Differential Reduction Algebras

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April, 2024

Basic Idea: From Centralizers to Normalizers

Consider an equation

$$Hf = 0$$

H some linear operator, f in some function space.
If Q is a linear operator commuting with H then Q acts on the solution space:

$$Hf = 0 \Longrightarrow H(Qf) = Q(Hf) = Q(0) = 0$$

Actually, it suffices that $HQ \in \mathcal{A}H$ for some algebra $\mathcal{A}$.
So let

- $\mathcal{A}$ be an algebra containing H,
- $I_+ = \mathcal{A}H$ be the left ideal generated by H
- $N = \{Q \in \mathcal{A} \mid I_+ Q \subset I_+\}$ be the normalizer of I_+ in $\mathcal{A}$.

Then N/I_+ is an algebra acting on the solution space

$$\{f \mid Hf = 0\}.$$

Examples: Laplace, Dirac, Maxwell

$$\Delta f = 0$$

$$\partial\!\!\!/ f = 0$$

$$\begin{cases} df = 0 \\ d^* f = 0 \end{cases}$$

Examples: Laplace, Dirac, Maxwell (cont'd)

Each of these operators are images of simple root vectors under an oscillator-type representation $\mathfrak{g} \to W$:

$$\Delta = \varphi(e_\alpha), \qquad \varphi : \mathfrak{sl}_2 \to W(2n)$$

$$\partial\!\!\!/ = \varphi(e_\delta), \qquad \varphi : \mathfrak{osp}_{1|2} \to C(n) \otimes W(2n)$$

(Zhelobenko?)

$$d = \varphi(e_{\delta_1}),\ d^* = \varphi(e_{\delta_2}), \qquad \varphi : \mathfrak{osp}_{2|2} \to C(2n) \otimes W(2n)$$

(Howe-Lu)

Here $W(m), C(m)$ are Weyl/Clifford algebras on m generators.

Reduction Algebras

Lie (super)algebra with triangular decomposition

$$\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h} \oplus \mathfrak{g}_+.$$

Associative (super)algebra A with a (super)algebra map

$$U(\mathfrak{g}) \to A.$$

View A as $U(\mathfrak{g})$-bimodule. Form the left ideal

$$I_+ = A\mathfrak{g}_+.$$

Consider the normalizer

$$N = \{a \in A \mid I_+ a \subset I_+\}.$$

The *reduction algebra* $Z(\mathfrak{g}, A)$ is

$$Z(\mathfrak{g}, A) = N/I_+.$$

Examples

- For each of Laplace, Dirac, Maxwell, the corresponding reduction algebra $Z(\mathfrak{g}, A)$ acts irreducibly on the solution space.
- A map $\mathfrak{g} \to \mathfrak{G}$ of Lie (super)algebras gives a reduction algebra

  $$Z(\mathfrak{g}, U(\mathfrak{G})).$$

  This algebra governs the branching rule $\mathfrak{G} \downarrow \mathfrak{g}$.
- In particular, the diagonal map $\mathfrak{g} \to \mathfrak{g} \times \mathfrak{g}$ yields the *diagonal reduction algebra*

  $$\Delta R(\mathfrak{g}) = Z(\mathfrak{g}, U(\mathfrak{g} \times \mathfrak{g})).$$

  $\Delta R(\mathfrak{gl}_n)$ was shown to be braided bialgebra by Khoroshkin and Ogievetsky, being a reflection equation algebra with dynamical R-matrix.
  $\Delta R(\mathfrak{osp}_{1|2})$ was studied by H* and Williams II (2021,2022).

“OK, but aren’t branching rules known?”

Yes, but reduction algebras

1. provide *explicit intertwining operators*, useful in physics eg. Clebsch-Gordan “addition of angular momentum”;
2. act on multiplicity spaces, not just compute their dimension;
3. are “symmetry algebras” for equations of motion in physics;
4. are interesting in their own right; precise relationship to dynamical quantum groups not fully understood.

History

The reduction algebras $Z(\mathfrak{g}, \mathfrak{G})$ have been studied extensively when $\operatorname{rk}\mathfrak{G} \leq 1 + \operatorname{rk}\mathfrak{g}$, including for

- $(\mathfrak{g}, \mathfrak{G}) = \big(\mathfrak{g}(n-1), \mathfrak{g}(n)\big)$ where $\mathfrak{g}(n) = \mathfrak{gl}(n), \mathfrak{sl}(n), \mathfrak{so}(n)$ (van den Hombergh 1976; Zhelobenko 1983–)
- $(\mathfrak{g}, \mathfrak{G}) = \big(\mathfrak{g}(n-1), \mathfrak{g}(n)\big)$ where $\mathfrak{g}(n) = \mathfrak{gl}(m|n), \mathfrak{osp}(n|2m)$, m fixed (Tolstoy 1986)
- $(\mathfrak{g}, \mathfrak{G}) = \big(\mathfrak{so}(n-2), \mathfrak{so}(n)\big)$ and $\big(\mathfrak{sp}(2n-2), \mathfrak{sp}(2n)\big)$ (Molev 2000).

The quantum analog of the reduction algebra, $Z_q(\mathfrak{g}, \mathfrak{G})$, associated to $U_q(\mathfrak{g}) \subset U_q(\mathfrak{G})$ has also been studied for $(\mathfrak{g}, \mathfrak{G}) = \big(\mathfrak{g}(n-1), \mathfrak{g}(n)\big)$ where

- $\mathfrak{g}(n) = \mathfrak{su}(n)$ (Tolstoy 1990)
- $\mathfrak{g}(n) = \mathfrak{su}(1|n)$ (Palev-Tolstoy 1991)
- $\mathfrak{g}(n) = \mathfrak{so}(n)$ and $\mathfrak{sp}(2n)$ (Ashton-Mudrov 2015)

Functoriality?

Given a commutative square

$$\begin{array}{ccc} \mathfrak{g} & \longrightarrow & A \\ \downarrow & & \downarrow \\ \mathfrak{G} & \longrightarrow & B \end{array}$$

what is the relationship between

$$Z(\mathfrak{g}, A) \quad \text{and} \quad Z(\mathfrak{G}, B) \quad ?$$

Some obvious partial answers; in general hard to say.

Stabilization and Cutting

Khoroshkin-Ogievetsky developed techniques of *stabilization* and *cutting* for commutation relations related to the square

$$\begin{array}{ccccc} \mathfrak{gl}_n & \longrightarrow & U(\mathfrak{gl}_n) \otimes U(\mathfrak{gl}_n) & & \rightsquigarrow \Delta R(\mathfrak{gl}_n) \\ \downarrow & & \downarrow & & \\ \mathfrak{gl}_{n+m} & \longrightarrow & U(\mathfrak{gl}_{n+m}) \otimes U(\mathfrak{gl}_{n+m}) & & \rightsquigarrow \Delta R(\mathfrak{gl}_{n+m}) \end{array}$$

key to their inductive computation of a complete presentation of the diagonal reduction algebra $\Delta R(\mathfrak{gl}_n)$.

- Open Problem: Find a presentation for $\Delta R(\mathfrak{g})$ for $\mathfrak{g} = \mathfrak{so}_n, \mathfrak{sp}_{2n}$.

Hard. Try easier cases first: Differential reduction algebras.

Differential Reduction Algebras

Oscillator representations provide homomorphisms

$$\mathfrak{g} \to W$$

W is some Weyl/Clifford (super)algebra.
Combine with diagonal map:

$$\mathfrak{g} \to U(\mathfrak{g}) \otimes U(\mathfrak{g}) \to W \otimes U(\mathfrak{g})$$

The corresponding *differential reduction algebra* is

$$D(\mathfrak{g}) = Z(\mathfrak{g}, W \otimes U(\mathfrak{g})).$$

It controls structure of $\mathfrak{g}$-modules of the form

$$\mathbb{C}[\underline{x}] \otimes V.$$

Squares of Interest

$$
\begin{array}{ccccc}
\mathfrak{gl}_n & \longrightarrow & W \otimes U(\mathfrak{gl}_n) & & \rightsquigarrow D(\mathfrak{gl}_n) \\
\downarrow & & \downarrow & & \\
\mathfrak{sp}_{2n} & \longrightarrow & W' \otimes U(\mathfrak{sp}_{2n}) & & \rightsquigarrow D(\mathfrak{sp}_{2n}) \\
\downarrow & & \downarrow & & \\
\mathfrak{osp}_{1|2n} & \longrightarrow & W'' \otimes U(\mathfrak{osp}_{1|2n}) & & \rightsquigarrow D(\mathfrak{osp}_{1|2n})
\end{array}
$$

- The algebra $D(\mathfrak{gl}_n)$ was studied in Herlemont's PhD thesis (2018).
- In our 2024 paper we give presentation of $D(\mathfrak{sp}_{2n})$ for $n = 2$.
- The case $n > 2$ still open, as is $D(\mathfrak{osp}_{1|2n})$.

Presentation of $D(\mathfrak{sp}_4)$

Let $\mathfrak{sp}_4 \to W(4)$ be the natural map and

$$D(\mathfrak{sp}_4) = R \otimes_{U(\mathfrak{h})} Z(\mathfrak{sp}_4, W(4) \otimes U(\mathfrak{sp}_4))$$

where R is the ring of "dynamical scalars"

$$R = \mathbb{C}[H_\alpha, H_\beta][(H_\gamma - n)^{-1} \mid \gamma \in Q^\vee, n \in \mathbb{Z}]$$

Theorem (H*, Williams II (2024))

$D(\mathfrak{sp}_4)$ is generated as an R-ring by $\bar{x}_1, \bar{\partial}_1, \bar{x}_2, \bar{\partial}_2$ subject to:

$$\bar{x}_i H_j = (H_j - a_{ij})\bar{x}_i, \; a_{ij} \in \mathbb{Z} \qquad \textit{+dual rel.}$$

$$\bar{x}_1 \bar{x}_2 = \Big(1 + \frac{1}{a}\Big)\bar{x}_2 \bar{x}_1 \qquad \textit{+dual rel.}$$

$$\bar{x}_1 \bar{\partial}_2 = \Big(1 + \frac{1}{c}\Big)\bar{\partial}_2 \bar{x}_1 \qquad \textit{+dual rel.}$$

$$\bar{x}_1 \bar{\partial}_1 = -1 + \frac{1}{a} + f_{11}\bar{\partial}_1 \bar{x}_1 + f_{12}\bar{\partial}_2 \bar{x}_2 \qquad +\bar{x}_2\bar{\partial}_2 \textit{ rel.}$$

where

$$f_{11} = \frac{(a+1)(a-1)(b+1)}{a^2 b} \qquad f_{12} = \frac{-(d+2)}{ac},$$

and $a = H_\alpha + 1, \;\; b = H_{\beta+2\alpha} + 1, \;\; c = H_{\beta+\alpha} + 1, \;\; d = H_\beta + 1.$

Limit as "$H_\gamma \to \infty$"

$$\bar{x}_1\bar{\partial}_1 = -1 + \frac{1}{a} + f_{11}\bar{\partial}_1\bar{x}_1 + f_{12}\bar{\partial}_2\bar{x}_2,$$

$$f_{11} = \frac{(a+1)(a-1)(b+1)}{a^2 b} \qquad f_{12} = \frac{-(d+2)}{ac},$$

Put

$$D^*(\mathfrak{sp}_4) = \mathbb{C}[\hbar, \hbar^{-1}] \otimes_{\mathbb{C}} D(\mathfrak{sp}_4)$$

Consider the integral form

$$D^0(\mathfrak{sp}_4) = \mathbb{C}[\hbar]\langle H'_\gamma = \hbar H_\gamma,\ \bar{x}_i,\ \bar{\partial}_j\rangle \subset D^*(\mathfrak{sp}_4)$$

Substituting $H_\gamma = \frac{1}{\hbar}H'_\gamma$ in above relations we see

$$D^0(\mathfrak{sp}_4)/(\hbar) \cong W(4) \otimes_{\mathbb{C}} U(\mathfrak{h}).$$

Generalized Weyl Algebra Realization

Generalized Weyl algebras were introduced by Bavula.

Theorem (H*, Williams II (2024))

The "normalized" generators of $D(\mathfrak{sp}_4)$:

$$\widehat{x}_1 = \bar{x}_1, \qquad \widehat{x}_2 = (H_\alpha + 2)\bar{x}_2,$$
$$\widehat{\partial}_1 = \bar{\partial}_1(H_\alpha + 1)(H_{\beta+\alpha} + 1), \qquad \widehat{\partial}_2 = \bar{\partial}_2(H_{\beta+\alpha} + 1)$$

satisfy

$$[\widehat{x}_1, \widehat{x}_2] = [\widehat{\partial}_1, \widehat{\partial}_2] = [\widehat{x}_1, \widehat{\partial}_2] = [\widehat{x}_2, \widehat{\partial}_1] = 0,$$
$$\widehat{x}_1\widehat{\partial}_1 = C_1 + g_{11}\widehat{\partial}_1\widehat{x}_1 + g_{12}\widehat{\partial}_2\widehat{x}_2$$
$$\widehat{x}_2\widehat{\partial}_2 = C_2 + g_{21}\widehat{\partial}_1\widehat{x}_1 + g_{22}\widehat{\partial}_2\widehat{x}_2$$

for certain $C_i, g_{ij} \in R$, making $D(\mathfrak{sp}_4)$ a non-trivial example of a generalized Weyl algebra.

Tools: Extremal Projector

- (Asherova, Smirnov, Tolstoy, ... 1971–) Every reductive Lie algebra $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h} \oplus \mathfrak{g}_+$ has a a unique *extremal projector*

$$P = P(\mathfrak{g}) \in TU(\mathfrak{g}) \quad \text{(certain completion of } U(\mathfrak{g}))$$

defined by

$$e_\alpha P = 0 = Pf_\alpha \ \forall \alpha \in \Phi_+$$

$$P \cong 1 \ (\text{mod } \mathfrak{g}_- TU(\mathfrak{g}) + TU(\mathfrak{g})\mathfrak{g}_+)$$

Moreover,

$$P^2 = P, \qquad [\mathfrak{h}, P] = 0.$$

$$P(\mathfrak{sl}_2) = \sum_{n \geq 0} \frac{(-1)^n}{n!} \frac{1}{(h+2)\cdots(h+n+1)} f^n e^n$$

$$P(\mathfrak{sp}_4) = P_\alpha P_{\beta+2\alpha} P_{\beta+\alpha} P_\beta = P_\beta P_{\beta+\alpha} P_{\beta+2\alpha} P_\alpha$$

Tools: Double Coset Algebra

- ($\sim$ Mickelsson 1973,...) Given $\mathfrak{g} \to A$, the map

$$Z(\mathfrak{g}, A) = N/I_+ \to I_- \backslash A / I_+$$

  is injective, where $I_+ = A\mathfrak{g}_+$, $I_- = \mathfrak{g}_- A$; and bijective after localization at coroots-minus-integers.

- (Zhelobenko; Khoroshkin-Ogievetsky) The product in $I_- \backslash A / I_+$ making the map an isomorphism is given by

$$\bar{a} \diamond \bar{b} = \overline{aPb}, \qquad \bar{a} = I_- + a + I_+ \in I_- \backslash A / I_+$$

  called the *diamond product*.

Future Directions

- Find presentation, center, irreps, etc for $D(\mathfrak{g})$, $\mathfrak{g} = \mathfrak{so}_n, \mathfrak{sp}_{2n}, \mathfrak{osp}_{1|2n}$, maybe using a version of Stabilization and Cutting.
- Do the same for $\Delta R(\mathfrak{g})$.
- $Z(\mathfrak{osp}(2|2), C(2n) \otimes W(2n))$ and Maxwell's equations?
- Coupled Dirac+Maxwell?
- ...

References

1. arXiv:2403.15968 [math.RT] *Symplectic differential reduction algebras and skew-affine generalized Weyl algebras*

Thank you for your attention.