Galois Orders and Deformation Quantization

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Rational Galois orders

- $\mathfrak{h}^* \simeq \mathbb{C}^n$ complex vector space; $\mathbb{C}[\mathfrak{h}^*] = \mathbb{C}[x_1, \ldots, x_n]$
- $T \simeq (\mathbb{C}^*)^m$ complex torus; $\mathbb{C}[T] = \mathbb{C}[z_1^{\pm 1}, \ldots, z_m^{\pm 1}] \simeq \mathbb{C}\mathbb{Z}^m$, $m \leq n$
- $\triangleright$ action of $\mathbb{C}[T]$ on $\mathbb{C}[\mathfrak{h}^*]$ by $\mathbb{Z}^m \hookrightarrow \mathrm{Aut}_{\mathbb{C}}(\mathbb{C}[\mathfrak{h}^*])$, $z_i \triangleright x_j = x_j + \delta_{ij}$
- $W < GL(\mathfrak{h}^*)$ complex reflection group normalizing $\mathbb{Z}^m$
- $\mathfrak{h}^*_{\mathrm{reg}}$ complement of reflecting hyperplanes of $\widehat{W} = \mathbb{Z}^n \rtimes W$
- $\mathscr{K} = \big(\mathbb{C}[\mathfrak{h}^*_{\mathrm{reg}}] \# \mathbb{C}[T]\big)^W$, $z_i x_j = (x_j + \delta_{ij}) z_i$
- $\Gamma = \mathbb{C}[\mathfrak{h}^*]^W$
- $\mathscr{K}_\Gamma = \{X \in \mathscr{K} \mid X \triangleright \Gamma \subset \Gamma\}$

Definition (Futorny-Ovsienko 2010; H* 2019)

A *rational Galois order* is a subalgebra $A \subset \mathscr{K}$ such that

(i) $\Gamma \subset A \subset \mathscr{K}_\Gamma$

(ii) $\mathbb{C}[\mathfrak{h}^*_{\mathrm{reg}}]^W \otimes_\Gamma A = \mathscr{K}$

Examples

- $\mathscr{K}_\Gamma$ is the maximal rational Galois order (the *standard Galois order*)
- $U(\mathfrak{gl}_n)$ (Futorny-Ovsienko 2010; Vishnyakova 2017)
- Generalized Weyl algebras (Futorny-Ovsienko 2010)
- Noncommutative Kleinian Singularities of type D_n (H* 202?)
- Finite W-algebras of type A (Futorny-Molev-Ovsienko 2010)
- Coulomb branches (Webster 2019)
- Rational Cherednik Algebras (LePage, Webster 2019)
- $U(\mathfrak{so}_n) \hookrightarrow \mathbb{C}[\mathfrak{h}^*_{\mathrm{reg}}]\#\mathbb{C}[T]$, $\Gamma \simeq \mathbb{C}[\mathfrak{h}^*]^W$ (Disch 2022)
- $U_q(\mathfrak{gl}_{n+1})$ are Galois orders (Futorny-H* 2011, H* 2019)
- $\imath$quantum group $U'_q(\mathfrak{so}_n) \hookrightarrow \mathbb{C}[H_{\mathrm{reg}}]\#\mathbb{C}[T]$, $\Gamma \simeq \mathbb{C}[H]^W$ (Disch 2022)
- q-OGZ algebras $U_q(\boldsymbol{r})$ (conjecturally q-analogs of certain Coulomb branches) are Galois orders (Mazorchuk 1999, H* 2016)

Galois order realization of $\mathfrak{gl}_n$

Let $\mathfrak{h} =$ CSA of $\mathfrak{gl}_1 \times \mathfrak{gl}_2 \times \cdots \times \mathfrak{gl}_n$,
$T =$ torus in $GL_1 \times GL_2 \times \cdots \times GL_{n-1}$, $W = S_1 \times S_2 \times \cdots \times S_n$
Futorny-Ovsienko (2010):

$$\varphi : U(\mathfrak{gl}_n) \hookrightarrow \left(\mathbb{C}[\mathfrak{h}^*_{\mathrm{reg}}]\#\mathbb{C}[T]\right)^W_\Gamma$$

$$e_k^\pm \mapsto \pm \sum_{i=1}^{k} \frac{\prod_{j=1}^{k\pm 1}(x_{k\pm 1,j} - x_{ki})}{\prod_{j:\, j\neq i}(x_{kj} - x_{ki})} z_{ki}^{\pm 1}$$

$$E_{kk} \mapsto \sum_{i=1}^{k} x_{ki} - \sum_{j=1}^{k-1} x_{k-1,j} + \text{const.}$$

Known property: $\varphi\big|_{Z_k} : Z_k \simeq \mathbb{C}[\mathfrak{h}_k^*]^{W_k}$ is the Harish-Chandra homomorphism for $\mathfrak{gl}_k$

Observation:

The map φ is $\mathbb{C}^*$-equivariant with natural actions on both sides.

Deformation

$U^\hbar(\mathfrak{gl}_n)$, $[E_{ij}, E_{kl}]_\hbar = \hbar[E_{ij}, E_{kl}]$.

- $U^\hbar(\mathfrak{gl}_n) \cong U(\mathfrak{gl}_n)$ for $\hbar \neq 0$, via $E_{ij} \mapsto \hbar E_{ij}$
- $U^0(\mathfrak{gl}_n) = S(\mathfrak{gl}_n) = \mathbb{C}[\mathfrak{gl}_n^*]$ Poisson algebra $\{E_{ij}, E_{kl}\} = \frac{1}{\hbar}[E_{ij}, E_{kl}]_\hbar$

$\mathbb{C}^*$ action on $\mathfrak{h}^*$ gives a deformation

$$\mathscr{K}^\hbar = \big(\mathbb{C}[\hbar][\mathfrak{h}^*_{\mathrm{reg}}]\#_\hbar\mathbb{C}[T]\big)^W$$

where $\#_\hbar$ refers to the deformed action

$$z_i \triangleright_\hbar x_j = x_j + \hbar\delta_{ij}$$

By $\mathbb{C}^*$-equivariance, the map φ can be deformed into a map of $\mathbb{C}[\hbar]$-algebras

$$\varphi^\hbar : U^\hbar(\mathfrak{gl}_n) \longrightarrow \big(\mathbb{C}[\hbar][\mathfrak{h}^*_{\mathrm{reg}}]\#_\hbar\mathbb{C}[T]\big)_\Gamma^W$$

Limit as $\hbar \to 0$

As $\hbar \to 0$ we obtain: $\varphi^0 : S(\mathfrak{gl}_n) = \mathbb{C}[\mathfrak{gl}_n^*] \longrightarrow \mathscr{K}_\Gamma^0$ where

$$\mathscr{K}_\Gamma^0 = \left(\mathbb{C}[\mathfrak{h}^*_{\mathrm{reg}}] \otimes \mathbb{C}[T]\right)_\Gamma^W$$
$$= \left\{ f \in \mathbb{C}\left[(\mathfrak{h}^*_{\mathrm{reg}} \times T)/W\right] \mid f \triangleright_0 \Gamma \subset \Gamma \right\} \quad z_{ki} \triangleright_0 x_{lj} = x_{lj}$$
$$= \left\{ f \in \mathbb{C}\left[(\mathfrak{h}^*_{\mathrm{reg}} \times T)/W\right] \mid f\big|_{\mathfrak{h}^*_{\mathrm{reg}}/W \times \{1_T\}} \in \mathbb{C}[\mathfrak{h}^*/W] \right\}$$

In other words, we have a pullback diagram

$$\begin{array}{ccc} \mathscr{K}_\Gamma^0 & \longrightarrow & \mathbb{C}\left[(\mathfrak{h}^*_{\mathrm{reg}} \times T)/W\right] \\ {\scriptstyle f \mapsto f|_{\mathfrak{h}^*_{\mathrm{reg}}/W \times \{1_T\}}} \big\downarrow & & \big\downarrow {\scriptstyle f \mapsto f|_{\mathfrak{h}^*_{\mathrm{reg}}/W \times \{1_T\}}} \\ \mathbb{C}[\mathfrak{h}^*/W] & \longrightarrow & \mathbb{C}[\mathfrak{h}^*_{\mathrm{reg}}/W] \end{array}$$

Furthermore, since $\mathscr{K}_\Gamma^0$ is commutative it is a Poisson algebra with

$$\{f, g\} = \lim_{\hbar \to 0} \frac{1}{\hbar}[f, g]_\hbar$$

The nontrivial relation is: $\{z_{lj}, x_{ki}\} = \delta_{kl}\delta_{ij} z_{lj}$

The Poisson Variety

$X = \mathrm{Spec}(\mathscr{K}_\Gamma^0)$ is an affine Poisson variety, and is the pushout

$$X = \mathfrak{h}^*/W \bigsqcup_{\mathfrak{h}^*_{\mathrm{reg}}/W} (\mathfrak{h}^*_{\mathrm{reg}} \times T)/W$$

with pushout diagram

$$\begin{array}{ccc} \mathfrak{h}^*_{\mathrm{reg}}/W & \longrightarrow & (\mathfrak{h}^*_{\mathrm{reg}} \times T)/W \\ \downarrow & & \downarrow \\ \mathfrak{h}^*/W & \longrightarrow & X \end{array}$$

The maps

$$\Gamma^\hbar \longrightarrow U^\hbar(\mathfrak{gl}_n) \longrightarrow \left(\mathbb{C}[\hbar][\mathfrak{h}^*_{\mathrm{reg}}]\#_\hbar \mathbb{C}[T]\right)^W_{\Gamma^\hbar} \longrightarrow \Gamma^\hbar$$

compose to identity. Letting $\hbar \to 0$ and taking Spec yields

$$\mathbb{C}^{n(n+1)/2} \longrightarrow X \longrightarrow \mathfrak{gl}_n^* \longrightarrow \mathbb{C}^{n(n+1)/2}$$

also composing to identity, where the last map is the Kostant-Wallach map.

Rational Poisson-Galois Orders

Put

- $\mathfrak{h}^* = \mathbb{C}^n,\ T = (\mathbb{C}^*)^m,\ W < GL(\mathfrak{h}^*)$ as before
- $\Gamma = \mathbb{C}[\mathfrak{h}^*/W]$
- $\mathscr{K}^0 = \mathbb{C}\big[(\mathfrak{h}^*_{\mathrm{reg}} \times T)/W\big]$
- $\mathscr{K}^0_\Gamma = \big\{ f \in \mathscr{K}^0 \mid f\big|_{\mathfrak{h}^*_{\mathrm{reg}}/W \times \{1_T\}} \in \mathbb{C}[\mathfrak{h}^*/W] \big\}$

Definition (H* 2022)

A *rational Poisson-Galois order* is a subalgebra $A^0 \subset \mathscr{K}^0$ such that

(i) $\Gamma \subset A^0 \subset \mathscr{K}^0_\Gamma$

(ii) $\mathbb{C}[\mathfrak{h}^*_{\mathrm{reg}}/W] \otimes_\Gamma A^0 = \mathscr{K}^0$

Example

- $\mathscr{K}^0_\Gamma$ is the *standard Poisson-Galois order*.
- $\mathbb{C}[\mathfrak{gl}_n^*]$ is a rational Poisson-Galois order via the map φ^0.

Toy Example

$\mathfrak{h} = \mathbb{C}$, $T = \mathbb{C}^*$, $W = S_2$ acting by negation on $\mathfrak{h}$ and inverse on T. Put $\Gamma = \mathbb{C}[x]^W = \mathbb{C}[x^2]$. Write $\mathbb{C}[\mathfrak{h}^* \times T] = \mathbb{C}[x, z, z^{-1}]$. In this case the subalgebra

$$\mathscr{K}_\Gamma \subset \left(\mathbb{C}[x]\#\mathbb{C}[z, z^{-1}]\right)^{S_2}, \qquad zx = (x+1)z$$

is the spherical nilHecke algebra of S_2, generated by

$$u = x^2 \qquad v = \frac{1}{2x}(z - z^{-1}) \qquad w = \frac{1}{2}(z + z^{-1})$$

Then

$$\mathscr{K}_\Gamma^0 \cong \frac{\mathbb{C}[u, v, w]}{(w^2 - uv^2 - 1)} \text{ (Kleinian singularity of "type } D_1\text{")}$$

$$\{v, u\} = 2w \qquad \{w, u\} = 2uv \qquad \{v, w\} = v^2$$

and $\Gamma = \mathbb{C}[u]$ is a maximal Poisson-commutative subalgebra in A^0.

Remark

Every Kleinian singularity of types A_n, D_n is a rational Poisson-Galois order.

Multiplicity-Free Branching Rules

- $\mathfrak{g}_n = \mathfrak{gl}_n$ or $\mathfrak{so}_{n+1}$
- $\mathfrak{g}_n \supset \mathfrak{g}_{n-1} \supset \cdots \supset \mathfrak{g}_1$
- $\mathfrak{g}_n \downarrow \mathfrak{g}_{n-1}$ is multiplicity-free
- $V_n = \bigoplus_{\lambda_n \in P_n^+} V(\lambda_n)$ sum of all fd irreps of $\mathfrak{g}_n$ (Gelfand model)
- $U_n = U(\mathfrak{g}_n)$, $Z_n = Z(U_n)$

Observe:

$$\mathrm{Hom}_{\mathfrak{g}_{n-1}}(V_{n-1}, V_n) \otimes_{Z_{n-1}} \mathrm{Hom}_{\mathfrak{g}_{n-2}}(V_{n-2}, V_{n-1}) \otimes_{Z_{n-2}} \cdots \otimes_{Z_2} \mathrm{Hom}_{\mathfrak{g}_1}(V_1, V_2)$$

$$\cong \bigoplus_{\lambda_1 \to \lambda_2 \to \cdots \to \lambda_n} \mathrm{Hom}_{\mathfrak{g}_{n-1}}(V(\lambda_{n-1}), V(\lambda_n)) \otimes_{Z_{n-1}} \mathrm{Hom}_{\mathfrak{g}_{n-2}}(V(\lambda_{n-2}), V(\lambda_{n-1}))$$

$$\otimes_{Z_{n-2}} \cdots \otimes_{Z_2} \mathrm{Hom}_{\mathfrak{g}_1}(V(\lambda_1), V(\lambda_2))$$

$$\cong V_n$$

On the other hand

$$\mathrm{Hom}_{\mathfrak{g}_{n-1}}(V_{n-1}, V_n) = \bigoplus_{\lambda_{n-1},\lambda_n} \mathrm{Hom}_{\mathfrak{g}_{n-1}}(V(\lambda_{n-1}), V(\lambda_n))$$

$$= \bigoplus_{\lambda_{n-1},\lambda_n} V(\lambda_n)^{\mathfrak{g}_{n-1}^+}[\lambda_{n-1}] = (V_n)^{\mathfrak{g}_{n-1}^+}$$

$\mathfrak{g}_n = \mathfrak{g}_n^- \oplus \mathfrak{h}_n \oplus \mathfrak{g}_n^+$ triangular decomposition.

Conclusion

The direct sum V_n of all finite-dimensional irreducible representations of $\mathfrak{g}_n$ can be factored as follows:

$$V_n \cong V_n^{\mathfrak{g}_{n-1}^+} \otimes_{Z_{n-1}} V_{n-1}^{\mathfrak{g}_{n-2}^+} \otimes_{Z_{n-2}} \cdots \otimes_{Z_2} V_2^{\mathfrak{g}_1^+}$$

Mickelsson's Step Algebra

(Mickelsson, van den Hombergh, Zhelobenko, Khoroshkin, Ogievetsky,...)

- $\mathfrak{g} \subset \mathfrak{G}$ reductive pair of fin-dim'l complex Lie algebras
- $\mathfrak{g} = \mathfrak{g}^- \oplus \mathfrak{h} \oplus \mathfrak{g}^+$ triangular decomposition
- $U = U(\mathfrak{G})$
- $I = U\mathfrak{g}_+$ left ideal
- $N = N_U(I) = \{u \in U \mid Iu \subset I\}$ normalizer
- $S(\mathfrak{G}, \mathfrak{g}) = N/I$ Mickelsson's *step algebra* (1973)

Main Property

If V is a $\mathfrak{G}$-module then $V^+ = \{v \in V \mid \mathfrak{g}^+ v = 0\}$ is an $S(\mathfrak{G}, \mathfrak{g})$-module.
(Indeed, if $u \in N$, $v \in V^+$ then $\mathfrak{g}^+ uv \subset Iuv \subset Iv = U\mathfrak{g}^+ v = 0$.)

Action of Mickelsson Step Algebras on V_n

Let

$$\boldsymbol{S}_n = S(\mathfrak{g}_n, \mathfrak{g}_{n-1}) \otimes_{Z_{n-1}} S(\mathfrak{g}_{n-1}, \mathfrak{g}_{n-2}) \otimes_{Z_{n-2}} \cdots \otimes_{Z_2} S(\mathfrak{g}_2, \mathfrak{g}_1)$$

Recall

$$V_n \cong V_n^{\mathfrak{g}_{n-1}^+} \otimes_{Z_{n-1}} V_{n-1}^{\mathfrak{g}_{n-2}^+} \otimes_{Z_{n-2}} \cdots \otimes_{Z_2} V_2^{\mathfrak{g}_1^+}$$

Observation

V_n is a left $\boldsymbol{S}_n$-module.

Conjecture (work in progress; known for $\mathfrak{gl}_n$)

- V_n is faithful as an $\boldsymbol{S}_n$-module.
- There exists an injective algebra map φ from U_n into a localization $\mathbb{K}_n$ of $\boldsymbol{S}_n$ such that the diagram commutes:

$$\begin{array}{ccc} & & U_n \\ & \overset{\varphi}{\swarrow} & \big\downarrow \rho \\ \boldsymbol{S}_n & \xrightarrow{\ \pi\ } & \mathrm{End}_{\mathbb{C}}(V_n) \end{array}$$

- The Weyl group $W_n \times W_{n-1} \times \cdots \times W_1$ acts on $\mathbb{K}_n$.
- Elements in the image of φ are $W_n \times W_{n-1} \times \cdots \times W_1$-invariants.
- $\mathbb{K}_n$ plays the role of $\mathbb{C}[\mathfrak{h}^*_{\mathrm{reg}}]\#\mathbb{C}[T]$.

Conclusion

- The theory of rational Galois orders has a Poisson-algebra analog.
- We found an affine Poisson variety X which maps to $\mathfrak{gl}_n^*$ related to the complex GZ integrable system.
- This picture generalizes to other rational Galois orders (Type A finite W-algebras, Coulomb branches, rational Cherednik algebras, ...): $X \to \operatorname{Spec}(\operatorname{gr} A) \to \operatorname{Spec}(\operatorname{gr} \Gamma)$
- (Conjectured) explanation of Galois orders that would enable $\mathfrak{so}_n$ analogs of the above.

Thank you for your attention.