r(f)\rangle = \langle a, b\rangle \circ f\,T_0, \tag{8.1c}$$

$$\langle ab, c\rangle = \sum_i \langle a, c_i'\rangle T_{\beta_i}\langle b, c_i''\rangle, \quad \Delta(c) = \sum_i c_i' \otimes c_i'', \quad c_i'' \in A_{\beta_i\gamma}, \tag{8.1d}$$

$$\langle a, bc\rangle = \sum_i \langle a_i'', b\rangle T_{\beta_i}\langle a_i', c\rangle, \quad \Delta(a) = \sum_i a_i' \otimes a_i'', \quad a_i'' \in A_{\beta_i\gamma}, \tag{8.1e}$$

$$\langle a, 1\rangle = \langle 1, a\rangle = \varepsilon(a), \tag{8.1f}$$

$$\sum_{ij} \mu_l\left(\langle a_i', b_j'\rangle 1\right) a_i'' b_j'' = \sum_{ij} \mu_r\left(\langle a_i'', b_j''\rangle 1\right) b_j' a_i'. \tag{8.1g}$$

Proposition 8.2. *Let $R : \mathfrak{h}^* \times \mathbb{C}^\times \to \mathrm{End}_{\mathfrak{h}}(V \otimes V)$ be a meromorphic function. Let A_R be the $\mathfrak{h}$-bialgebroid associated to R as in Section 4.2. Assume that $\varphi : \mathbb{C}^\times \to \mathbb{C}$ is a holomorphic function, not vanishing identically, such that, for each $x, y, a, b \in X$, $z \in \mathbb{C}^\times$, the limit $\lim_{w\to z}\left(\varphi(w)R_{xy}^{ab}(\zeta, w)\right)$ exists and defines a meromorphic function in $M_{\mathfrak{h}^*}$. Then the following statements are equivalent:*

(i) there exists a cobraiding $\langle\cdot,\cdot\rangle : A_R \times A_R \to D_{\mathfrak{h}}$ satisfying

$$\langle L_{ij}(z_1), L_{kl}(z_2)\rangle = \lim_{w\to z_1/z_2}\left(\varphi(w)R_{ik}^{jl}(\zeta, w)\right) T_{-\omega(i)-\omega(k)}, \tag{8.2}$$

(ii) R satisfies the QDYBE (3.1).

Remark 8.3. a) The identity (8.1g) is not necessary when proving that (i) implies (ii). Without assuming (8.1g), $\langle\cdot,\cdot\rangle$ is a *paring on $A^{cop} \times A$.* See [R04].

b) Without the factor $\varphi(w)$, the cobraiding is not well-defined if $R(\zeta, z)$ has poles in the z variable. We also remark that the residual relations (4.10) are necessary for (ii) to imply (i).

Proof. The proof is straightforward and is carried out in [N05], Lemma 2.2.5, under the assumption that the R-matrix is regular in the spectral variable. □

Corollary 8.4. *The $\mathfrak{h}$-bialgebroid $\mathscr{F}_{ell}(M(n))$ carries a cobraiding $\langle\cdot,\cdot\rangle$ satisfying*

$$\langle e_{ij}(z), e_{kl}(w)\rangle = \tilde{R}_{ik}^{jl}(\zeta, z/w) T_{-\omega(i)-\omega(k)} \qquad \forall z, w \in \mathbb{C}^\times, i, j \in [1, n], \tag{8.3}$$

where

$$\tilde{R}_{ik}^{jl}(\zeta, z) = \lim_{w\to z}\left(\theta(q^2 w)R_{ik}^{jl}(\zeta, w)\right). \tag{8.4}$$

Proof. It suffices to notice that, by (5.3), (3.4),(3.5), $\tilde{R}$ is regular in z, and apply Proposition 8.2. □

8.2 Properties of the quantum determinant

Let $A = \mathscr{F}_{\mathrm{ell}}(M(n))$. When $I = J = [1,n]$ we set

$$\det(z) = \xi_I^J(z) \tag{8.5}$$

for $z \in \mathbb{C}^\times$, where $\xi_I^J(z)$ is the elliptic minor given in Theorem 7.10. Thus one possible expression for $\det(z)$ is

$$\det(z) = \sum_{\sigma \in S_n} \frac{F^{[1,n]}(\rho)}{\sigma\big(F^{[1,n]}(\lambda)\big)} e_{\sigma(1)1}(z) e_{\sigma(2)2}(q^2 z) \cdots e_{\sigma(n)n}(q^{2(n-1)}z). \tag{8.6}$$

Theorem 8.5. *a)* $\det(z)$ *is a grouplike element of* A *for each* $z \in \mathbb{C}^\times$*, i.e.*

$$\Delta(\det(z)) = \det(z) \otimes \det(z), \qquad \varepsilon(\det(z)) = 1.$$

b) $\det(z)$ *is almost central in the following sense:*

$$[e_{ij}(z), \det(w)] = [f(\lambda), \det(w)] = [f(\rho), \det(w)] = 0 \tag{8.7}$$

for all $f \in M_{\mathfrak{h}^*}$*,* $i, j \in [1,n]$ *and all* $z, w \in \mathbb{C}^\times$ *such that*

$$z/w \notin p^{\mathbb{Z}} \cdot \{q^2, q^4, \ldots, q^{2(n-2)}\}. \tag{8.8}$$

Proof. Let $\Lambda^n(z) = M_{\mathfrak{h}^*} v_I(z)$, where $I = [1,n]$. It is a one-dimensional subcorepresentation of the left exterior corepresentation Λ. Its matrix element is $\det(z)$, i.e.

$$\Delta(v_I(z)) = \det(z) \otimes v_I(z).$$

From the coassociativiy and counity axioms for a corepresentation follows that $\det(z)$ is grouplike, proving part a).

Let us prove part b). It follows from the definition that $\det(z) \in A_{00}$ and thus it commutes with $f(\rho)$ and $f(\lambda)$ for any $f \in M_{\mathfrak{h}^*}$. Applying the cobraiding identity (8.1g) to $a = e_{ij}(z)$ and $b = \det(w)$ and using that $\det(w)$ is grouplike we get

$$\sum_{x=1}^n \mu_l\big(\langle e_{ix}(z), \det(w)\rangle 1\big) e_{xj}(z) \det(w) = \sum_{x=1}^n \mu_r\big(\langle e_{xj}(z), \det(w)\rangle 1\big) \det(w) e_{ix}(z). \tag{8.9}$$

By condition (8.1a), $\langle e_{ix}(z), \det(w)\rangle$ vanishes unless $x = i$ and similarly in the right hand side. Hence

$$\mu_l\big(\langle e_{ii}(z), \det(w)\rangle 1\big) e_{ij}(z) \det(w) = \mu_r\big(\langle e_{jj}(z), \det(w)\rangle 1\big) \det(w) e_{ij}(z). \tag{8.10}$$

It is enough to show that $e_{11}(z)$ commutes with $\det(w)$ since we can apply an automorphism from the $S_n \times S_n$-action on A and use that $\det(z)$ is fixed by those, by relation (7.15). In view of (8.10) it remains to show that $\langle e_{11}(z), \det(w)\rangle 1$ is nonzero and independent of the dynamical variable $\zeta \in \mathfrak{h}^*$. For this we need a lemma.

Lemma 8.6. *Let $\sigma \in S_n$. If $\sigma \neq \mathrm{Id}$ then*

$$\langle e_{11}(z), e_{\sigma(1)1}(w)\cdots e_{\sigma(n)n}(q^{2(n-1)}w)\rangle = 0,$$

while for $\sigma = \mathrm{Id}$ we have

$$\langle e_{11}(z), e_{11}(w)\cdots e_{nn}(q^{2(n-1)}w)\rangle = \theta(q^2z/w)\prod_{j=2}^{n}\tilde{\alpha}\left(\zeta_{1j}-1, \frac{z}{q^{2(j-1)}w}\right)\cdot T_{-\omega(1)},$$

where $\tilde{\alpha}(\lambda, z)$ was given in (7.29).

Proof. Let $\sigma \in S_n$. By the property (8.1e) we have

$$\begin{aligned}\langle e_{11}(z), e_{\sigma(1)1}(w)\cdots e_{\sigma(n)n}(q^{2(n-1)}w)\rangle =\\ = \sum_{x=1}^{n}\langle e_{x1}(z), e_{\sigma(1)1}(w)\rangle T_{\omega(x)}\langle e_{1x}(z), e_{\sigma(2)2}(q^2w)\cdots e_{\sigma(n)n}(q^{2(n-1)}w)\rangle\end{aligned}$$

By (8.1a) it follows that $x = \sigma(1) = 1$ if the expression is to be nonzero. Repeating this argument and moving all the T's to the right the claim follows. □

Put $I = [1, n]$. We get

$$\begin{aligned}\langle e_{11}(z), \det(w)\rangle 1 &= \langle e_{11}(z), \sum_{\sigma\in S_n}\frac{F^I(\rho)}{\sigma(F^I(\lambda))}e_{\sigma(1)1}(w)\cdots e_{\sigma(n)n}(q^{2(n-1)}w)\rangle 1 =\\ &= \sum_{\sigma\in S_n}\sigma(F^I(\zeta))^{-1}\langle e_{11}(z), e_{\sigma(1)1}(w)\cdots e_{\sigma(n)n}(q^{2(n-1)}w)\rangle F^I(\zeta) =\\ &= F^I(\zeta)^{-1}\theta(q^2\frac{z}{w})\prod_{1<j}\tilde{\alpha}\left(\zeta_{1j}-1, \frac{z}{q^{2(j-1)}w}\right)F^I(\zeta-\omega(1))\\ &= \theta(q^2\frac{z}{w})\prod_{1<j}\frac{E(\zeta_{1j}-1)}{E(\zeta_{1j})}\tilde{\alpha}(\zeta_{1j}-1, \frac{z}{q^{2(j-1)}w}) =\\ &= q^{n-1}\theta(q^2\frac{z}{w})\theta(\frac{z}{q^2w})\theta(\frac{z}{q^4w})\cdots\theta(\frac{z}{q^{2(n-1)}w})\end{aligned}\tag{8.11}$$

using the definitions, (7.29) and (2.5), of $\tilde{\alpha}$ and E respectively. Hence we have proved that $[e_{ij}(z), \det(w)] = 0$ if $z/w \notin p^{\mathbb{Z}}\cdot\{q^{-2}, q^2, q^4, \ldots q^{2(n-1)}\}$. We must show that this also holds when $z/w \in p^{\mathbb{Z}}\{q^{-2}, q^{2(n-1)}\}$.

For this we note that relations (5.4),(5.5) imply that there is a $\mathbb{C}$-linear map $T : \mathscr{F}_{\mathrm{ell}}(M(n)) \to \mathscr{F}_{\mathrm{ell}}(M(n))$ such that $T(ab) = T(b)T(a)$ for all $a, b \in \mathscr{F}_{\mathrm{ell}}(M(n))$, given by

$$T\big(e_{ij}(z)\big) = e_{ij}(z^{-1}), \quad T\big(f(\lambda)\big) = f(-\lambda), \quad T\big(f(\rho)\big) = f(-\rho),$$

for all $f \in M_{\mathfrak{h}^*}$, $i, j \in [1, n]$ and $z \in \mathbb{C}^\times$. One verifies that $T(\det(z)) = \det(q^{-2(n-1)}z^{-1})$. Thus if $z/w \in p^{\mathbb{Z}}\{q^{-2}, q^{2(n-1)}\}$ we have

$$T\big([e_{ij}(z), \det(w)]\big) = -[e_{ij}(z^{-1}), \det(q^{-2(n-1)}w^{-1})] = 0$$

by what was proved above, since

$$z^{-1}/q^{-2(n-1)}w^{-1} = q^{2(n-1)}w/z \in p^{\mathbb{Z}}\{q^{2n}, 1\}.$$

This finishes the proof. □

9 Laplace expansions and the antipode

9.1 Laplace expansions

For subsets $I, J \subseteq [1, n]$ we define $S_l(I, J; \zeta), S_r(I, J; \zeta) \in M_{\mathfrak{h}^*}$ by

$$v_I(q^{2\#J}z)v_J(z) = S_l(I, J; \zeta)v_{I\cup J}(z), \tag{9.1}$$

$$w^I(z)w^J(q^{2\#I}z) = S_r(I, J; \zeta)w^{I\cup J}(z). \tag{9.2}$$

That this is possible follows from the definitions (7.8),(7.9) of $v_I(z), w^I(z)$ and the commutation relations (6.3b)-(6.3d), (6.8a)-(6.8b). In particular $S_l(I, J; \zeta) = 0 = S_r(I, J; \zeta)$ if $I \cap J \neq \emptyset$.

Theorem 9.1. *(i) Let $I_1, I_2, J \subseteq [1, n]$ and set $I = I_1 \cup I_2$. Then*

$$S_l(I_1, I_2; \lambda)\xi_I^J(z) = \sum_{J_1\cup J_2=J} S_l(J_1, J_2; \rho)\xi_{I_1}^{J_1}(q^{2\#I_2}z)\xi_{I_2}^{J_2}(z). \tag{9.3}$$

(ii) Let $J_1, J_2, I \subseteq [1, n]$ and set $J = J_1 \cup J_2$. Then

$$S_r(J_1, J_2; \rho)\xi_I^J(z) = \sum_{I_1\cup I_2=I} S_r(I_1, I_2; \lambda)\xi_{I_1}^{J_1}(z)\xi_{I_2}^{J_2}(q^{2\#J_1}z). \tag{9.4}$$

Proof. We have

$$\begin{aligned}\Delta_\Lambda(v_{I_1}(q^{2\#I_2}z))\Delta_\Lambda(v_{I_2}(z)) &= \sum_{J_1,J_2} \xi_{I_1}^{J_1}(q^{2\#I_2}z)\xi_{I_2}^{J_2}(z) \otimes v_{J_1}(q^{2\#I_2}z)v_{J_2}(z) = \\ &= \sum_{J_1,J_2} \xi_{I_1}^{J_1}(q^{2\#I_2}z)\xi_{I_2}^{J_2}(z) \otimes S_l(J_1, J_2; \zeta)v_J(z) = \\ &= \sum_J \left(\sum_{J_1\cup J_2=J} S_l(J_1, J_2; \rho)\xi_{I_1}^{J_1}(q^{2\#I_2}z)\xi_{I_2}^{J_2}(z)\right) \otimes v_J(z).\end{aligned}$$

On the other hand,

$$\begin{aligned}\Delta_\Lambda(v_{I_1}(q^{2\#I_2}z))\Delta_\Lambda(v_{I_2}(z)) &= \Delta_\Lambda(v_{I_1}(q^{2\#I_2}z)v_{I_2}(z)) = \\ &= \Delta_\Lambda(S_l(I_1, I_2; \zeta)v_I(z)) = \sum_J S_l(I_1, I_2; \lambda)\xi_I^J(z) \otimes v_J(z).\end{aligned}$$

Equating these expressions proves (9.3) since, by Proposition 6.5, the set $\{v_J(z) : J \subseteq [1, n]\}$ is linearly independent over $M_{\mathfrak{h}^*}$. The second part is completely analogous, using the right comodule algebra Λ' in place of Λ. □

We need the following lemma, relating the left and right signums $S_l(I,J;\zeta)$ and $S_r(I,J;\zeta)$, defined in (9.1),(9.2).

Lemma 9.2. *Let I,J be two disjoint subsets of $[1,n]$. Then*

$$S_l(I,J;\zeta+\omega(I)) = S_r(J,I;\zeta)^{-1} \tag{9.5}$$

where $\omega(I)=\sum_{i\in I}\omega(i)$.

Proof. First we claim that, we have the following explicit formulas:

$$S_l(I,J;\zeta) = \prod_{i\in I, j\in J} E(\zeta_{ji}+1), \tag{9.6}$$

$$S_r(I,J;\zeta) = \prod_{i\in I, j\in J} E(\zeta_{ij})^{-1}. \tag{9.7}$$

Recall the definition, (7.8), of $v_I(z)$. Since E is odd, relation (6.3b) implies that

$$v_i(q^2z)v_j(z) = \frac{E(\zeta_{ji}+1)}{E(\zeta_{ij}+1)} v_j(q^2z)v_i(z).$$

Also, $F_J(\zeta)$ only involves ζ_{ij} with $i,j\in J$ so it commutes with any $v_k(z)$ with $k\in I$ (since $I\cap J=\emptyset$). From these facts we obtain

$$\begin{aligned}
v_I(q^{2\#J}z)v_J(z) &= \frac{F_I(\zeta)^{-1}F_J(\zeta)^{-1}}{F_{I\cup J}(\zeta)^{-1}} \prod_{\substack{i\in I, j\in J\\ i<j}} \frac{E(\zeta_{ji}+1)}{E(\zeta_{ij}+1)} v_{I\cup J}(z) = \\
&= \prod_{\substack{(i,j)\in K\\ i<j}} E(\zeta_{ij}+1) \prod_{\substack{i\in I, j\in J\\ i<j}} \frac{E(\zeta_{ji}+1)}{E(\zeta_{ij}+1)} v_{I\cup J}(z) = \\
&= \prod_{i\in I, j\in J} E(\zeta_{ji}+1) v_{I\cup J}(z),
\end{aligned}$$

where $K=(I\times J)\cup(J\times I)$. This proves (9.6). Similarly one proves (9.7). Now we have

$$S_l(J,I;\zeta+\omega(J))^{-1} = \prod_{i\in I, j\in J} E((\zeta+\omega(J))_{ij}+1)^{-1} = \prod_{i\in I, j\in J} E(\zeta_{ij})^{-1} = S_r(I,J;\zeta).$$

Here we used that for any $i\in I, j\in J$ we have $\omega(J)(E_{ii})=0$ and $\omega(J)(E_{jj})=1$ and hence $(\omega(J))_{ij}=-1$. □

9.2 The antipode

We use the following definition for the antipode, given in [KR01].

Definition 9.3. An $\mathfrak{h}$-*Hopf algebroid* is an $\mathfrak{h}$-bialgebroid A equipped with a $\mathbb{C}$-linear map $S : A \to A$, called the antipode, such that

$$S(\mu_r(f)a) = S(a)\mu_l(f), \quad S(a\mu_l(f)) = \mu_r(f)S(a), \quad a \in A, f \in M_{\mathfrak{h}^*}, \tag{9.8}$$

$$\begin{aligned} m \circ (\mathrm{Id} \otimes S) \circ \Delta(a) &= \mu_l(\varepsilon(a)1), \quad a \in A, \\ m \circ (S \otimes \mathrm{Id}) \circ \Delta(a) &= \mu_r(T_\alpha(\varepsilon(a)1)), \quad a \in A_{\alpha\beta}, \end{aligned} \tag{9.9}$$

where m denotes the multiplication and $\varepsilon(a)1$ is the result of applying the difference operator $\varepsilon(a)$ to the constant function $1 \in M_{\mathfrak{h}^*}$.

Let $\mathscr{F}_{\mathrm{ell}}(M(n))\langle \det(z)^{-1} : z \in \mathbb{C}^\times \rangle$ be the algebra obtained from $\mathscr{F}_{\mathrm{ell}}(M(n))$ by adjoining infinitely many indeterminates $\det(z)^{-1}$, $z \in \mathbb{C}^\times$, which do not commute with each other or with the elements in $\mathscr{F}_{\mathrm{ell}}(M(n))$. We define $\mathscr{F}_{\mathrm{ell}}(GL(n))$ to be

$$\mathscr{F}_{\mathrm{ell}}(M(n))\langle \det(z)^{-1} : z \in \mathbb{C}^\times \rangle / J$$

where J is the ideal generated by the relations $\det(z)\det(z)^{-1} = 1 = \det(z)^{-1}\det(z)$ for each $z \in \mathbb{C}^\times$. We extend the bigrading of $\mathscr{F}_{\mathrm{ell}}(M(n))$ to $\mathscr{F}_{\mathrm{ell}}(M(n))\langle \det(z)^{-1} : z \in \mathbb{C}^\times \rangle$ by requiring that $\det(z)^{-1}$ has bidegree $0,0$ for each $z \in \mathbb{C}^\times$. Then J is homogenous and the bigrading descends to $\mathscr{F}_{\mathrm{ell}}(GL(n))$. We extend the comultiplication and counit by requiring that $\det(z)^{-1}$ is grouplike for each $z \in \mathbb{C}^\times$, i.e. that

$$\Delta(\det(z)^{-1}) = \det(z)^{-1} \otimes \det(z)^{-1}, \qquad \varepsilon(\det(z)^{-1}) = 1.$$

Here 1 denotes the identity operator in $D_{\mathfrak{h}}$. One verifies that J is a coideal and that $\varepsilon(J) = 0$, which induces operations Δ, ε on $\mathscr{F}_{\mathrm{ell}}(GL(n))$. In this way $\mathscr{F}_{\mathrm{ell}}(GL(n))$ becomes an $\mathfrak{h}$-bialgebroid. This algebra is nontrivial since $\varepsilon(J) = 0$ implies that J is a proper ideal.

For $i \in [1, n]$ we set $\hat{\imath} = \{1, \ldots, i-1, i+1, \ldots, n\}$.

Theorem 9.4. *$\mathscr{F}_{ell}(GL(n))$ is an $\mathfrak{h}$-Hopf algebroid with antipode S given by*

$$S(f(\lambda)) = f(\rho), \qquad S(f(\rho)) = f(\lambda), \tag{9.10}$$

$$S(e_{ij}(z)) = \frac{S_r(\hat{\jmath}, \{j\}; \lambda)}{S_r(\hat{\imath}, \{i\}; \rho)} \det(q^{-2(n-1)}z)^{-1} \xi_{\hat{\jmath}}^{\hat{\imath}}(q^{-2(n-1)}z), \tag{9.11}$$

$$S(\det(z)^{-1}) = \det(z), \tag{9.12}$$

for all $f \in M_{\mathfrak{h}^*}$, $i, j \in [1, n]$ and $z \in \mathbb{C}^\times$.

Proof. We proceed in steps.

Step 1. Define S on the generators of $\mathscr{F}_{\mathrm{ell}}(M(n))$ by (9.10), (9.11). We show that the antipode axiom (9.9) holds if a is a generator. Indeed for $a = f(\lambda)$ or $a = f(\rho)$, $f \in M_{\mathfrak{h}^*}$ this is easily checked. Let $a = e_{ij}(z)$. Using the right Laplace expansion (9.4) with $J_1 = \hat{\imath}$, $J_2 = \{j\}$, $I = [1, n]$ and z replaced by $q^{-2(n-1)}z$ we obtain

$$\sum_{x=1}^{n} S(e_{ix}(z))e_{xj}(z) = \delta_{ij}. \tag{9.13}$$

Similarly, using the left Laplace expansion (9.3) with $I_1 = \{i\}$, $I_2 = \hat{\imath}$, $J = [1,n]$ and z replaced by $q^{-2(n-1)}z$, together with the identity (9.5), we get

$$\sum_{x=1}^{n} e_{ix}(z)S(e_{xj}(z)) = \delta_{ij}, \tag{9.14}$$

using also the crucial fact that, by Theorem 8.5, $e_{ij}(z)$ commutes in $\mathscr{F}_{\mathrm{ell}}(M(n))$ with $\det(q^{-2(n-1)}z)$ and hence in $\mathscr{F}_{\mathrm{ell}}(GL(n))$ with $\det(q^{-2(n-1)}z)^{-1}$. This proves that the antipode axiom (9.9) is satisfied for $a = e_{ij}(z)$.

Step 2. We show that S extends to a $\mathbb{C}$-linear map $S : \mathscr{F}_{\mathrm{ell}}(M(n)) \to \mathscr{F}_{\mathrm{ell}}(GL(n))$ satisfying $S(ab) = S(b)S(a)$. For this we must verify that S preserves the relations, (5.1),(5.2), (5.5) of $\mathscr{F}_{\mathrm{ell}}(M(n))$. Since $S(e_{ij}(z)) \in \mathscr{F}_{\mathrm{ell}}(GL(n))_{\omega(\hat{\jmath}),\omega(\hat{\imath})}$ and $\omega(i) + \omega(\hat{\imath}) = 0$, we have

$$\begin{aligned} S(e_{ij}(z))S(f(\lambda)) &= S(e_{ij}(z))f(\rho) = f(\rho - \omega(\hat{\imath}))S(e_{ij}(z)) = \\ &= f(\rho + \omega(i))S(e_{ij}(z)) = S\big(f(\lambda + \omega(i))\big)S(e_{ij}(z)) \end{aligned}$$

similarly, $S(e_{ij}(z))S(f(\rho)) = S\big(f(\rho + \omega(j))\big)S(e_{ij}(z))$ so relations (5.1) are preserved. Next, consider the RLL relation

$$\sum_{x,y=1}^{n} R_{ac}^{xy}(\lambda, \frac{z_1}{z_2})e_{xb}(z_1)e_{yd}(z_2) = \sum_{x,y=1}^{n} R_{xy}^{bd}(\rho, \frac{z_1}{z_2})e_{cy}(z_2)e_{ax}(z_1). \tag{9.15}$$

Multiply (9.15) from the left by $S(e_{ic}(z_2))$ and from the right by $S(e_{dk}(z_2))$, sum over c, d and use (9.13),(9.14) to obtain

$$\sum_{x,c} R_{ac}^{xk}(\lambda - \omega(\hat{c}), \frac{z_1}{z_2})S(e_{ic}(z_2))e_{xb}(z_1) = \sum_{x,d} R_{xi}^{bd}(\rho - \omega(\hat{\imath}), \frac{z_1}{z_2})e_{ax}(z_1)S(e_{dk}(z_2)).$$

Then multiply from the left by $S(e_{ja}(z_1))$ and from the right by $S(e_{bl}(z_1))$, sum over a, b and use (9.13),(9.14) again to get

$$\begin{aligned} &\sum_{a,c} R_{ac}^{lk}(\lambda - \omega(\hat{a}) - \omega(\hat{c}), \frac{z_1}{z_2})S(e_{ja}(z_1))S(e_{ic}(z_2)) = \\ &\qquad = \sum_{b,d} R_{ji}^{bd}(\rho - \omega(\hat{\jmath}) - \omega(\hat{\imath}), \frac{z_1}{z_2})S(e_{dk}(z_2))S(e_{bl}(z_1)). \end{aligned} \tag{9.16}$$

Since $S(e_{ij}(z)) \in \mathscr{F}_{\mathrm{ell}}(GL(n))_{\hat{\jmath},\hat{\imath}}$ and $R_{ji}^{bd}(\rho - \omega(\hat{\jmath}) - \omega(\hat{\imath}), \frac{z_1}{z_2}) = R_{ji}^{bd}(\rho - \omega(\hat{b}) - \omega(\hat{d}), \frac{z_1}{z_2})$ by the $\mathfrak{h}$-invariance of R, (9.16) can be rewritten

$$\sum_{a,c} S(e_{ja}(z_1))S(e_{ic}(z_2))R_{ac}^{lk}(\lambda, \frac{z_1}{z_2}) = \sum_{b,d} S(e_{dk}(z_2))S(e_{bl}(z_1))R_{ji}^{bd}(\rho, \frac{z_1}{z_2}).$$

This is the result of formally applying S to the RLL-relations, proving that S preserves (5.2). Similarly (5.5) is preserved.

Step 3. Since, by the above steps, (9.9) holds on the generators of $\mathscr{F}_{\mathrm{ell}}(M(n))$ and $S(ab) = S(b)S(a)$ for all $a, b \in \mathscr{F}_{\mathrm{ell}}(M(n))$, it follows that (9.9) holds for any $a \in \mathscr{F}_{\mathrm{ell}}(M(n))$. By taking in particular $a = \det(z)$ we get

$$\det(z)S(\det(z)) = 1 \qquad \text{and} \qquad S(\det(z))\det(z) = 1$$

respectively. Thus, definining S on $\det(z)^{-1}$ by (9.12), the relations $\det(z)\det(z)^{-1} = 1 = \det(z)^{-1}\det(z)$ are preserved by S. Hence S extends to an anti-multiplicative $\mathbb{C}$-linear map $S : \mathscr{F}_{\mathrm{ell}}(GL(n)) \to \mathscr{F}_{\mathrm{ell}}(GL(n))$ satisfying the antipode axiom (9.9) on $\mathscr{F}_{\mathrm{ell}}(M(n))$ and on $\det(z)^{-1}$. Hence (9.9) holds for any $a \in \mathscr{F}_{\mathrm{ell}}(GL(n))$. □

10 Discussion

We suspect that relations of the form (4.10) are not enough to prove that $\det(z)$ is central and that one may have to add some "higher order" residual relations for this to be true. However, in a representation of $\mathscr{F}_{\mathrm{ell}}(M(n))$ where $e_{ij}(z)$ act as meromorphic functions of z, the element $[e_{ij}(z), \det(w)]$ acts as zero since it vanishes for (z, w) in a dense subset of $\mathbb{C}^\times \times \mathbb{C}^\times$. This essentially means that $\det(z)$ is central in the operator algebra from [FV97]. To define the antipode we only needed that $e_{ij}(z)$ commutes with $\det(q^{-2(n-1)}z)$. This can also be proved using the Laplace expansions.

Perhaps one could avoid problems with spectral poles and zeros of the R-matrix by thinking of the algebra as generated by meromorphic sections of a $M_{\mathfrak{h}^*\oplus\mathfrak{h}^*}$-line bundle over the elliptic curve $\mathbb{C}^\times/\{z \sim pz\}$. In this direction we found that the relation $e_{ij}(pz) = q^{2(\lambda_i - \rho_j)}e_{ij}(z)$ respects the RLL-relation (here $\mathfrak{h}$ should be the Cartan subalgebra of $\mathfrak{gl}_n$). This relation should most likely be added to the algebra.

It would be interesting to develop harmonic analysis for the elliptic $GL(n)$ quantum group, similarly to [KR01]. In this context it is valuable to have an abstract algebra to work with, and not only a collection of representations. For example the analogue of the Haar measure seems most naturally defined as a certain linear functional on the algebra.

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References

[EV98] P. Etingof, A. Varchenko, *Solutions of the quantum dynamical Yang-Baxter equation and dynamical quantum groups*, Comm. Math. Phys. **196** No. 3 (1998) 591–640.

[F95] G. Felder, *Elliptic quantum groups*, In *XIth International Congress of Mathematical Physics (Paris, 1994)*, Int. Press, Cambridge, MA, (1995) 211–218.

[FV96] G. Felder, A. Varchenko, *On representations of the elliptic quantum group* $E_{\tau,\eta}(\mathrm{sl}_2)$, Comm. Math. Phys. **181** No. 3 (1996) 741–761.

[FV97] G. Felder, A. Varchenko, *Elliptic quantum groups and Ruijsenaars models*, J. Statist. Phys. **89** No. 5-6 (1997) 963–980.

[GN84] J.-L. Gervais, A. Neveu, *Novel triangle relation and absence of tachyons in Liouville string field theory*, Nuclear Phys. B **238** No. 1 (1984) 125–141.

[JKMO88] M. Jimbo, A. Kuniba, T. Miwa, M. Okado, *The* $A_n^{(1)}$ *face models*, Commun. Math. Phys. **119** (1988) 543–565.

[KN06] E. Koelink, Y. van Norden, *The dynamical* $\mathrm{U}(n)$ *quantum group*, Int. J. Math. Math. Sci. (2006) Art. ID 65279.

[KNR04] E. Koelink, Y. van Norden, H. Rosengren, *Elliptic* $\mathrm{U}(2)$ *quantum group and elliptic hypergeometric series*, Comm. Math. Phys. **245** No. 3 (2004) 519–537.

[KR01] E. Koelink, H. Rosengren, *Harmonic analysis on the* $\mathrm{SU}(2)$ *dynamical quantum group*, Acta Appl. Math. **69** No. 2 (2001) 163–220.

[L96] J.-H. Lu, *Hopf algebroids and quantum groupoids*, Internat. J. Math. **7** No. 1 (1996) 47–70.

[N05] Y. van Norden, *Dynamical quantum groups, duality and special functions*, PhD Thesis, Thomas Stieltjes Institute for Mathematics, 2005.

[R04] H. Rosengren, *Duality and self-duality for dynamical quantum groups*, Algebr. Represent. Theory **7** No. 4 (2004) 363–393.

[TV01] V. Tarasov, A. Varchenko, *Small elliptic quantum group* $e_{\tau,\gamma}(\mathfrak{sl}_N)$, Mosc. Math. J. **1** No. 2 (2001) 243–286.

[ZSY03] S.-Y. Zhao, K.-J. Shi, R.-H. Yue, *Center for elliptic quantum group* $E_{r,\eta}(\mathrm{sl}_n)$, Commun. Theor. Phys. (Beijing) **39** No. 1 (2003) 67–72.

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