Harish-Chandra modules over orders in smash products

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Outline

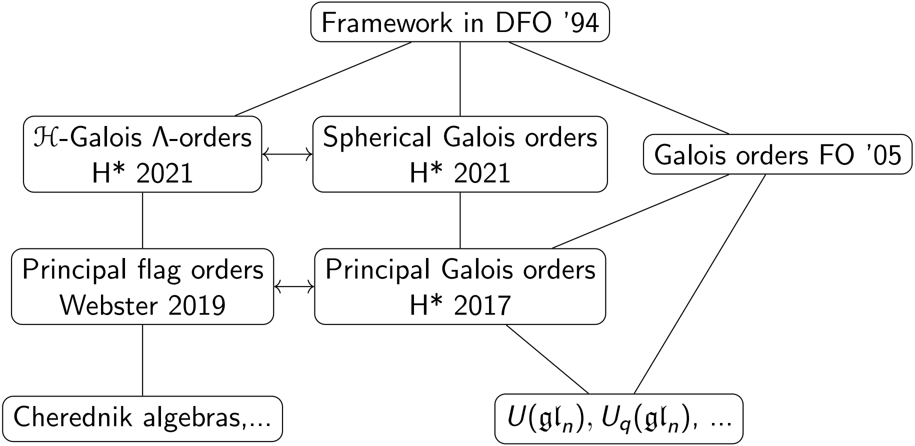

Figure: (D)FO = (Drozd-)Futorny-Ovsienko

Harish-Chandra subalgebras

A — associative algebra over a field $\mathbb{k}$
Γ — subalgebra of A
$\mathrm{cfs}(\Gamma)$ — set of maximal ideals of Γ of finite co-dimension
$S_{\mathfrak{m}}$ — the unique simple (left) $\Gamma/\mathfrak{m}$-module for $\mathfrak{m} \in \mathrm{cfs}(\Gamma)$

Definition (DFO 1994)

- Γ is *quasi-commutative* if $\mathrm{Ext}^1_\Gamma(S_{\mathfrak{m}}, S_{\mathfrak{n}}) = 0$ for all $\mathfrak{m}, \mathfrak{n} \in \mathrm{cfs}(\Gamma)$, $\mathfrak{m} \neq \mathfrak{n}$.
- Γ is *quasi-central in A* if every finitely generated Γ-subbimodule of A is finitely generated as a left and right Γ-module.
- Γ is a *Harish-Chandra subalgebra of A* if Γ is both quasi-commutative and quasi-central in A.

Harish-Chandra subalgebras, contd.

Example

If $\operatorname{char} \mathbb{k} = 0$ and $\mathfrak{g}$ is a finite-dimensional Lie algebra and $\mathfrak{k}$ is a reductive or nilpotent Lie subalgebra of $\mathfrak{g}$, then $\Gamma = U(\mathfrak{k})$ is a Harish-Chandra subalgebra of $U(\mathfrak{g})$.

Example

If Γ is commutative noetherian, and A is generated over Γ by an element x such that $\Gamma x + \Gamma = x\Gamma + \Gamma$, then Γ is a Harish-Chandra subalgebra of A.

Example

$\Gamma = \langle Z_1, \ldots, Z_n \rangle \subset U(\mathfrak{gl}_n),\ Z_k = Z(U(\mathfrak{gl}_k)),\ \mathfrak{gl}_1 \subset \mathfrak{gl}_2 \subset \cdots \subset \mathfrak{gl}_n$
Then Γ is a (commutative) Harish-Chandra subalgebra of $U(\mathfrak{gl}_n)$.

Harish-Chandra modules

For the rest of this talk, Γ is commutative.

Definition (DFO 1994)

Suppose $\Gamma \subset A$ is a Harish-Chandra subalgebra.

- An A-module V is a *Harish-Chandra module (with respect to* Γ*)* if

$$V = \bigoplus_{\mathfrak{m} \in \mathrm{cfs}(\Gamma)} V^{\mathfrak{m}}, \quad V^{\mathfrak{m}} = \{ v \in V \mid \mathfrak{m}^n v = 0, n \gg 0 \}.$$

- $\mathrm{HC}(A, \Gamma)$ — category of Harish-Chandra A-modules with respect to Γ.
- The *fiber over* $\mathfrak{m} = \mathrm{Irr}(A, \mathfrak{m})$ — set of isomorphism classes of simple Harish-Chandra A-modules with respect to Γ, such that $V^{\mathfrak{m}} \neq 0$.

Equivalence of categories

Definition (DFO 1994)

Category $\mathcal{A}$ (associated to $\Gamma \subset A$) is given by

- Objects: $\mathrm{cfs}(\Gamma)$
- Morphisms:

$$\mathcal{A}(\mathfrak{m}, \mathfrak{n}) = \varprojlim \frac{A}{\mathfrak{n}^n A + A\mathfrak{m}^n}$$

An $\mathcal{A}$-module M is *discrete* if all $\mathcal{A}(\mathfrak{m}, \mathfrak{n}) \times M(\mathfrak{m}) \to M(\mathfrak{n})$ are continuous.

Theorem (DFO 1994)

There is an equivalence of categories

$$\mathrm{HC}(A, \Gamma) \simeq \mathrm{Mod}^{\mathrm{d}}\!-\!\mathcal{A}$$

and $\mathrm{Irr}(A, \mathfrak{m})$ *is in bijection with simple discrete* $\mathcal{A}(\mathfrak{m}, \mathfrak{m})$*-modules.*

Problem 1: Existence/construction

- Existence problem: Given $\mathfrak{m}$, when is $\mathrm{Irr}(A, \mathfrak{m})$ nonempty? Explicit construction of some V?

Example: History of Gelfand-Tsetlin modules

$\mathfrak{gl}_1 \subset \mathfrak{gl}_2 \subset \cdots \subset \mathfrak{gl}_n$
$\Gamma = \langle Z_1, \ldots, Z_n \rangle \subset U(\mathfrak{gl}_n),\ Z_k = Z(U(\mathfrak{gl}_k))$
Given a character λ of the Gelfand-Tsetlin subalgebra Γ with kernel $\mathfrak{m}_\lambda$, does there exist a simple Harish-Chandra module V for $U(\mathfrak{gl}_n)$ with respect to Γ such that $V^{\mathfrak{m}_\lambda} \neq 0$? If so, how can one be constructed?

- DFO 1994 - existence for generic λ (from 1950's formulas)
- Ovsienko 2003 - existence for any λ
- Futorny-Grantcharov-Ramírez 2014: derivative tableaux
- Vishnyakova 2017: local distributions
- Ramírez-Zadunaisky 2017: explicit construction for any λ
- Early-Mazorchuk-Vishnyakova 2017: canonical simple modules
- Webster 2019: classification by goodly Lyndon words
- Silverthorne-Webster 2020: any simple GZ module is canonical

Problem 2: Finiteness

- Finiteness problem: When is $\mathrm{Irr}(A, \mathfrak{m})$ finite?

Put

$$\widehat{A}_{\mathfrak{m}} = \mathcal{A}(\mathfrak{m}, \mathfrak{m}) = \varprojlim \frac{A}{\mathfrak{m}^n A + A\mathfrak{m}^n} \qquad \widehat{\Gamma}_{\mathfrak{m}} = \varprojlim \Gamma/\mathfrak{m}^n.$$

Theorem (DFO 1994)

If $\widehat{A}_{\mathfrak{m}}$ is finitely generated as a right $\widehat{\Gamma}_{\mathfrak{m}}$-module then

(i) $\mathrm{Irr}(A, \mathfrak{m})$ *is finite,*

(ii) *for any* $[V] \in \mathrm{Irr}(A, \mathfrak{m})$, $V^{\mathfrak{m}}$ *is finite-dimensional.*

Theorem (FO 2006)

For any character λ of $\Gamma \subset U = U(\mathfrak{gl}_n)$, $\widehat{U}_{\mathfrak{m}_\lambda}$ is finitely generated as a right $\widehat{\Gamma}_{\mathfrak{m}_\lambda}$-module, hence $\mathrm{Irr}(U, \mathfrak{m}_\lambda)$ *is finite.*

Settings

Definition

A *setting* is a pair $(\mathcal{H}, \Lambda)$ where $\mathcal{H}$ Hopf algebra with invertible antipode, Λ a left $\mathcal{H}$-module algebra such that

(i) Λ is a noetherian integral domain,

(ii) the action of $\mathcal{H}$ on Λ extends to $L = \operatorname{Frac} \Lambda$,

(iii) Λ is faithful as a left module over the smash product $\Lambda \# \mathcal{H}$.

- Condition (ii) holds if the coradical of $\mathcal{H}$ is finite-dimensional or co-commutative (Skryabin - Van Oystaeyen 2006), or if $\mathcal{H}$ acts locally finitely on Λ (Skryabin 2020).

Examples

1. $\mathcal{H} = \Bbbk G$, G acting faithfully on a noetherian integral domain Λ.
2. V finite-dimensional complex vector space, $\mathcal{H} = S(V) \rtimes W$, $\Lambda = \mathbb{C}[V]$, $W \leq \mathrm{GL}(V)$.
3. G connected complex affine algebraic group, $\mathcal{H} = U(\mathfrak{g}) \rtimes W$, $\Lambda = \mathbb{C}[G]$, W group acting on $\mathbb{C}[G]\# U(\mathfrak{g})$, faithfully on $\mathbb{C}[G]$ and by Hopf algebra automorphisms on $U(\mathfrak{g})$.
4. $\mathcal{H}$ Hopf algebra, L a field and a right $\mathcal{H}$-comodule algebra. Suppose the Galois map
$$\beta : L \otimes_{L^{\mathrm{co}\mathcal{H}}} L \to L \otimes \mathcal{H}, \quad a \otimes b \mapsto ab_{(0)} \otimes b_{(1)}$$
is surjective. Let $\Lambda \subset L$ be a noetherian $\mathcal{H}^\circ$-module subalgebra with fraction field L. Then $(\mathcal{H}^\circ, \Lambda)$ is a setting.
5. If $(\mathcal{H}, \Lambda)$ is a setting and $\mathcal{H}'$ is a Hopf subalgebra of $\mathcal{H}$, then $(\mathcal{H}', \Lambda)$ is a setting.
6. If $(\mathcal{H}, \Lambda)$ is a setting, then $(\mathcal{H}[\frac{\partial}{\partial t}], \Lambda[t])$ is a setting.

$\mathcal{H}$-Galois Λ-orders

Definition (H* 2021)

Let $(\mathcal{H}, \Lambda)$ be a setting. An $\mathcal{H}$-*Galois* Λ-*order* is a subalgebra $F \subset L\#\mathcal{H}$ such that

(i) $\Lambda \subset F$,

(ii) $LF = L\#\mathcal{H}$,

(iii) $\widehat{X}(\Lambda) \subset \Lambda$ for all $X \in F$.

- If $X = \sum_i f_i h_i \in L\#\mathcal{H}$, where $f_i \in L$, $h_i \in \mathcal{H}$, then $\widehat{X} \in \mathrm{End}(L)$ is given by
$$\widehat{X}(f) = \sum_i f_i \cdot (h_i \triangleright f) \in L, \qquad \forall f \in L.$$
- F is a "noncommutative" order in the sense that Λ is not contained in the center of F. In many important examples, $Z(F) = \mathbb{k}$.

Examples

- The *standard* $\mathcal{H}$*-Galois* Λ*-order* is defined by

$$\mathcal{F}(\mathcal{H},\Lambda)=\{X\in L\#\mathcal{H}\mid \widehat{X}(\Lambda)\subset\Lambda\}.$$

- If G is a group acting faithfully on Λ, a $\Bbbk G$-Galois Λ-order is roughly the same as Webster's principal flag orders.
- Rational Cherednik algebras are examples of $\mathcal{H}$-Galois Λ-orders in the setting $(\mathcal{H},\Lambda)=(S(V)\rtimes W,\mathbb{C}[V])$ (via Dunkl-Opdam polynomial representation).

First results

Lemma

(i) *The short exact sequence of left* Λ*-module*

$$0 \to \Lambda \to F \to F/\Lambda \to 0$$

splits. A splitting map $F \to \Lambda$ *is given by* $X \mapsto \widehat{X}(1_\Lambda)$*.*

(ii) Λ *is maximal commutative in* F*.*

(iii) $Z(F) = \Lambda^{\mathcal{H}}$*.*

(iv) Λ *is a Harish-Chandra subalgebra of* F*.*

Proof.

(ii) If $X \in F$ satisfies $Xa = aX$ for all $a \in \Lambda$, put $Y = X - \widehat{X}(1_\Lambda)$. Then

$$\widehat{Y}(a) = \widehat{X}(a) - \widehat{X}(1_\Lambda)a = (\widehat{X}\widehat{a} - \widehat{a}\widehat{X})(1_\Lambda) = 0 \implies Y = 0.$$

□

Canonical modules of local distributions

For a maximal ideal $\mathfrak{m}$ of Λ, put

$$\mathrm{Dist}(\Lambda, \mathfrak{m}) = \{\xi \in \Lambda^* \mid \mathfrak{m}^n \subset \ker \xi, n \gg 0\}.$$

The *space of local distributions* is

$$\mathrm{Dist}(\Lambda) = \bigoplus_{\mathfrak{m}} \mathrm{Dist}(\Lambda, \mathfrak{m})$$

where $\mathfrak{m}$ ranges over maximal ideals of finite codimension.

Theorem (H* 2021)

Let F be an $\mathcal{H}$-Galois Λ-order and consider Λ^* as a left F^{op}-module.

(i) $\mathrm{Dist}(\Lambda)$ *is an F^{op}-submodule of Λ^*,*

(ii) $\mathrm{Dist}(\Lambda)$ *is a Harish-Chandra module with respect to Λ,*

(iii) *If $\lambda : \Lambda \to \Bbbk$ is an algebra map with kernel $\mathfrak{m}_\lambda$, then the cyclic F^{op}-submodule of $\mathrm{Dist}(\Lambda)$ generated by λ has a unique simple quotient $V(\lambda)$. Moreover, $V(\lambda)$ is a simple Harish-Chandra module with $V(\lambda)^{\mathfrak{m}_\lambda} \neq 0$.*

The stabilizer

Analogous but incomparable to Schneider 1990:

Definition (H* 2021)

Let $(\mathcal{H}, \Lambda)$ be a setting and $\mathfrak{m}$ be a maximal ideal of Λ. The *stabilizer* of $\mathcal{H}$ at $\mathfrak{m}$ is defined as $\mathrm{Stab}(\mathcal{H}, \mathfrak{m}) = \mathcal{H}/\mathcal{S}(\mathcal{H}, \mathfrak{m})$ where $\mathcal{S}(\mathcal{H}, \mathfrak{m})$ is the unique maximal subcoalgebra $\mathcal{C}$ such that for every finite-dimensional subcoalgebra $\mathcal{C}'$ of $\mathcal{C}$ there exists $R = \sum_i r_i \otimes s_i \in \Lambda \otimes \Lambda$ which is invertible mod $\mathfrak{m} \otimes \Lambda + \Lambda \otimes \mathfrak{m}$ and $\sum_i r_i \cdot (x \triangleright s_i) = 0$ for all $x \in \mathcal{C}'$.

Example

If $\mathcal{H} = \mathcal{H}' \rtimes G$ where $\mathcal{H}'$ is a connected Hopf algebra and G is a group, then $\mathcal{S}(\mathcal{H}, \mathfrak{m}) = \mathcal{H}' \otimes \bigoplus_{g \in G, g(\mathfrak{m}) \neq \mathfrak{m}} \Bbbk g$ and hence

$$\mathrm{Stab}(\mathcal{H}, \mathfrak{m}) \cong \mathcal{H}' \otimes \Bbbk\, \mathrm{Stab}(G, \mathfrak{m})$$

where $\mathrm{Stab}(G, \mathfrak{m}) = \{g \in G \mid g(\mathfrak{m}) = \mathfrak{m}\}$.

Finiteness theorem

Theorem (H* 2021)

Let $(\mathcal{H}, \Lambda)$ *be a setting and* $\mathfrak{m}$ *be a maximal ideal of* Λ *of finite codimension. Assume that* $\mathrm{Stab}(\mathcal{H}, \mathfrak{m})$ *is finite-dimensional. Then for any* $\mathcal{H}$*-Galois* Λ*-order* F, $\widehat{F}_{\mathfrak{m}}$ *is finitely generated as a left and right* $\widehat{\Lambda}_{\mathfrak{m}}$*-module. Thus*

(i) *There are only finitely many isomorphism classes of simple Harish-Chandra* F*-modules* V *such that* $V^{\mathfrak{m}} \neq 0$*.*

(ii) *For any simple Harish-Chandra* F*-module* V*, the generalized weight space* $V^{\mathfrak{m}}$ *is finite-dimensional,*

and the same holds for F^{op}*.*

Definition (H* 2021)

A *spherical Galois order* with respect to $(\mathcal{H}', W, \Lambda)$ is a subalgebra U of $(L\#\mathcal{H})^W$ such that

(i) $\Lambda^W \subset U$,

(ii) $L^W U = (L\#\mathcal{H})^W$,

(iii) $\widehat{X}(\Lambda^W) \subset \Lambda^W$ for all $X \in U$.

Example

- Principal Galois orders (H* 2017): $\mathcal{H}' = \Bbbk\mathcal{M}$ group algebra. This includes
  - $U(\mathfrak{gl}_n) \hookrightarrow \big(\mathbb{C}(x_{ki} \mid 1 \le i \le k \le n)\#\mathbb{C}\mathbb{Z}^{n(n-1)/2}\big)^{S_1 \times S_2 \times \cdots \times S_n}$ (Futorny-Ovsienko 2010)
  - $U_q(\mathfrak{gl}_n)$ (Hartwig 2017)
  - finite W-algebras of type A (Futorny-Molev-Ovsienko 2010)
  - Coulomb branches (Webster 2019)
- Spherical subalgebras of rational Cherednik algebras

Lemma (H* 2021)

Let F be an $\mathcal{H}$-Galois Λ-order where $\mathcal{H} = \mathcal{H}' \rtimes W$, $|W| \in \Bbbk^\times$. Put $e = \frac{1}{|W|}\sum_{w\in W} w$. Then the centralizer subalgebra eFe is isomorphic to a spherical Galois order U. Conversely, any spherical Galois order occurs this way.

Corollary (H* 2021)

Let U be a spherical Galois order with respect to $(\mathcal{H}', W, \Lambda)$. Let $\mathfrak{m}$ be a maximal ideal of Λ of finite codimension such that the stabilizer $\mathrm{Stab}(\mathcal{H}' \rtimes W, \mathfrak{m})$ is finite-dimensional. Let $\mathfrak{n} = \Lambda^W \cap \mathfrak{m}$. Then

(i) $\mathrm{Irr}(U, \mathfrak{n})$ *is finite,*

(ii) *for any* $[V] \in \mathrm{Irr}(U, \mathfrak{n})$, $V^{\mathfrak{n}}$ *is finite-dimensional,*

and the same holds for U^{op}.

Open problems

- What can be said in settings $(\mathcal{H}^\circ, \Lambda)$ where Λ is possibly noncommutative but $\Lambda^{\text{co}\,\mathcal{H}} \subset \Lambda$ is a Hopf-Galois extension?
- When is F free/flat/projective as a left Λ-module? (Ovsienko's theorem says $U(\mathfrak{gl}_n)$ is free as a left Γ-module.)
- Study various examples in more detail (eg. compute $\dim V^{\mathfrak{m}}$).

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- What can be said in settings $(\mathcal{H}^\circ, \Lambda)$ where Λ is possibly noncommutative but $\Lambda^{\mathrm{co}\,\mathcal{H}} \subset \Lambda$ is a Hopf-Galois extension?
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Thank you for your attention.