LOCALLY FINITE MODULES OVER COMMUTATIVE ALGEBRAS

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Abstract. This is a quick overview, intended for students, on the basics of weight modules, generalized weight modules, and locally finite modules over commutative algebras. These concepts often serve as a basis for representation theory of *non*-commutative algebras which contain a "large" commutative subalgebra.

1. Generalized eigenspace decomposition

1.1. **Linear algebra perspective.** Let V be a finite-dimensional complex vector space and let $L : V \to V$ be a linear transformation. Choose a basis $\{v_1, v_2, \ldots, v_n\}$ such that the matrix of L is in Jordan normal form. Then each v_i is a generalized eigenvector for L. For each eigenvalue λ of L, let $V(\lambda)$ be the corresponding span of generalized eigenvectors [1]:

$$V(\lambda) = \{v \in V \mid \exists N \geq 0 : (L - \lambda \operatorname{Id}_V)^N v = 0\}. \tag{1.1}$$

If $\lambda \in \mathbb{C}$ isn't an eigenvalue of L then $V(\lambda)$ is the zero subspace of V. Therefore we have

$$V = \bigoplus_{\lambda \in \mathbb{C}} V(\lambda). \tag{1.2}$$

Note that from this direct sum we don't see what the structure of L restricted to $V(\lambda)$ is like. It could be diagonalizable or a single Jordan block, or something in between.

1.2. **Module perspective.** Let $A = \mathbb{C}[x]$ be the algebra of complex polynomials in one variable and let V be a finite-dimensional A-module. Then the action of x on V is a linear transformation of V and thus

$$V = \bigoplus_{\lambda \in \mathbb{C}} V(\lambda) \tag{1.3}$$

where now (letting . stand for the A-module action)

$$V(\lambda) = \{v \in V \mid (x - \lambda)^N.v = 0,\ N \gg 0\}. \tag{1.4}$$

1.3. **Module perspective with maximal ideals.** For $\lambda \in \mathbb{C}$, the principal ideal $\mathfrak{m}_\lambda = (x - \lambda)$ is a maximal ideal of $A = \mathbb{C}[x]$. In fact, by the weak Nullstellensatz (Cor. 7.10 in [1]), every maximal ideal of A has this form. Furthermore, $(x - \lambda)^N.v = 0$ iff $\mathfrak{m}_\lambda^N.v = 0$. Thus we have

$$V = \bigoplus_{\mathfrak{m} \in \operatorname{MaxSpec(A)}} V(\mathfrak{m}) \tag{1.5}$$

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[1]The statement $\exists N \geq 0 : (L - \lambda \operatorname{Id}_V)^N v = 0$ is equivalent to that $\exists N \geq 0\, \forall n \geq N : (L - \lambda \operatorname{Id}_V)^n v = 0$ which is usually written $(L - \lambda \operatorname{Id}_V)^N v = 0$, $N \gg 0$ where $N \gg 0$ means "for sufficiently large N". We will use this shorthand henceforth.

where MaxSpec(A) is the *maximal spectrum* of A, defined as the set of maximal ideals of A, and

$$V(\mathfrak{m}) = \{v \in V \mid \mathfrak{m}^N.v = 0,\ N \gg 0\}. \tag{1.6}$$

This decomposition can be generalized to arbitrary commutative $\Bbbk$-algebras A (where $\Bbbk$ is any field) and finite-dimensional A-modules V. The goal of these notes is to show how this is achieved.

2. Basics from commutative algebra

We will frequently reference [1] but most books on commutative algebra will contain the results cited. Let A be a commutative ring. The *nilradical* $\mathcal{N}(A)$ is the set of nilpotent elements in A:

$$\mathcal{N}(A) = \{a \in A \mid a^N = 0,\ N \gg 0\}.$$

Lemma 2.1 (Prp. 1.8 in [1]). *$\mathcal{N}(A)$ equals the intersection of all prime ideals of A.*

The *Jacobson radical* $\mathcal{R}(A)$ is the intersection of all maximal ideals of A.

Two ideals I and J of A are *coprime* if $I + J = A$.

Theorem 2.2 (Remainder Theorem, Prp 1.10 in [1]). *Let $I_1, I_2, \ldots, I_n$ be ideals of a commutative ring A and let*

$$\varphi : A \to \prod_{k=1}^{n} A/I_k$$

be the ring homomorphism $\varphi(a) = (a + I_1, a + I_2, \ldots, a + I_n)$.

(i) *φ is surjective iff I_j and I_k are coprime whenever $j \neq k$, in which case $\prod_{k=1}^n I_k = \cap_{k=1}^n I_k$.*
(ii) *φ is injective iff $\cap_{k=1}^n I_k = (0)$.*

An ideal of A is *nil* if it consists of nilpotent elements. An ideal I is *nilpotent* if $I^n = 0$ for some positive integer n. Every nilpotent ideal is nil, but the converse fails in general. (For example in $A = \Bbbk[x_1, x_2, \ldots]/(x_1, x_2^2, x_3^3, \ldots)$ the ideal $I = (\bar{x}_1, \bar{x}_2, \ldots)$ is nil but not nilpotent.) However:

Lemma 2.3. *Every finitely-generated nil ideal in a commutative ring is nilpotent.*

The *radical* of an ideal $I \subset A$ is $\sqrt{I} = \{a \in A \mid a^N \in I,\ N \gg 0\}$. If $\mathfrak{m}$ is a maximal ideal of A then $\sqrt{\mathfrak{m}^k} = \mathfrak{m}$ for all positive integers k.

Lemma 2.4 (Prp. 1.16 in [1]). *Let I and J be ideals of A such that $\sqrt{I}$ and $\sqrt{J}$ are coprime. Then I and J are coprime.*

Theorem 2.5 (Weak Nullstellensatz, Cor. 7.10 in [1]). *If $\mathfrak{m}$ is a maximal ideal of a finitely generated commutative $\Bbbk$-algebra ($\Bbbk$ a field), then $A/\mathfrak{m}$ is a finite field extension of $\Bbbk$.*

3. Generalized weight space decomposition for finite-dimensional modules over commutative $\Bbbk$-algebras

Lemma 3.1. *Let $\Bbbk$ be a field and A be a finite-dimensional commutative $\Bbbk$-algebra.*

(i) *Every prime ideal of A is maximal.*
(ii) *The Jacobson radical $\mathcal{R}(A)$ is nilpotent.*
(iii) *A has only finitely many maximal ideals.*
(iv) *There exists a positive integer k such that*

$$A \cong \frac{A}{\mathfrak{m}_1^k} \times \frac{A}{\mathfrak{m}_2^k} \times \cdots \times \frac{A}{\mathfrak{m}_n^k} \tag{3.1}$$

as $\Bbbk$-algebras where $\{\mathfrak{m}_1, \mathfrak{m}_2, \ldots, \mathfrak{m}_n\}$ is the set of all maximal ideals of A.

Proof. (i) Let $\mathfrak{p}$ be a prime ideal of A. Then $B = A/\mathfrak{p}$ is a finite-dimensional integral domain. Let x be a nonzero element of B. Let $L_x : B \to B$ be the linear map of left multiplication by x: $L_x(y) = xy$ for $y \in B$. Since B is an integral domain, L_x is injective. Since B is finite-dimensional, L_x is surjective. Hence there exists $y \in B$ such that $xy = 1_B$. Therefore B is a field. Hence $\mathfrak{p}$ is a maximal ideal of A.

(ii) Immediate by (i), Lemma 2.1 and Lemma 2.3.

(iii) Among the collection of finite intersections of maximal ideals of A, pick one, say $\mathfrak{m}_1 \cap \mathfrak{m}_2 \cap \cdots \cap \mathfrak{m}_n$, of minimal dimension. Then for any other maximal ideal $\mathfrak{m}$ of A, we have $\mathfrak{m} \cap \mathfrak{m}_1 \cap \mathfrak{m}_2 \cap \cdots \cap \mathfrak{m}_n = \mathfrak{m}_1 \cap \mathfrak{m}_2 \cap \cdots \cap \mathfrak{m}_n$ (otherwise we'd get a contradiction to the minimality of the dimension). This means that $\mathfrak{m} \supseteq \mathfrak{m}_1 \cap \mathfrak{m}_2 \cap \cdots \cap \mathfrak{m}_n$. Since $\mathfrak{m}$ is maximal hence prime, $\mathfrak{m} \supseteq \mathfrak{m}_i$ for some i. (Here we're using that a prime ideal containing an intersection of ideals must contain one the ideals.) Since $\mathfrak{m}_i$ is maximal, $\mathfrak{m} = \mathfrak{m}_i$.

(iv) Let $\{\mathfrak{m}_1, \mathfrak{m}_2, \ldots, \mathfrak{m}_n\}$ be the set of all maximal ideals of A. By (ii) there exists a positive integer k such that $\mathcal{R}(A)^k = 0$. Thus $\prod_i \mathfrak{m}_i^k \subset (\cap_i \mathfrak{m}_i)^k = \mathcal{R}(A)^k = 0$. By Lemma 2.4, the ideals $\mathfrak{m}_i^k$ are pairwise coprime. Thus the statement follows from the Remainder Theorem. □

Definition 3.2. Let A be a commutative algebra over a field $\Bbbk$ and let V be an A-module. We call V a *generalized weight module with respect to A* (or a *generalized A-weight module*) if

$$V = \bigoplus_{\mathfrak{m} \in \mathrm{MaxSpec}(A)} V(\mathfrak{m}), \tag{3.2}$$

where

$$V(\mathfrak{m}) = \{v \in V \mid \mathfrak{m}^N v = 0,\, N \gg 0\}. \tag{3.3}$$

We are now ready to prove:

Theorem 3.3. *Let $\Bbbk$ be a field, A be any commutative $\Bbbk$-algebra, and V be a finite-dimensional A-module. Then V is a generalized weight module with respect to A.*

Proof. Let I be the annihilator of V. Then A/I injects into $\mathrm{End}(V)$ hence is a finite-dimensional algebra. By Lemma 3.1(iv), $A/I \cong \prod A/\mathfrak{m}_i^k$ where $\mathfrak{m}_i$ are the (finitely many) maximal ideals of A containing I. Let e_i be the corresponding idempotents of A/I. Then $V = \bigoplus_i V_i$ where $V_i = e_i V$. Furthermore $\mathfrak{m}_i^k V_i = 0$. □

4. Generalization to locally finite case

Definition 4.1. Let A be a commutative algebra over a field $\Bbbk$ and let V be an A-module. We say that V is *locally finite-dimensional for A* (or just *locally finite*) if every cyclic A-submodule of V is finite-dimensional: $\forall v \in V : \dim_{\Bbbk}(A.v) < \infty$.

Theorem 4.2. *Let A be a commutative algebra over a field $\Bbbk$ and let V be an A-module. If V is locally finite then V is a generalized weight module with respect to A. The converse holds if A is noetherian.*

Proof. Suppose V is locally finite-dimensional for A. Then any cyclic A-submodule of V is finite-dimensional, hence a generalized weight module by Theorem 3.3. Since V (like any module) is the sum of its cyclic submodules, V is itself a generalized weight module.

Conversely, suppose A is noetherian and that V is a generalized weight module. Let $v \in V$. Then v is a sum of finitely many generalized weight vectors. So it suffices to show that each generalized weight vector w generates a finite-dimensional submodule. If $\mathfrak{m}^N w = 0$ where $\mathfrak{m}$ is a cofinite maximal ideal of A, then Aw is a quotient of $A/\mathfrak{m}^N$. Since A is noetherian, each ideal $\mathfrak{m}^k$ is finitely generated. Therefore $\mathfrak{m}^k/\mathfrak{m}^{k+1}$ is finite-dimensional as a vector space over $A/\mathfrak{m}$. By the weak Nullstellensatz, $A/\mathfrak{m}$ is finite-dimensional over $\Bbbk$. Since $A/\mathfrak{m}^N$ has a filtration $A/\mathfrak{m}^N \supseteq \mathfrak{m}/\mathfrak{m}^N \supseteq \mathfrak{m}^2/\mathfrak{m}^N \supseteq \cdots \mathfrak{m}^{N-1}/\mathfrak{m}^N$ whose subquotients are isomorphic to $\mathfrak{m}^k/\mathfrak{m}^{k+1}$ each of which is finite-dimensional as a vector space over $A/\mathfrak{m}$, we conclude $A/\mathfrak{m}^N$ is finite-dimensional over $\Bbbk$. Therefore Aw is also finite-dimensional over $\Bbbk$. □

Example 4.3. Let $A = \mathbb{C}[x_1, x_2, \ldots]$ be a polynomial algebra in a countably infinite set of variables x_i. Let $\mathfrak{m} = (x_1, x_2, \ldots)$ be the maximal ideal generated by the variables. Then $V = A/\mathfrak{m}^2$ is a generalized weight module with a single weight space because every element of V is annihilated by $\mathfrak{m}^2$. On the other hand, V contains $\mathfrak{m}/\mathfrak{m}^2$ which is infinite-dimensional with basis $\{\bar{x}_i\}_{i=1}^{\infty}$, $\bar{x}_i = x_i + \mathfrak{m}^2$. This shows that the phrase "if A is noetherian" cannot be removed from the statement of Theorem 4.2.

5. Exercise

Let $0 \to U \to V \to W \to 0$ be a short exact sequence of A-modules, where A is a commutative $\Bbbk$-algebra, $\Bbbk$ a field. Prove that

1. V is locally finite iff U and W are locally finite.
2. V is a generalized weight module iff U and W are generalized weight modules.

References

[1] M. F. Atiyah I. G. Macdonald, *Introduction to Commuative Algebra*, Addison-Wesley.

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