On the Diagonal Reduction Algebra for $\mathfrak{osp}(1|2)$

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Extremal Projector

(Asherova, Smirnov, Tolstoy, ... 1971–)

- $H = H_- \otimes H_0 \otimes H_+$ Hopf superalgebra with triangular decomposition
- $\mathcal{C}$ a subcategory of H-Mod
- $I_\pm = \ker \varepsilon_{H_\pm}$ augmentation ideals
- functors $(-)^+, (-)_- : \mathcal{C} \to \mathrm{Vec}$:

$$V^+ = \text{“}\ker I_+\text{”} = \{v \in V \mid I_+ v = 0\} \qquad (\textit{invariants})$$

$$V_- = \text{“}\mathrm{coker}\, I_-\text{”} = V/I_- V \qquad (\textit{coinvariants})$$

- Inclusion $\iota_V : V^+ \to V$ and projection $\pi_V : V \to V_-$ compose to

$$\mathrm{Q}_V : V^+ \to V_- \quad v \mapsto v + I_- \quad \rightsquigarrow \mathrm{Q} : (-)^+ \Rightarrow (-)_-$$

Definition (HW 2021)

The *extremal projector* P *for* H *in* $\mathcal{C}$ is the inverse of Q (if it exists). Then $P_V := \iota_V \circ \mathrm{P}_V \circ \pi_V$ is a linear map $V \to V$ for any $V \in H$-Mod, satisfying

$$I_+ P_V = 0 = P_V I_- \qquad P^2 = P \qquad P_V \circ \iota_V = \iota_V \qquad \pi_V \circ P_V = \pi_V$$

Examples

Theorem (Tolstoy 1985)

If $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h} \oplus \mathfrak{g}_+$ is a fin-dim'l complex reductive classical Lie superalgebra with a non-degenerate Killing form, then $H = U(\mathfrak{g})$ has an extremal projector in the category $\mathcal{C}$ of locally $\mathfrak{g}_+$-finite weight modules with non-integral support.

Example

For $\mathfrak{g} = \mathfrak{sl}(2)$:

$$P = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(h+\rho(h)+n)_n} f^n e^n$$

where $\rho(h) = 1$ and $(x)_n = x(x-1)\cdots(x-n+1)$ is the falling factorial.

Remark

If $V \in \mathcal{C}$ is semisimple then weight bases for V^+ are in bijection with decompositions of V into irreducible $\mathfrak{g}$-submodules.

Mickelsson's Step Algebra

(Mickelsson, van den Hombergh, Zhelobenko, Khoroshkin, Ogievetsky,...)

- $\mathfrak{g} \subset \mathfrak{G}$ reductive pair of fin-dim'l complex Lie (super)algebras
- $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h} \oplus \mathfrak{g}_+$ triangular decomposition
- $U = U(\mathfrak{G})$
- $I = U\mathfrak{g}_+$ left ideal
- $N = N_U(I) = \{u \in U \mid Iu \subset I\}$ normalizer
- $S(\mathfrak{G}, \mathfrak{g}) = N/I$ Mickelsson's *step algebra* (1973)
- If V is a $U(\mathfrak{G})$-module then $V^+ = \{v \in V \mid \mathfrak{g}_+ v = 0\}$ is an $S(\mathfrak{G}, \mathfrak{g})$-module: $\mathfrak{g}_+ uv \subset Iuv \subset Iv = 0$ for $u \in N$, $v \in V^+$.

Theorem (van den Hombergh 1975)

If V is a locally $\mathfrak{g}$-finite irreducible $U(\mathfrak{G})$-module, then V^+ is an irreducible $S(\mathfrak{G}, \mathfrak{g})$-module.

Difficulties with $S(\mathfrak{G}, \mathfrak{g})$

1. $S(\mathfrak{G}, \mathfrak{g})$ is not a finitely generated $\mathbb{C}$-algebra. How can one write down elements of $S(\mathfrak{G}, \mathfrak{g})$?
2. How can one effectively find relations among elements?

Remarkable Observation

(Zhelobenko 1985)

Let $V = U/I = U(\mathfrak{G})/U(\mathfrak{G})\mathfrak{g}_+$, regarded as a left $\mathfrak{g}$-module. Then

$$V^+ = \{u + I \mid \mathfrak{g}_+ u \subset I\} = N/I = S(\mathfrak{G}, \mathfrak{g}).$$

In other words, *the step algebra itself is the space of $\mathfrak{g}_+$-invariants in the universal relative Verma module U/I.*

Therefore, if we can use the extremal projector P we can describe $S(\mathfrak{G}, \mathfrak{g})$ and resolve the difficulties.

Zhelobenko's Reduction Algebra

To deploy P one replaces U by $U' = U(\mathfrak{G})[(h_\alpha - n)^{-1} \mid n \in \mathbb{Z}, \alpha \in \Phi(\mathfrak{g})]$ in the construction of $S(\mathfrak{G}, \mathfrak{g})$ to obtain

$$Z(\mathfrak{G}, \mathfrak{g}) = N_{U'}(I')/I', \quad I' = U'\mathfrak{g}_+$$

which is called the *reduction algebra* of the pair $\mathfrak{g} \subset \mathfrak{G}$. This ensures that U'/I' is an object of $\mathcal{C}$ so that we have (assuming now $\mathfrak{g}$ is as in Tolstoy 1985):

$$P_{U'/I'} : U'/I' \twoheadrightarrow (U'/I')^+ = Z(\mathfrak{G}, \mathfrak{g})$$

The following addresses the "difficulties":

Theorem

1. *Decompose $\mathfrak{G} = \mathfrak{g} \oplus \mathfrak{p}$ as $\mathfrak{g}$-modules. Then the image of $\mathfrak{p}$ in $Z(\mathfrak{G}, \mathfrak{g})$ generates $Z(\mathfrak{G}, \mathfrak{g})$ as a $U'(\mathfrak{h})$-ring. (Mickelsson 1973)*
2. *The bijection $Q_{U/I} : Z(\mathfrak{G}, \mathfrak{g}) \to (U/I)_- = \mathfrak{g}_- U \backslash U / U\mathfrak{g}_+$ equips the double coset space with a product $\bar{u} \Diamond \bar{v} = \overline{uPv}$. (Khoroshkin-Ogievetsky 2008)*

Previous Work

The reduction algebras $Z(\mathfrak{G}, \mathfrak{g})$ have been studied extensively when $\operatorname{rk} \mathfrak{G} \leq 1 + \operatorname{rk} \mathfrak{g}$, including for

- $(\mathfrak{G}, \mathfrak{g}) = \big(\mathfrak{g}(n), \mathfrak{g}(n-1)\big)$ where $\mathfrak{g}(n) = \mathfrak{gl}(n), \mathfrak{sl}(n), \mathfrak{so}(n)$ (van den Hombergh 1976; Zhelobenko 1983–)
- $(\mathfrak{G}, \mathfrak{g}) = \big(\mathfrak{g}(n), \mathfrak{g}(n-1)\big)$ where $\mathfrak{g}(n) = \mathfrak{gl}(m|n), \mathfrak{osp}(n|2m)$, m fixed (Tolstoy 1986)
- $(\mathfrak{G}, \mathfrak{g}) = \big(\mathfrak{so}(n), \mathfrak{so}(n-2)\big)$ and $\big(\mathfrak{sp}(2n), \mathfrak{sp}(2n-2)\big)$ (Molev 2000).

The quantum analog of the reduction algebra, $Z_q(\mathfrak{G}, \mathfrak{g})$, associated to $U_q(\mathfrak{g}) \subset U_q(\mathfrak{G})$ has also been studied for $(\mathfrak{G}, \mathfrak{g}) = \big(\mathfrak{g}(n), \mathfrak{g}(n-1)\big)$ where

- $\mathfrak{g}(n) = \mathfrak{su}(n)$ (Tolstoy 1990)
- $\mathfrak{g}(n) = \mathfrak{su}(1|n)$ (Palev-Tolstoy 1991)
- $\mathfrak{g}(n) = \mathfrak{so}(n)$ and $\mathfrak{sp}(2n)$ (Ashton-Mudrov 2015)

Diagonal Reduction Algebras

For a reductive Lie superalgebra $\mathfrak{g}$, take $\mathfrak{G} = \mathfrak{g} \times \mathfrak{g}$. The diagonal embedding $\mathfrak{g} \subset \mathfrak{g} \times \mathfrak{g}$ gives rise to $DR(\mathfrak{g}) = Z(\mathfrak{g} \times \mathfrak{g}, \mathfrak{g})$ called the *diagonal reduction algebra of* $\mathfrak{g}$.

Theorem (Khoroshkin-Ogievetsky 2011, 2017)

For $\mathfrak{g} = \mathfrak{gl}(n)$*:*

1. *Complete presentations of* $DR(\mathfrak{g})$ *including one in terms of the reflection equation from R-matrix formalism.*
2. *Construction of* $2n$ *central elements of* $DR(\mathfrak{g})$ *that conjecturally generate the whole center.*
3. $DR(\mathfrak{g})$ *has the structure of a braided bialgebra.*

The Orthosymplectic Lie Superalgebra $\mathfrak{osp}(1|2)$

$$\mathfrak{g} = \mathfrak{osp}(1|2) = \mathfrak{osp}(1|2)_{\bar{0}} \oplus \mathfrak{osp}(1|2)_{\bar{1}} = (\mathbb{C}x_{-2\alpha} \oplus \mathbb{C}h \oplus \mathbb{C}x_{2\alpha}) \oplus (\mathbb{C}x_{\alpha} \oplus \mathbb{C}x_{-\alpha})$$

$$\mathfrak{osp}(1|2)_{\bar{0}} \cong \mathfrak{sl}(2, \mathbb{C})$$

$$\mathfrak{osp}(1|2)_{\bar{1}} \cong \mathbb{C}^2 \qquad \text{(as an } \mathfrak{sl}(2, \mathbb{C})\text{-module)}$$

$$\mathfrak{h} = \mathbb{C}h \qquad \mathfrak{n}_{\pm} = \mathbb{C}x_{\pm\alpha} \oplus \mathbb{C}x_{\pm 2\alpha}$$

Main Result 1

Theorem (HW 2021)

1. *Complete presentation of* $DR(\mathfrak{osp}(1|2))$ *by generators and relations.*
2. *PBW type basis.*

Gorelik's Ghost Center

The *center* of an associative superalgebra A consists of all sums of homogeneous z satisfying

$$za = (-1)^{|z||a|} az$$

for all homogeneous $a \in A$.

Definition (Gorelik 2000)

1. The *anti-center* $\Sigma = \Sigma(A)$ of an associative superalgebra A is given by all sums of homogeneous z satisfying

   $$za = (-1)^{(|z|+\bar{1})|a|} az$$

   for all homogeneous $a \in A$.
2. The *ghost center* is $\mathbb{Z}(A) = Z(A) \oplus \Sigma(A)$.

Main Result 2: Ghost Center of $DR\big(\mathfrak{osp}(1|2)\big)$

Put

- $\mathfrak{g} = \mathfrak{osp}(1|2)$
- $C \in Z(U(\mathfrak{g}))$ the Casimir element
- $Q \in \Sigma\big(U(\mathfrak{g})\big)$ the Scasimir element (Leśniewski 1995)

Theorem (HW 2022)

Let $A = DR\big(\mathfrak{osp}(1|2)\big)$. The ghost center $\mathbb{Z}(A)$ is generated by the three elements

$$\mathbb{C}_{\pm} := C \otimes 1 \pm 1 \otimes C + I \in Z(A),$$

$$\mathbb{Q} := Q \otimes Q + I \in \Sigma(A).$$

Moreover, there is an injective algebra map

$$\varphi : \mathbb{Z}(A) \to \mathbb{C}[x, y]$$

such that $\varphi(\mathbb{C}_+) = x^2 + y^2, \quad \varphi(\mathbb{C}_-) = 2xy, \quad \varphi(\mathbb{Q}) = x^2 - y^2.$

Main Result 3: Irreps of $DR\big(\mathfrak{osp}(1|2)\big)$

Theorem (HW 2022)

Let $A = DR\big(\mathfrak{osp}(1|2)\big)$.

1. *For every odd positive integer* n *and every* $(\lambda, \mu) \in \mathbb{C} \times (\mathbb{C} \setminus \mathbb{Z})$ *satisfying*
$$\lambda^2 = (\mu + n)^2$$
there is an irreducible n*-dimensional representation* $L(\lambda, \mu)$ *of* A *such that the action of the ghost center on* $L(\lambda, \mu)$ *is given by*
$$\mathbb{C}_+ \mapsto \lambda^2 + \mu^2 \qquad \mathbb{C}_- \mapsto 2\lambda\mu \qquad \mathbb{Q} \mapsto (\lambda^2 - \mu^2)(-1)^{|\cdot|}$$
where $(-1)^{|\cdot|} \in \mathrm{End}_{\mathbb{C}}\big(L(\lambda, \mu)\big)$ *sends homogeneous* v *to* $(-1)^{|v|}$.
2. *Every finite-dimensional irreducible representation of* A *has odd dimension and is isomorphic to* $L(\lambda, \mu)$ *for a unique pair* (λ, μ) *satisfying* $\lambda^2 = (\mu + \dim V)^2$.

Application to Tensor Product Decompositions

Let $\mathfrak{g} = \mathfrak{osp}(1|2)$. For $\ell \in \mathbb{Z}_{\geq 0}$, let $V(\ell)$ be the $(1+2\ell)$-dimensional irrep of $\mathfrak{g}$, and $\mathbb{C}[x] = V(-1/2)$ be the polynomial irrep of $\mathfrak{g}$. We know:

$$V(\ell) \otimes V(\ell') \cong \bigoplus_{j=-|\ell-\ell'|}^{\ell+\ell'} V(j) \quad \text{(Scheunert-Nahm-Rittenberg 1977)}$$

$$\mathbb{C}[x] \otimes V(1) \cong \bigoplus_{j=0}^{2} V(1 - \frac{1}{2} - j) \quad \text{(special case of Coulembier 2013)}$$

Theorem (HW 2022)

$$\mathbb{C}[x] \otimes V(\ell) = \bigoplus_{j=0}^{2\ell} U(\mathfrak{g}_-) L^j (1 \otimes v_\ell)$$

where $L \in N \subset U(\mathfrak{g} \times \mathfrak{g})$ is a lowering operator explicitly given in a PBW basis for $\mathfrak{g} \times \mathfrak{g}$.

Future directions

- Can $DR(\mathfrak{osp}(1|2n))$ be presented using R-matrix formalism, analogous to the reflection equation for $DR(\mathfrak{gl}(n))$?
- One can define $Z(A, \mathfrak{g})$ where A is an associative superalgebra and $\mathfrak{g} \to A$. We are interested in $Z\big(A_n(\mathbb{C}) \otimes U(\mathfrak{osp}(1|2n)), \mathfrak{osp}(1|2n)\big)$ and applications to intertwining operators for $\mathbb{C}[x_1, \dots, x_n] \otimes V(\lambda)$.

References

1. arXiv:2106.04380 [math.RT] *Diagonal reduction algebra for* $\mathfrak{osp}(1|2)$ (in Theoretical and Mathematical Physics), with D.A. Williams II
2. arXiv:2203.08068 [math.RT] *Ghost center and representations of the diagonal reduction algebra of* $\mathfrak{osp}(1|2)$, with D.A. Williams II

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References

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Thank you for your attention.