Higher Spin Fields and Orthosymplectic Reduction Algebras

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$\mathfrak{sl}(2)$-triples in the Weyl algebra

$$(x^2 f(x))'' = (2xf(x) + x^2 f'(x))' = 2(xf(x))' + 2xf'(x) + x^2 f''(x)$$

Put $\partial = \frac{d}{dx}$. We have proved the operator identity:

$$\partial^2 x^2 - x^2 \partial^2 = 2(x\partial + \partial x)$$

More variables:

$$(\sum_i \partial_i^2)(\sum_i x_i^2) - (\sum_i x_i^2)(\sum_i \partial_i^2) = 2\sum_i (x_i \partial_i + \partial_i x_i)$$

We can make this an $\mathfrak{sl}(2)$-triple! Put

$$E = \frac{1}{2}\sum_i \partial_i^2, \quad F = -\frac{1}{2}\sum_i x_i^2, \quad H = -\frac{1}{2}\sum_i (x_i \partial_i + \partial_i x_i)$$

Then $[E, F] = H, \quad [H, E] = 2E, \quad [H, F] = -2F.$

One more tweak

Denote variables by $x^1, \ldots, x^D$ and let $(\eta_{\mu\nu})$ be a (constant) metric, i.e. a non-degenerate symmetric matrix with inverse $(\eta^{\mu\nu})$ and put

$$E = \frac{1}{2}\partial_\mu\partial^\mu = \frac{1}{2}\sum_{\mu,\nu}\eta^{\mu\nu}\partial_\mu\partial_\nu$$

$$F = -\frac{1}{2}x_\mu x^\mu = -\frac{1}{2}\sum_{\mu,\nu}\eta_{\mu\nu}x^\mu x^\nu$$

$$H = -\frac{1}{2}\{x^\mu, \partial_\mu\} = \sum_{\mu}(x^\mu\partial_\mu + \partial_\mu x^\mu)$$

Upshot: When $D = 4$ and $(\eta_{\mu\nu}) = \mathrm{diag}(-1, 1, 1, 1)$, we have $E = \frac{1}{2}\Box$ where $\Box$ is the d'Alembert operator used in the wave equation (massless Klein-Gordon equation) $\Box\phi = 0$.

Observations so far

- $\mathfrak{sl}(2)$ Lie algebra of all complex traceless 2 × 2-matrices
- $W(2D) = A_D$ the D:th Weyl algebra
- There is a Lie algebra homomorphism $\mathfrak{sl}(2) \to W(2D)$
- Under this map, $e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is sent to (a multiple of) the differential operator $\Box = \partial_\mu \partial^\mu = \sum_{\mu,\nu} \eta^{\mu\nu} \partial_\mu \partial_\nu$ used in classical field theory.
- Let $\mathcal{F} = \mathbb{C}[x^1, x^2, \ldots, x^D]$ be the natural $W(2D)$-module
- Polynomial solutions to the massless Klein-Gordon equation $\Box\phi = 0$ is the subspace of "highest weight vectors":

$$\mathcal{F}^+ = \{\phi \in \mathcal{F} \mid e \cdot \phi = 0\}.$$

Reduction Algebras

Let $\mathfrak{g}$ be a simple (or reductive) Lie algebra with triangular decomposition

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$$

(For example, $\mathfrak{g} = \mathfrak{sl}(2) = \mathbb{C}f \oplus \mathbb{C}h \oplus \mathbb{C}e$.)
Suppose we have a Lie algebra map into an associative algebra:

$$\varphi : \mathfrak{g} \to A, \quad \varphi([x,y]) = \varphi(x)\varphi(y) - \varphi(y)\varphi(x)$$

Regard $\mathfrak{g} \subset A$. If $\mathcal{F}$ is a representation of A, put

$$\mathcal{F}^+ = \{v \in V \mid \mathfrak{n}_+ v = 0\} \tag{1}$$

Which operators from A preserve this space? Let

$$N = \{a \in A \mid \mathfrak{n}_+ a \subset A\mathfrak{n}_+\} \tag{2}$$

If $v \in \mathcal{F}^+$ and $a \in N$ then $av \in \mathcal{F}^+$:

$$\mathfrak{n}_+ av \overset{(2)}{\subset} A\mathfrak{n}_+ v \overset{(1)}{=} 0$$

$$\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$$

$$\varphi : \mathfrak{g} \to A$$

$$N = \{a \in A \mid \mathfrak{n}_+ a \subset A\mathfrak{n}_+\}$$

Note that N is the normalizer of the left ideal $I_+ = A\mathfrak{n}_+$ in A.

Definition

$\mathcal{Z} = N/I_+$ is the reduction algebra associated to $\mathfrak{g}$, A, and φ.

The reduction algebra is also known as (in various contexts): idealizer ring, transvector algebra, generalized Mickelsson algebra, Zhelobenko algebra, quantum Hamiltonian reduction, finite W-algebra, ...

Punchline

Reduction algebras (for appropriate $\mathfrak{g}, A, \varphi$) acts spaces $\mathcal{F}^+$ of solutions to equations of motion in classical field theories.

Theorem (H*, Uhl, Williams II, 2025)

Let $D > 2$ *and* $\mathfrak{sl}(2) \to W(2D)$ *acting on* $\mathcal{F} = \mathbb{C}[x^1, x^2, \ldots, x^D]$. *Then* $\mathcal{Z}$ *acts irreducibly on* $\mathcal{F}^+ = \{\phi \in \mathcal{F} \mid \Box\phi = 0\}$.

Theorem (H*, Uhl, Williams II, 2025)

$\mathbb{C}(H) \otimes_{\mathbb{C}[H]} \mathcal{Z}$ *is generated as a* $\mathbb{C}(H)$*-ring by* $\bar{x}^\mu$, $\bar{\partial}_\mu$ *subject to*

(1) $[\bar{x}^\mu, \bar{x}^\nu] = 0$, $[\bar{\partial}_\mu, \bar{\partial}_\nu] = 0$

(2) $\bar{\partial}_\mu \bar{x}^\nu - \bar{x}^\nu \bar{\partial}_\mu = 1 + \frac{1}{H+1} \bar{x}_\mu \bar{\partial}^\nu$

(3) $\bar{x}^\mu H = (H+1)\bar{x}^\mu$, $\bar{\partial}_\mu H = (H-1)\bar{\partial}_\mu$

(4) $\bar{x}^\mu \bar{x}_\mu = 0$, $\bar{\partial}^\mu \bar{\partial}_\mu = 0$, $\bar{x}^\mu \bar{\partial}_\mu = (-H + \frac{D}{2})$

Other equations

In 1985, Howe connected Maxwell's equations with the Lie superalgebra $\mathfrak{osp}(2|2)$, and the Dirac equation with $\mathfrak{osp}(1|2)$.

Lie (super)alg	Algebra	Field Eq
$\mathfrak{sl}_2$	$W(2D)$	Klein–Gordon eq
$\mathfrak{osp}_{2\|2}$	$W(2D) \otimes \mathbb{C}\langle dx^\mu, \lrcorner_\mu \rangle$	Maxwell eq
$\mathfrak{osp}_{1\|2}$	$W(2D) \otimes \mathbb{C}\langle \gamma_\mu \rangle$	Dirac eq

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What is the pattern? Answer:

Lie superalg	Weyl superalg	Field Eq	Spin
$\mathfrak{sl}_2 \cong \mathfrak{osp}_{0\|2}$	$W(2D\|0D)$	Klein–Gordon eq	0
$\mathfrak{osp}_{1\|2}$	$W(2D\|1D)$	Dirac eq	1/2
$\mathfrak{osp}_{2\|2}$	$W(2D\|2D)$	Maxwell eq	1
$\mathfrak{osp}_{3\|2}$	$W(2D\|3D)$	Rarita–Schwinger eq	3/2
$\mathfrak{osp}_{m\|2}$	$W(2D\|mD)$	(field eq?)	m/2

Higher spin?

“Higher spin Rarita-Schwinger operators” have been studied in Clifford analysis (D. Eelbode, T. Raeymaekers) but they seem to be related to $\mathfrak{osp}(1|2n)$.

- $\omega = (\cdot,\cdot)$ non-deg. anti-symmetric bil. form on $V = \mathbb{C}^{2D}$
- D:th Weyl algebra

$$W(2D) = T(V)/J, \quad J = \langle x \otimes y - y \otimes x - \omega(x,y) \mid x, y \in V \rangle$$

$$W(2D) \cong \mathbb{C}\langle x_1, \partial_1, \ldots, x_D, \partial_D \rangle / \langle \ldots \rangle$$

- V carries a rep. of the symplectic Lie algebra $\mathfrak{sp}(2D)$
- $\mathfrak{sp}(2D)$ acts by derivations on $T(V)$, preserves J, so

$$\mathfrak{sp}(2D) \to \mathrm{Der}(W(2D)) \cong W(2D)_{\mathrm{Lie}}$$

- Combined with

$$\mathfrak{sl}(2) \cong \mathfrak{sp}(2) \to \mathfrak{sp}(2)^{\oplus D} \to \mathfrak{sp}(2D)$$

this gives a Lie algebra map

$$\varphi : \mathfrak{sl}(2) \to W(2D)$$

Clifford algebra

- The complex Clifford algebra $C(n)$ has n generators γ_μ satisfying $\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}$.
- $C(n)$ is independent, up to isomorphism, of choice of metric. One could take $\eta_{\mu\nu} = \delta_{\mu\nu}$.
- When $n = 2D$, we can instead choose generators and relations

$$dx^\mu,\ \lrcorner_\mu \qquad \{\lrcorner_\mu, dx^\nu\} = \delta_\mu^\nu$$

Weyl-Clifford superalgebra $W(2D|3D)$

$W(2D|3D)$ has $2D$ even generators x^μ, ∂_μ satisfying

$$[\partial_\mu, x^\nu] = \delta_\mu^\nu$$

and $3D$ odd generators $\gamma^\mu, \lrcorner_\mu, x^\mu$ satisfying

$$\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}$$

$$\{\lrcorner_\mu, dx^\nu\} = \delta_\mu^\nu$$

Generalizing $\mathfrak{sp}(2) \to W(2D)$, there is a natural Lie superalgebra map

$$\mathfrak{osp}(3|2) \to W(2D|3D)$$

Spin 3/2: The Rarita-Schwinger Equation

Lagrangian $\mathscr{L}_{\mathrm{RS}} = \overline{\psi}_\mu \gamma^{\mu\nu\lambda}\partial_\nu \psi_\lambda$
Field equations are the Euler-Lagrange obtained by varying $\overline{\psi}_\mu$:
$\gamma^{\mu\nu\lambda}\partial_\nu\psi_\lambda = 0$
One can show that this is equivalent to

$$\gamma^\mu(\partial_\mu\psi_\nu - \partial_\nu\psi_\mu) = 0$$

Here

$$\psi = \psi_\mu dx^\mu = \psi_\mu^\alpha dx^\mu \otimes v_\alpha \in \mathbb{C}[x^1, \ldots, x^D] \otimes \wedge(\mathbb{C}^D) \otimes S$$

and $S = \oplus_\alpha \mathbb{C} v_\alpha$ is an (irreducible) representation of the Clifford algebra $C(D)$
γ_μ are generators of the Clifford algebra $\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}$

Field strength RS

The LHS of the RS eq can be rewritten

$$\gamma^\mu(\partial_\mu\psi_\nu - \partial_\nu\psi_\mu)dx^\nu = (\gamma^\lambda \lrcorner_\lambda \circ d)(\psi_\nu dx^\nu)$$

Introduce the field strength ($d = d_{\mathrm{dR}}$)

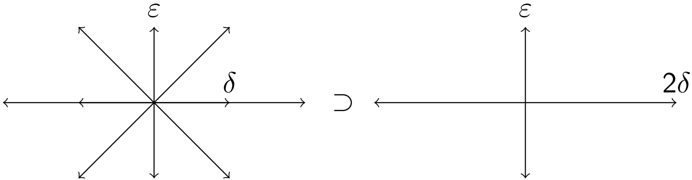
$$F = d\psi = \frac{1}{2}F^\alpha_{\mu\nu}(dx^\mu \wedge dx^\nu) \otimes v_\alpha \in \mathscr{F}$$

Then the RS equation for ψ is equivalent to the two equation system for F:

$$dF = 0, \qquad \gamma^\lambda \lrcorner_\lambda F = 0$$

(because $H^2 = 0$)

Root System of $\mathfrak{osp}(3|2)$

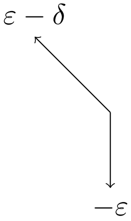


ε is even, δ is odd

Even part:

$\mathfrak{osp}(3|2)_{\overline{0}} = \mathfrak{o}(3) \oplus \mathfrak{sp}(2) = \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$

Choice of simple roots:



Under

$$\mathfrak{osp}(3|2) \to W(2D|3D)$$

we have:

$$e_{\varepsilon-\delta} \mapsto dx^{\mu}\partial_{\mu} = d_{\mathrm{dR}} \quad \text{de Rham coboundary}$$

$$e_{-\varepsilon} \mapsto \gamma^{\mu}\lrcorner_{\mu}$$

Conclusion: The solutions to the field strength RS equations for F form precisely the degree two piece of the highest weight space

$$\mathcal{F}^{+} = \{F \in \mathcal{F} \mid \mathfrak{n}_{+} \cdot F = 0\}$$

It means there is an irreducible action of the associated reduction algebra!

General procedure

Start with natural action of $\mathfrak{osp}(m|2n)$ on $\mathbb{C}^{m|2n}$.
Apply parity-swap functor to get $\mathbb{C}^{2n|m}$
Construct associated Weyl-Clifford superalgebra $W(2n|m)$
Replace m by mD and n by nD, compose with diagonal map

$$\mathfrak{osp}(m|2n) \to W(2nD|mD)$$

This acts on natural function space $\mathcal{F}$
Conjecture: For appropriate choice of simple roots, the image of $\mathfrak{n}_+$ become operators for some associated field theory.

Howe dual pairs

Definition

(1) The *centralizer* of a subalgebra $\mathfrak{k}$ in a Lie (super)algebra $\mathfrak{g}$ is

$$\mathfrak{c}_{\mathfrak{g}}(\mathfrak{k}) = \{x \in \mathfrak{g} \mid [x, \mathfrak{k}] = 0\}$$

(2) Let $\mathfrak{g}$ be a Lie (super)algebra. A pair of subalgebras $(\mathfrak{k}, \mathfrak{l})$ of $\mathfrak{g}$ is a *Howe dual pair* if each is the centralier of the other:

$$\mathfrak{c}_{\mathfrak{g}}(\mathfrak{k}) = \mathfrak{l}, \qquad \mathfrak{c}_{\mathfrak{g}}(\mathfrak{l}) = \mathfrak{k}$$

Example

$\mathfrak{osp}(m|2) \subset \mathfrak{osp}(mD|2D) \supset \mathfrak{so}(D)$

(1) Howe dual pairs provide multiplicity-free decompositions

(2) $\mathfrak{so}(D)$ is contained in the reduction algebra.

Conformal Algebra

The complexified conformal algebra in $D > 2$ dimensions is isomorphic to $\mathfrak{so}(D+2)$.
Lorentz alg: $\mathfrak{so}(D) = \mathrm{span}\{M_{\mu\nu}\}$
Poincaré alg: $\mathfrak{iso}(D) = \mathrm{span}\{M_{\mu\nu},\, P_\mu\}$
Conformal alg: $\mathfrak{co}(D) = \mathrm{span}\{M_{\mu\nu},\, P_\mu,\, D,\, K_\mu\} \cong \mathfrak{so}(D+2)$

- The field equations discussed possess not only Poincaré symmetry but, also on-shell conformal symmetry.
- This means that the conformal algebra should be a subalgebra of the reduction algebra (checked this in the spin 0 case).

More research needs to be done here.

Papers

- *Exact Solutions to the Klein–Gordon Equation via Reduction Algebras*, J.T. Hartwig, L.R. Uhl, D.A. Williams II, arXiv:2507.04572 [math.RT]
- *Dirac Reduction Algebra*, M.T. Dorang, J.T. Hartwig, D.A. Williams II, arXiv:2507.21730 [math.RT]
- *Maxwell Reduction Algebra (tentative)*, M.T. Dorang, J.T. Hartwig, D.A. Williams II, (ongoing work)
- *Rarita-Schwinger and* $\mathfrak{osp}(3|2)$ *(tentative)*, J.T. Hartwig, (ongoing work)

Tool: Extremal Projector

$\mathfrak{n}_+P = 0, P\mathfrak{n}_- = 0, P = 1 \mod \mathfrak{n}_-U \cap U\mathfrak{n}_+, P^2 = P, P^* = P$
Example: $\mathfrak{sl}(2)$

$$P = 1 + \frac{-1}{h+2}fe + \frac{1}{(h+2)(h+3)}f^2e^2 + \cdots$$

Because sum is infinite, it belongs to a completion of $U(\mathfrak{sl}(2))$
Under assumptions on A, the natural map
$S = N/I \to A/I \to A/II$ where $II = \mathfrak{n}_-A + A\mathfrak{n}_+$
is a bijection, with inverse $a + II \mapsto Pa + I = \sum_{\lambda \in Q_+}(P_\lambda a + I)$
Furthermore, the multiplication on A/II that makes the bijection an algebra isomorphism is
$(a + II) \diamond (b + II) = \sum_\lambda (aP_\lambda b + II)$