Generalized Derivations on Algebras and Highest Weight Representations of the Virasoro Algebra

Jonas T. Hartwig

Master's thesis
2002:E16

Lund University

Centre for Mathematical Sciences
Mathematics

CENTRUM SCIENTIARUM MATHEMATICARUM

Generalized Derivations on Algebras and Highest Weight Representations of the Virasoro Algebra

Jonas T. Hartwig
Supervisors: Arne Meurman, Sergei Silvestrov

May 22, 2002

Preface

This master thesis consists of two parts:

I. Generalized derivations on algebras;

II. Highest weight representations of the Virasoro algebra.

In the first part we investigate (σ, τ)-derivations on some classes of algebras, for example on unique factorization domains and on the Witt algebra. This part will be included in the joint article [HaS]. In the second part we study the Witt and Virasoro algebras, and their representation theory. In particular Kac determinant formula is stated and some consequences examined.

The two parts are connected in two ways. First, we find in the first part a generalization of the Witt algebra to an algebra of σ-derivations. In the second part we show that a supersymmetric extension of the Witt algebra can be obtained as a super Lie algebra of superderivations, which are σ-derivations with a special choice of σ.

Acknowledgements

I would like to express my gratitude to my two supervisors, Arne Meurman and Sergei Silvestrov, for interesting and helpful discussions during the writing of this work. I would also like to thank my high school teacher the late Lars-Eric Björk for encouraging me to continue with mathematical studies.

Generalized Derivations on Algebras

Jonas T. Hartwig
Department of Mathematics
Lund University
Box 118
SE-221 00 Lund
Sweden
hartwig@math.lu.se

Sergei D. Silvestrov
Department of Mathematics
Lund Institute of Technology
Box 118
SE-221 00 Lund
Sweden
Sergei.Silvestrov@math.lth.se

May 22, 2002

Abstract. In this paper we study (σ,τ)-derivations on algebras from an abstract point of view. After some definitions and examples, we derive Leibniz type formulas and introduce a module structure on spaces of (σ,τ)-derivations. Then we find all (σ,τ)-derivations on unique factorization domains when σ and τ are different endomorphisms. We also prove necessary equations for σ-derivations on the quantum plane. Conditions for products and Jacobi type identities for (σ,τ)-derivations on associative algebras are considered. Then follows an investigation of homogenous (σ,τ)-derivations on the Witt algebra of degree zero. Finally we generalize the Witt algebra to a skew-symmetric algebra of σ-derivations on a commutative associative algebra.

Contents

1 Introduction

Differential and difference operators of various kinds play fundamental role in many parts of mathematics and its applications – from algebra, non-commutative geometry and non-commutative analysis, functional analysis, classical analysis, function theory and differential equations to specific numerical and statistical methods and algorithms. One of the key properties shared by all these differential and difference operators and making them so important and useful is that they satisfy some versions of the Leibniz rule saying how to calculate the operator on the product of functions given its action on each function. Therefore it is desirable to have a single unifying differentiation theory, which would be concerned with operators of certain general class satisfying general Leibniz rule and containing the well-known derivation and differentiation as examples.

The main objects considered in this article are general operators satisfying Leibniz rule of the form

$$D(xy) = D(x)\tau(y) + \sigma(x)D(y).$$

These operators act on some algebra which in general can be non-commutative. Such operators are called (σ, τ)-derivations. In the special case when τ is the identity mapping sending any x to itself, (σ, τ)-derivations are called σ-derivations. This is a very important general class of operators containing various well-known classes of differential or difference operators obtained by specifying the algebra and the mappings σ and τ acting on it. Some of them are usual differential operators, derivations on commutative and non-commutative algebras, usual difference operators associated with additive shifts, q-difference operators, superderivations on Lie superalgebras, colored (graded) derivations on colored (graded) Lie algebras and Lie superalgebras, derivations and super-derivations on graded associative algebras. We will provide many such examples throughout this article.

Since 1930's there has appeared a large number of publications specifically containing results on general (σ, τ)-derivations or σ-derivations. We have included some of the articles and books known to us in the reference list. At the end of each reference we indicated approximate direction of the reference in relation to (σ, τ)-derivations. We hope that this would be useful for the readers. These works show that (σ, τ)-derivations or their specific classes play important role for the theory of Ore rings and algebras, skew polynomial algebras and skew fields, theory of Noetherian rings, differential and difference algebra, homological algebra, Lie algebras and groups, Lie superalgebras, colored Lie algebras, general ring theory, operator algebras, differential geometry, symbolic and algorithmic algebra, non-commutative geometry, quantum groups and quantum algebras, q-analysis and q-special functions and numerical analysis.

In this article we begin a systematic study of (σ, τ)-derivations on algebras. Many results we present are very fundamental for the theory. However, most of them seems to be not directly available in the literature on the subject known to us. Thus we included proofs of almost all results for completeness of exposition and convenience of the reader. We hope that this article will be a useful contribution to the subject.

We begin by introducing some notations and definitions in Section 2 which will be used in the article. In Section 3 we give some examples of (σ,τ)-derivations to motivate the theory. General formulas for (σ,τ)-derivations on associative algebras, which are useful for explicit calculations, are stated and proved in Section 4. Section 5 contains some general propositions on (σ,τ)-derivations of associative algebras. The structure of (σ,τ)-derivations on algebras which are unique factorization domains as rings are investigated in Section 6. In Section 7 we describe equations which must be satisfied for a linear operator on the quantum plane to be a σ-derivation. In Section 8 we state some conditions which a semigroup must satisfy if it can be the grading semigroup of a semigroup-graded associative algebra on which there exists a homogenous (σ,τ)-derivation. Section 9 contains some general conditions which are necessary and sufficient for the existence of various products for (σ,τ)-derivations. We also consider generalized versions of the Jacobi identity corresponding to these products. Our main motivating example for these results is Γ-graded ϵ-Lie algebras. In Section 10 we consider homogenous (σ,τ)-derivations on the Witt algebra consisting of derivations of Laurent polynomials. Finally, Section 11 contains a generalization of the construction of the Witt algebra to σ-derivations. Using results from previous sections, we show that in particular the space of σ-derivations, $(\sigma \neq \mathrm{id})$, on a unique factorization domain can be given the structure of a skew-symmetric algebra which satisfies a generalized Jacobi identity.

2 Definitions and notations

In this section we introduce some notation and recall some basic definitions which will be used throughout the paper. All algebras and linear spaces under consideration are assumed to be defined over the field of complex numbers $\mathbb{C}$. Many of the results presented are true if $\mathbb{C}$ is replaced by an arbitrary field, or at least a field of characteristic zero.

Let us start by defining the concept of a derivation in an algebra.

Definition 1 (derivation)**.** Let A be an algebra. A linear operator $D : A \to A$ is called a *derivation* in A if

$$D(a_1a_2) = D(a_1)a_2 + a_1D(a_2) \tag{1}$$

for all $a_1, a_2 \in A$.

Definition 2 (σ-derivation)**.** Let σ be a linear operator on an algebra A. Then a linear operator D on A is called a *σ-derivation* in A if

$$D(a_1a_2) = D(a_1)a_2 + \sigma(a_1)D(a_2) \tag{2}$$

for all $a_1, a_2 \in A$.

We shall sometimes write D_σ to denote that D is a σ-derivation. When $\sigma = \mathrm{id}_A$ is the identity operator on A, then the equality (2) coincides with (1) and thus derivations are special cases of σ-derivations with $\sigma = \mathrm{id}_A$.

Definition 3 ((σ, τ)-derivation)**.** Let σ and τ be two linear operators on an algebra A. Then a linear operator D on A is a (σ, τ)-*derivation* on A if

$$D(ab) = D(a)\tau(b) + \sigma(a)D(b) \tag{3}$$

for all $a, b \in A$.

Note that this generalizes σ-derivations in the sence that (σ, id_A)-derivations are precisely σ-derivations, where id_A is the identity operator on A. For an algebra A, and linear operators σ, τ on A we let $\mathfrak{D}_{(\sigma,\tau)}(A)$ denote the set of all (σ, τ)-derivations on A. Also we set $\mathfrak{D}_\sigma(A) = \mathfrak{D}_{(\sigma,\mathrm{id}_A)}(A)$ and $\mathfrak{D}(A) = \mathfrak{D}_{\mathrm{id}_A}(A)$.

Let A be an associative algebra, and let σ and τ be endomorphisms on A. Define for each $a \in A$ a map

$$\delta_a : A \to A,$$

$$\delta_a(x) = a\tau(x) - \sigma(x)a.$$

Then δ_a is a (σ, τ)-derivation on A for each $a \in A$. In fact, for each $x, y \in A$ we have

$$\begin{aligned}
\delta_a(xy) &= a\tau(xy) - \sigma(xy)a = \\
&= a\tau(x)\tau(y) - \sigma(x)\sigma(y)a = \\
&= a\tau(x)\tau(y) - \sigma(x)a\tau(y) + \sigma(x)a\tau(y) - \sigma(x)\sigma(y)a = \\
&= \delta_a(x)\tau(y) + \sigma(x)\delta_a(y).
\end{aligned}$$

Definition 4 (inner (σ, τ)-derivation)**.** The (σ, τ)-derivations δ_a for $a \in A$ are called the *inner* (σ, τ)-derivations of A.

In Section 6.4 we will consider associative algebras A which are unique factorization domains as rings. Given two different endomorphisms σ and τ on A we will give a sufficient and necessary condition (see Corollary 29) on σ and τ under which any (σ, τ)-derivation on A is inner.

For the remainder of this section we shall define graded linear spaces and algebras. For this, let S denote an arbitrary semigroup, written multiplicatively, and let S^{op} denote the opposite semigroup of S, obtained from S by reversing multiplication, i.e. setting $a \star b = b \cdot a$ for all $a, b \in S$, where $\star$ denotes the multiplication in S^{op} and $\cdot$ the multiplication in S.

Definition 5 (S-graded linear space)**.** A linear space V is called *S-graded* if we have associated to V a family $\{V_s\}_{s\in S}$ of subspaces $V_s \subseteq V$ satisfying

$$V = \bigoplus_{s\in S} V_s. \tag{4}$$

An element a of an S-graded linear space V is called *homogenous of degree* $s \in S$, if $a \in V_s$. Note that the zero element $0 \in V$ is homogenous of every degree, and that if $a \in V$ is nonzero and homogenous there is a unique $s \in S$ such that $a \in V_s$. This element s is called the *degree* of a, and is denoted $\deg a$. A subspace $W \subseteq V$ of an S-graded linear space $V = \oplus_{s\in S} V_s$ is called *graded* if $W = \oplus_{s\in S}(W \cap V_s)$.

Definition 6 (S-graded algebra)**.** An algebra A is called *S-graded* if it is S-graded as a linear space, $A = \oplus_{s\in S} A_s$, in such a way that

$$A_s \cdot A_t \subseteq A_{st} \quad \text{for all } s, t \in S \tag{5}$$

From (5) follows that the product of two homogenous elements is again homogenous. A subalgebra $B \subseteq A$ of an S-graded algebra $A = \oplus_{s\in S} A_s$ is called *graded* if $B = \oplus_{s\in S}(B \cap A_s)$.

Definition 7 (homogenous linear operator)**.** Let $V = \oplus_{s\in S} V_s$ and $W = \oplus_{s\in S} W_s$ be two S-graded linear spaces. A linear mapping

$$L : V \to W$$

is called *homogenous* if for each $s \in S$ there is some $t \in S$ such that

$$L(V_s) \subseteq W_t \tag{6}$$

If U, V and W are S-graded linear spaces and

$$L : U \to V \quad \text{and} \quad K : V \to W$$

are homogenous linear mappings, then for every $s \in S$ there are some $t, u \in S$ such that

$$(KL)(U_s) = K(L(U_s)) \subseteq K(V_t) \subseteq W_u. \tag{7}$$

Thus the composition KL is again a homogenous linear mapping.

Definition 8 (right and left degree)**.** Let $V = \oplus_{s\in S} V_s$ and $W = \oplus_{s\in S} W_s$ be two S-graded linear spaces. Let $t \in S$. A homogenous linear operator $L : V \to W$ is said to be *of right degree* t if

$$L(V_s) \subseteq W_{st} \quad \text{for all } s \in S \tag{8}$$

and *of left degree* t if

$$L(V_s) \subseteq W_{ts} \quad \text{for all } s \in S \tag{9}$$

If S is commutative, we simply say that L is *of degree* t when (8) and (9) are true.

For two linear spaces V and W, let $\mathcal{L}(V,W)$ denote the linear space of all linear mappings $V \to W$, and set $\mathcal{L}(V) = \mathcal{L}(V,V)$. If V and W are S-graded linear spaces, define for each $s \in S$ the linear subspace $\mathrm{Lgr}_\ell(V,W)_s$ of $\mathcal{L}(V,W)$ consisting of all homogenous linear mappings $V \to W$ of left degree s. Form

$$\mathrm{Lgr}_\ell(V,W) = \bigoplus_{s\in S} \mathrm{Lgr}_\ell(V,W)_s \tag{10}$$

Then $\mathrm{Lgr}_\ell(V,W)$ is an S-graded linear space.

Take now $W = V$, set $\mathrm{Lgr}_\ell(V)_s = \mathrm{Lgr}_\ell(V,V)_s$ for $s \in S$ and $\mathrm{Lgr}_\ell(V) = \mathrm{Lgr}_\ell(V,V)$. Let $K \in \mathrm{Lgr}_\ell(V)_s$ and $L \in \mathrm{Lgr}_\ell(V)_t$. Then the composition KL of the linear mappings K and L satisfies

$$KL(V_u) \subseteq K(V_{tu}) \subseteq V_{s(tu)} = V_{(st)u} \tag{11}$$

for all $u \in S$, since S is associative. Thus $KL \in \mathrm{Lgr}_\ell(V)_{st}$. Therefore we have shown that $\mathrm{Lgr}_\ell(V)$ is an S-graded associative algebra.

Similarly we can construct the S^{op}-graded associative algebra $\mathrm{Lgr}_r(V)$ as the direct sum of the linear spaces $\mathrm{Lgr}_r(V)_s$ of homogenous linear mappings $V \to V$ of right degree s for $s \in S$. If S is commutative, $\mathrm{Lgr}_r(V)_s = \mathrm{Lgr}_\ell(V)_s$ for all $s \in S$ so that $\mathrm{Lgr}_r(V) = \mathrm{Lgr}_\ell(V)$. Then we drop the indices ℓ and r and simply write $\mathrm{Lgr}(V)_s$ and $\mathrm{Lgr}(V)$ respectively.

In many important examples of graded linear spaces and algebras, S is an abelian group, but the definitions can be made for a general semigroup S. Associativity of S is however a reasonable assumption, since otherwise, in general, $\mathrm{Lgr}_\ell(V)$ will not be an S-graded algebra and $\mathrm{Lgr}_r(V)$ will not be an S^{op}-graded algebra.

3 Examples

In this section we consider some important examples of (σ,τ)-derivations to motivate the theory. Our first example is simple, but it illustrates the fact that (σ,τ)-derivations form a very general class of operators.

Example 1. Let A be some algebra, and σ an algebra endomorphism on A. Then

$$\sigma(xy) = \sigma(x)\sigma(y) = \sigma(x)\cdot\frac{1}{2}\sigma(y) + \frac{1}{2}\sigma(x)\cdot\sigma(y).$$

Thus σ is a $(\frac{1}{2}\sigma, \frac{1}{2}\sigma)$-derivation. More generally, for any two complex numbers λ, μ such that $\lambda + \mu = 1$ we have that an endomorphism σ is a $(\lambda\sigma, \mu\sigma)$-derivation.

3.1 Superderivations

Let $A = A_0 \oplus A_1$ be an associative algebra which is $\mathbb{Z}_2$-graded, that is, such that

$$A_\alpha A_\beta \subseteq A_{\alpha+\beta}$$

for $\alpha, \beta \in \mathbb{Z}_2 = \{0, 1\}$. Define the $\mathbb{Z}_2$-grading operator as a linear operator $\Theta : A \to A$ acting on homogenous elements by

$$\Theta(a) = \begin{cases} a, & \text{when } a \in A_0 \\ -a, & \text{when } a \in A_1 \end{cases} \tag{12}$$

or equivalently, we set $\Theta\big|_{A_0} = \mathrm{id}_{A_0}$ and $\Theta\big|_{A_1} = -\mathrm{id}_{A_1}$. Note also that Θ is a homogenous operator of degree zero, since $\Theta(A_\alpha) = A_\alpha$ for $\alpha \in \mathbb{Z}_2$. We may also introduce a $\mathbb{Z}_2$-grading on $\mathcal{L}(A)$ as follows:

$$\mathcal{L}(A) = \mathcal{L}(A)_0 \oplus \mathcal{L}(A)_1$$

where

$$\begin{aligned} \mathcal{L}(A)_0 &= \mathcal{L}(A_0, A_0) \oplus \mathcal{L}(A_1, A_1) = \{A \in \mathcal{L}(A) \bigm| A\Theta - \Theta A = 0\} \\ \mathcal{L}(A)_1 &= \mathcal{L}(A_0, A_1) \oplus \mathcal{L}(A_1, A_0) = \{A \in \mathcal{L}(A) \bigm| A\Theta + \Theta A = 0\} \end{aligned}$$

and a corresponding $\mathbb{Z}_2$-grading operator $\widetilde{\Theta} : \mathcal{L}(A) \to \mathcal{L}(A)$.

A homogenous linear operator $D \in \mathcal{L}(A)_\alpha$ is called a *superderivation* on A if it is a Θ^α-derivation, where $\Theta^0 = \mathrm{id}_A$ is the identity operator on A and $\Theta^1 = \Theta$ is the $\mathbb{Z}_2$-grading operator (12). So, if $D \in \mathcal{L}(A)_0$ is a superderivation, then $D(ab) = D(a)b + aD(b)$ for all homogenous $a, b \in A$ and by bilinearity for all $a, b \in A$. Thus the set of superderivations in $\mathcal{L}(A)_0$ is just the set of all derivations of A which are homogenous of degree 0. If $D \in \mathcal{L}(A)_1$ is a superderivation, then $D(ab) = D(a)b + \Theta(a)D(b)$ for all homogenous $a, b \in A$ and by bilinearity for all $a, b \in A$. So the set of all superderivations in $\mathcal{L}(A)_1$ is the set of all Θ-derivations of A which are homogenous of degree 1. Denote the $\mathbb{Z}_2$-graded subspace of all superderivations on A by $\mathfrak{D}(A) = \mathfrak{D}(A)_0 \oplus \mathfrak{D}(A)_1$. Let $D \in \mathfrak{D}(A)_\delta$ and $E \in \mathfrak{D}(A)_\epsilon$ be superderivations on A and define

$$\langle D, E \rangle = DE - (-1)^{\delta\epsilon} ED.$$

Then $\langle D, E \rangle$ is also a superderivation of A, and $\langle D, E \rangle \in \mathfrak{D}(A)_{\delta+\epsilon}$. This bracket $\langle \cdot, \cdot \rangle$ makes $\mathfrak{D}(A)$ into a super Lie algebra, that is, it satisfies the following two identities:

$$(-1)^{\gamma\epsilon} \langle \langle C, D \rangle, E \rangle + (-1)^{\epsilon\delta} \langle \langle E, C \rangle, D \rangle + (-1)^{\delta\gamma} \langle \langle D, E \rangle, C \rangle = 0 \tag{13}$$

$$\langle D, E \rangle = -(-1)^{\delta\epsilon} \langle E, D \rangle \tag{14}$$

where $C \in \mathfrak{D}(A)_\gamma$, $D \in \mathfrak{D}(A)_\delta$ and $E \in \mathfrak{D}(A)_\epsilon$. Equation (13) is the super Jacobi identity and property (14) is super skewsymmetry.

3.2 ϵ-Derivations

Throughout this this subsection, Γ will always denote an abelian group. A *commutation factor* on the group Γ is a mapping

$$\epsilon : \Gamma \times \Gamma \to \mathbb{C} \tag{15}$$

such that

$$\epsilon(\alpha, \beta)\epsilon(\beta, \alpha) = 1 \tag{16}$$
$$\epsilon(\alpha, \beta + \gamma) = \epsilon(\alpha, \beta)\epsilon(\alpha, \gamma) \tag{17}$$
$$\epsilon(\alpha + \beta, \gamma) = \epsilon(\alpha, \gamma)\epsilon(\beta, \gamma) \tag{18}$$

for all $\alpha, \beta, \gamma \in \Gamma$.

Example 2. Let $\Gamma = \mathbb{Z}_2$ and define

$$\epsilon_{\mathbb{Z}_2} : \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathbb{C} \tag{19}$$

by

$$\epsilon_{\mathbb{Z}_2}(\alpha, \beta) = (-1)^{\alpha\beta} \tag{20}$$

Then the axioms (16)-(18) are easily verified when $\epsilon = \epsilon_{\mathbb{Z}_2}$. Thus $\epsilon_{\mathbb{Z}_2}$ is a commutation factor on $\mathbb{Z}_2$.

Let ϵ be a commutation factor on an abelian group Γ. A Γ-graded algebra

$$A = \oplus_{\gamma \in \Gamma} A_\gamma$$

whose product mapping is denoted by a bracket $\langle \cdot, \cdot \rangle$, is called a Γ-graded ϵ-*Lie algebra* if the following identities are satisfied:

$$\langle A, B \rangle = -\epsilon(\alpha, \beta)\langle B, A \rangle \qquad (\epsilon\text{-skew-symmetry}) \tag{21}$$

$$\epsilon(\gamma, \alpha)\langle A, \langle B, C \rangle\rangle + \epsilon(\beta, \gamma)\langle C, \langle A, B \rangle\rangle + \epsilon(\alpha, \beta)\langle B, \langle C, A \rangle\rangle = 0 \tag{22}$$

(ϵ-Jacobi identity)

for all homogenous elements $A \in A_\alpha$, $B \in A_\beta$, and $C \in A_\gamma$.

Remark 1. The $\mathbb{Z}_2$-graded $\epsilon_{\mathbb{Z}_2}$-Lie algebras are precisely the super Lie algebras. This can be seen by substituting the definition (19)-(20) of $\epsilon_{\mathbb{Z}_2}$ into (21) and (22) and comparing with (14) and (13). Thus super Lie algebras is a special case of Γ-graded ϵ-Lie algebras with $\Gamma = \mathbb{Z}_2$ and $\epsilon = \epsilon_{\mathbb{Z}_2}$.

Let A be an associative Γ-graded algebra. On the Γ-graded linear space A we define a new multiplication $\langle\cdot,\cdot\rangle$, the *$\epsilon$-commutator*, by setting

$$\langle a,b\rangle = ab - \epsilon(\alpha,\beta)ba \tag{23}$$

for all homogenous elements $a\in A_\alpha$ and $b\in A_\beta$ and the extending linearly. The bracket $\langle\cdot,\cdot\rangle$ turns A into a Γ-graded ϵ-Lie algebra which is said to be associated with A and which will be denoted by $A(\epsilon)$.

Let V be a Γ-graded linear space. We know that $\mathrm{Lgr}(V,V)$ is an associative Γ-graded algebra. The Γ-graded ϵ-Lie algebra associated with $\mathrm{Lgr}(V,V)$ is called the *general linear Γ-graded ϵ-Lie algebra* of V and will be denoted by $\mathrm{gl}(V,\epsilon)$.

Let $A=\oplus_{\gamma\in\Gamma}A_\gamma$ be a Γ-graded algebra. For any $\delta\in\Gamma$, let $D(A,\epsilon)_\delta$ denote the subspace of $\mathrm{gl}(A,\epsilon)_\delta$ consisting of all elements D such that

$$D(ab) = D(a)b + \epsilon(\delta,\alpha)aD(b) \tag{24}$$

for all homogenous elements $a\in A_\alpha$, $b\in A_\beta$. Then

$$D(A,\epsilon)=\bigoplus_{\delta\in\Gamma} D(A,\epsilon)_\delta$$

is a graded subalgebra of $\mathrm{gl}(A,\epsilon)$ in the sence of Γ-graded ϵ-Lie algebras. The proof of this simple fact is contained in Example 3, Section 9, page 58 and Example 5, Section 9.5, page 64. The elements of $D(A,\epsilon)$ are called *ϵ-derivations* of A. Note that a homogenous ϵ-derivation is a σ-derivation, with a special choice of σ. Namely, if $D\in D(A,\epsilon)_\delta$, that is, D is a homogenous ϵ-derivation on A of degree $\delta\in\Gamma$, we define a linear operator $\sigma_\delta : A\to A$ by setting $\sigma_\delta(a)=\epsilon(\alpha,\delta)a$ for homogenous $a\in A_\alpha$ of degree α, and extend linearly. Then D is an σ_δ-derivation on A.

Further in Section 9, some generalizations of Γ-graded ϵ-Lie algebras, ϵ-commutators (23), and of ϵ-Jacobi identities (22) will be considered.

3.3 Ore Extensions, difference and shift type operators

Many of the most important and frequently studied operators in analysis and its applications in physics, engineering, statistics, numerical analysis and other subjects are expressed using some basic difference and shift operators which satisfy Leibniz type rules, making them into (σ,τ)-derivations for various choices of σ and τ. These operators act on functions.

Let A be an associative algebra with unit, and $A[t]$ be the free left A-module on one generator t consisting of all polynomials of the form

$$P = a_nt^n + a_{n-1}t^{n-1} + \cdots + a_0t^0$$

Table 1: (σ, τ)-derivations and their Leibniz rules.

Operator ∂	$(\partial \cdot f)(x)$	$(\partial \cdot fg)(x)$	(σ, τ)
Differentiation	$f'(x)$	$f(x)(\partial \cdot g)(x) + (\partial \cdot f)(x)g(x)$	$(\mathrm{id}, \mathrm{id})$
Shift S	$f(x+1)$	$f(x+1)(\partial \cdot g)(x)$	$(S, 0)$
Difference	$f(x+1) - f(x)$	$f(x+1)(\partial \cdot g)(x) + (\partial \cdot f)(x)g(x)$	(S, id)
q-Dilation $T_{x,q}$	$f(qx)$	$f(qx)(\partial \cdot g)(x)$	$(T_{x,q}, 0)$
Continuous q-difference	$f(qx) - f(x)$	$f(qx)(\partial \cdot g)(x) + (\partial \cdot f)(x)g(x)$	$(T_{x,q}, \mathrm{id})$
Jackson q-differentiation $D_{x,q}$	$\frac{f(qx)-f(x)}{(q-1)x}$	$f(qx)(\partial \cdot g)(x) + (\partial \cdot f)(x)g(x)$	$(T_{x,q}, \mathrm{id})$
Eulerian operator	$xf'(x)$	$f(x)(\partial \cdot g)(x) + (\partial \cdot f)(x)g(x)$	$(\mathrm{id}, \mathrm{id})$
Mahlerian operator C_p	$f(x^p)$	$f(x^p)(\partial \cdot g)(x)$	$(C_p, 0)$
Divided differences	$\frac{f(x)-f(a)}{x-a}$	$f(a)(\partial \cdot g)(x) + (\partial \cdot f)(x)g(x)$	$(\mathrm{id}, \mathrm{id})$

Table 2: Ore algebras and their commutation rules and representations.

Operator ∂	$\sigma(P)(X)$	$\delta(P)(X)$	$\partial P(X)$	∂X
Differentiation	$P(X)$	$P'(X)$	$P(X)\partial + P'(X)$	$X\partial + 1$
Shift	$P(X+1)$	0	$P(X+1)\partial$	$(X+1)\partial$
Difference	$P(X+1)$	$(\Delta \cdot P)(X)$	$P(X+1)\partial + (\Delta \cdot P)(X)$	$(X+1)\partial + 1$
q-Dilation	$P(qX)$	0	$P(qX)\partial$	$qX\partial$
Cont. q-difference	$P(qX)$	$P(qX) - P(X)$	$P(qX)\partial + P(qX) - P(X)$	$qX\partial + (q-1)X$
Jackson q-diff.	$P(qX)$	$\frac{P(qX)-P(X)}{(q-1)X}$	$P(qX)\partial + \frac{P(qX)-P(X)}{(q-1)X}$	$qX\partial + 1$
q-Shift	$P(qX)$	0	$P(qX)\partial$	$qX\partial$
Discrete q-difference	$P(qX)$	$P(qX) - P(X)$	$P(qX)\partial + P(qX) - P(X)$	$qX\partial + (q-1)X$
Eulerian operator	$P(X)$	$XP'(X)$	$P(X)\partial + XP'(X)$	$X\partial + X$
e^x-Differentiation	$P(X)$	$XP'(X)$	$P(X)\partial + XP'(X)$	$X\partial + X$
Mahlerian operator	$P(X^p)$	0	$P(X^p)\partial$	$X^p\partial$
Divided differences	$P(a)$	$\frac{P(X)-P(a)}{X-a}$	$P(a)\partial + \frac{P(X)-P(a)}{X-a}$	$a\partial + 1$

Operator ∂	$(\partial \cdot f)(x)$	$(X \cdot f)(x)$
Differentiation	$f'(x)$	$xf(x)$
Shift	$f(x+1)$	$xf(x)$
Difference	$f(x+1) - f(x)$	$xf(x)$
q-Dilation	$f(qx)$	$xf(x)$
Cont. q-difference	$f(qx) - f(x)$	$xf(x)$
Jackson q-diff.	$\frac{f(qx)-f(x)}{(q-1)x}$	$xf(x)$
q-Shift	$f(x+1)$	$q^x f(x)$
Discrete q-difference	$f(x+1) - f(x)$	$q^x f(x)$
Eulerian operator	$xf'(x)$	$xf(x)$
e^x-Differentiation	$f'(x)$	$e^x f(x)$
Mahlerian operator	$f(x^p)$	$xf(x)$
Divided differences	$\frac{f(x)-f(a)}{x-a}$	$xf(x)$

with coefficients a_i in A. If $a_n \neq 0$ we say that the degree $\deg(P)$ of P is equal to n; by convention, we set $\deg(0) = -\infty$. We state a theorem which classifies all algebra structures on $A[t]$ compatible with the algebra structure on A and with the degree. The proof of the theorem can be found for example in [Ka].

Theorem 1. (a) *Assume that $A[t]$ has an algebra structure such that the natural inclusion of A into $A[t]$ is a homomorphism of algebras, and we have $\deg(PQ) = \deg(P) + \deg(Q)$ for any $P, Q \in A[t]$. Then A has no zero-divizors and there exists a unique injective algebra endomorphism σ of A and a unique σ-derivation D of A such that*

$$ta = \sigma(a)t + D(a) \tag{25}$$

(b) *Conversely, let A be an algebra without zero-divizors. Given an injective algebra endomorphism σ of A and an σ-derivation D of A, there exists a unique algebra structure on $A[t]$ such that the inclusion of A into $A[t]$ is an algebra homomorphism and relation (25) holds for all a in A.*

The algebra defined by Theorem 1 (b) is denoted by $A[t, \sigma, D]$, and is called the *Ore Extension* attached to the data (A, σ, D).

4 General Leibniz type formulas

In this section we state and prove some formulas, all of which are generalizations of the Leibniz formula for the derivation of a product of functions. Throughout, A will denote an algebra.

4.1 Formulas for derivations

Theorem 2 (Leibniz formula)**.** *Let D be a derivation in A and $a_1, a_2 \in A$. Then*

$$D^n(a_1a_2) = \sum_{k=0}^{n} \binom{n}{k} D^{n-k}(a_1)D^k(a_2) \tag{26}$$

for all integers $n \geq 1$.

Proof. We will prove this statement by induction on n. For $n = 1$ the equality reduces to the derivation property (1) of D required in Definition 1. Assume (26) holds for

$n = m \geq 1$. Then we have

$$\begin{aligned}
D^{m+1}(a_1a_2) &= D\bigl(D^m(a_1a_2)\bigr) = D\Biggl(\sum_{k=0}^{m}\binom{m}{k}D^{m-k}(a_1)D^k(a_2)\Biggr) = \\
&= \sum_{k=0}^{m}\binom{m}{k}\Bigl(D^{m-k+1}(a_1)D^k(a_2) + D^{m-k}(a_1)D^{k+1}(a_2)\Bigr) = \\
&= \sum_{k=0}^{m}\binom{m+1}{k}\frac{m+1-k}{m+1}D^{(m+1)-k}(a_1)D^k(a_2) + \\
&\quad + \sum_{k=0}^{m}\binom{m+1}{k+1}\frac{k+1}{m+1}D^{(m+1)-(k+1)}(a_1)D^{k+1}(a_2) = \\
&= \sum_{k=0}^{m}\binom{m+1}{k}\left(1-\frac{k}{m+1}\right)D^{(m+1)-k}(a_1)D^k(a_2) + \\
&\quad + \sum_{k=1}^{m+1}\binom{m+1}{k}\frac{k}{m+1}D^{(m+1)-k}(a_1)D^k(a_2) = \\
&= D^{m+1}(a_1)a_2 + \\
&\quad + \sum_{k=1}^{m}\binom{m+1}{k}\left(1-\frac{k}{m+1}+\frac{k}{m+1}\right)D^{(m+1)-k}(a_1)D^k(a_2) + \\
&\quad + a_1D^{m+1}(a_2) \\
&= \sum_{k=0}^{m+1}\binom{m+1}{k}D^{(m+1)-k}(a_1)D^k(a_2)
\end{aligned}$$

So (26) holds for $n = m + 1$ as well. Thus by the principle of mathematical induction (26) holds for every integer $n \geq 1$. □

Suppose now that A is associative, and replace a_2 by a_2a_3 in (26). Then we can use the formula once more to get

$$\begin{aligned}
D^n(a_1a_2a_3) &= \sum_{k=0}^{n}\binom{n}{k}D^{n-k}(a_1)D^k(a_2a_3) = \\
&= \sum_{k=0}^{n}\binom{n}{k}D^{n-k}(a_1)\sum_{i=0}^{k}\binom{k}{i}D^{k-i}(a_2)D^i(a_3) = \\
&= \sum_{k=0}^{n}\sum_{i=0}^{k}\frac{n!}{(n-k)!\,k!}\,\frac{k!}{(k-i)!\,i!}\,D^{n-k}(a_1)D^{k-i}(a_2)D^i(a_3) = \\
&= \sum_{n\geq k\geq i\geq 0} n!\cdot\frac{D^{n-k}(a_1)}{(n-k)!}\cdot\frac{D^{k-i}(a_2)}{(k-i)!}\cdot\frac{D^i(a_3)}{i!}
\end{aligned}$$

This process can be repeated, and we come to the following generalization of the Leibniz formula (26).

Theorem 3 (Generalized Leibniz formula)**.** *If A is associative, D is a derivation in A, and $n \geq 1$ a positive integer. Then for every integer $m \geq 1$ and every m-tuple $(a_1, \ldots, a_m) \in A^m$,*

$$D^n(a_1 a_2 \ldots a_m) = \sum_{\substack{n=k_0 \geq k_1 \geq \cdots \\ \cdots \geq k_j \geq k_{j+1} \geq \cdots \\ \cdots \geq k_{m-1} \geq k_m = 0}} n! \cdot \frac{D^{n-k_1}(a_1)}{(n-k_1)!} \cdots \frac{D^{k_{j-1}-k_j}(a_j)}{(k_{j-1}-k_j)!} \cdots \frac{D^{k_{m-1}}(a_m)}{k_{m-1}!} \tag{27}$$

Proof. To prove this we use induction over m. For $m = 1$, the formula (27) reduces to

$$D^n(a_1) = n! \frac{D^n(a_1)}{n!}$$

which is true for all $a_1 \in A$. Also we know that (27) holds for $m = 2$ and every $(a_1, a_2) \in A^2$ as it becomes (26). Assume (27) it holds for $m = p$, where $p \geq 1$ is a positive integer, and every $(a_1, \ldots, a_p) \in A^p$. Then for any $(a_1, \ldots, a_{p+1}) \in A^{p+1}$,

$$\begin{aligned}
&D^n(a_1 a_2 \ldots a_{p+1}) = \\
&\quad = \sum_{\substack{n=k_0 \geq k_1 \geq \cdots \\ \cdots \geq k_j \geq k_{j+1} \geq \cdots \\ \cdots \geq k_{p-1} \geq k_p = 0}} n! \cdot \frac{D^{n-k_1}(a_1)}{(n-k_1)!} \cdots \frac{D^{k_{j-1}-k_j}(a_j)}{(k_{j-1}-k_j)!} \cdots \frac{D^{k_{p-1}}(a_p a_{p+1})}{k_{p-1}!} = \\
&= \sum_{\substack{n=k_0 \geq k_1 \geq \cdots \\ \cdots \geq k_j \geq k_{j+1} \geq \cdots \\ \cdots \geq k_{p-1} \geq 0}} \Bigg(n! \cdot \frac{D^{n-k_1}(a_1)}{(n-k_1)!} \cdots \frac{D^{k_{j-1}-k_j}(a_j)}{(k_{j-1}-k_j)!} \cdots \\
&\qquad\qquad \cdots \sum_{k_p=0}^{k_{p-1}} \frac{\binom{k_{p-1}}{k_p}}{k_{p-1}!} D^{k_{p-1}-k_p}(a_p) D^{k_p}(a_{p+1}) \Bigg) = \\
&= \sum_{\substack{n=k_0 \geq k_1 \geq \cdots \\ \cdots \geq k_j \geq k_{j+1} \geq \cdots \\ \cdots \geq k_p \geq k_{p+1} = 0}} n! \cdot \frac{D^{n-k_1}(a_1)}{(n-k_1)!} \cdots \frac{D^{k_{j-1}-k_j}(a_j)}{(k_{j-1}-k_j)!} \cdots \frac{D^{k_{p-1}-k_p}(a_p)}{(k_{p-1}-k_p)!} \frac{D^{k_p}(a_{p+1})}{k_p!}
\end{aligned}$$

where we used (26) in the second equality. This calculation shows that (27) holds for $m = p + 1$ and every $(a_1, \ldots, a_{p+1}) \in A^{p+1}$. Thus, by the principle of mathematical induction, (27) holds for any integer $m \geq 1$ and any m-tuple $(a_1, \ldots, a_m) \in A^m$. $\square$

If we rewrite the coefficient in each term of the sum in (27) as follows:

$$\frac{n!}{(n-k_1)!\cdot\ldots\cdot(k_j-k_{j+1})!\cdot\ldots\cdot k_{m-1}!}=\prod_{j=0}^{m-1}\frac{k_j!}{(k_j-k_{j+1})!\cdot k_{j+1}!}=\prod_{j=0}^{m-1}\binom{k_j}{k_{j+1}},$$

where $k_0=n$ and $k_m=0$, we see that the formula (27) also can be written

$$D^n(a_1a_2\ldots a_m)=\sum_{\substack{n=k_0\geq k_1\geq\cdots\\ \cdots\geq k_j\geq k_{j+1}\geq\cdots\\ \cdots\geq k_{m-1}\geq k_m=0}}\left(\prod_{l=0}^{m-1}\binom{k_l}{k_{l+1}}\cdot D^{k_l-k_{l+1}}(a_{l+1})\right)$$

4.2 Formulas for σ-derivations

In this subsection, A will be an algebra,and σ will be a linear operator $A\to A$. The Leibniz formula (26) for derivations can be generalized to σ-derivations as follows.

Theorem 4 (Leibniz formula for σ-derivations)**.** *Let D_σ be a σ-derivation in A. Then*

$$D_\sigma^n(a_1a_2)=\sum_{k=0}^{n}\left(\sum_{\{\gamma:\{1,\ldots,n\}\to\{\sigma,D_\sigma\}|\#\gamma^{-1}(\sigma)=k\}}\gamma(1)\ldots\gamma(n)(a_1)\cdot D_\sigma^k(a_2)\right)\tag{28}$$

for all $a_1,a_2\in A$, and all integers $n\geq 1$.

Remark 2. In (28) and throughout this paper $\#A$ will denote the number of elements or cardinality of the set A.

Remark 3. The formula (28) becomes $a_1a_2=a_1a_2$ and thus is valid for $n=0$, provided the following usual conventions are used: the set of functions with empty domain of definition is the set $\{\emptyset\}$ of the empty set; the notation $\{1,...,n\}=\{j\in\mathbb{Z}|1\leq j\leq n\}$ is used for any $n\in\mathbb{Z}$, and in particular $\{1,...,n\}$ is the empty set if $n\leq 0$; the product $\gamma(1)\cdot\ldots\cdot\gamma(n)$ is defined to be the identity transformation id_A for $n\leq 0$ as product over the empty set. These conventions and also the convention that sums over the empty set is zero will be used in formulas throughout the paper.

Proof. We use induction over n. For $n=1$ we get (2) which is true, since D_σ is a σ-derivation. Now assume (28) holds for $n=p$, where $p\geq 1$ is some positive integer. Then

$$D_\sigma^{p+1}(a_1a_2)=D_\sigma\left(\sum_{k=0}^{p}\left(\sum_{\{\gamma:\{1,\ldots,p\}\to\{\sigma,D_\sigma\}|\#\gamma^{-1}(\sigma)=k\}}\gamma(1)\ldots\gamma(p)(a_1)\cdot D_\sigma^k(a_2)\right)\right)=$$

$$=\sum_{k=0}^{p}\left(\sum_{\{\gamma:\{1,\ldots,p\}\to\{\sigma,D_\sigma\}|\#\gamma^{-1}(\sigma)=k\}}\Big(D_\sigma\gamma(1)\ldots\gamma(p)(a_1)\cdot D_\sigma^k(a_2)+\sigma\gamma(1)\ldots\gamma(p)(a_1)\cdot D_\sigma^{k+1}(a_2)\Big)\right)=$$

$$= \sum_{k=0}^{p} \left(\sum_{\{\gamma:\{1,\ldots,p\}\to\{\sigma,D_\sigma\} | \#\gamma^{-1}(\sigma)=k\}} \sigma\gamma(1)\ldots\gamma(p)(a_1)\cdot D_\sigma^{k+1}(a_2) \right) +$$

$$+ \sum_{k=0}^{p} \left(\sum_{\{\gamma:\{1,\ldots,p\}\to\{\sigma,D_\sigma\} | \#\gamma^{-1}(\sigma)=k\}} D_\sigma\gamma(1)\ldots\gamma(p)(a_1)\cdot D_\sigma^{k}(a_2) \right) =$$

$$= \sum_{k=1}^{p+1} \left(\sum_{\{\gamma:\{2,\ldots,p+1\}\to\{\sigma,D_\sigma\} | \#\gamma^{-1}(\sigma)=k-1\}} \sigma\gamma(1)\ldots\gamma(p)(a_1)\cdot D_\sigma^{k}(a_2) \right) +$$

$$+ \sum_{k=0}^{p} \left(\sum_{\{\gamma:\{2,\ldots,p+1\}\to\{\sigma,D_\sigma\} | \#\gamma^{-1}(\sigma)=k\}} D_\sigma\gamma(2)\ldots\gamma(p+1)(a_1)\cdot D_\sigma^{k}(a_2) \right) \overset{(*)}{=}$$

$$\overset{(*)}{=} \sum_{k=0}^{p+1} \left(\sum_{\{\gamma:\{1,\ldots,p+1\}\to\{\sigma,D_\sigma\} | \#\gamma^{-1}(\sigma)=k\}} \gamma(1)\ldots\gamma(p+1)(a_1)\cdot D_\sigma^{k}(a_2) \right)$$

where the last equality $(*)$ is obtained by replacing the set $\{2, ..., p+1\}$ of p elements by the set $\{1, ..., p\}$ in the index of both sums, and then changing the summation index k to $k+1$ in the first sum. Thus (28) holds for $n = p+1$ also. By the principle of mathematical induction, (28) holds for every integer $n \geq 1$. □

Suppose A is associative, and D_σ a σ-derivation on A. Then we want to find a formula for $D_\sigma^n(a_1, \ldots, a_m)$, generalizing (28). For $m = 3$ we can use theorem 4 twice to get the following:

$$D_\sigma^n(a_1a_2a_3) = \sum_{k_1=0}^{n} \left(\sum_{\{\gamma_1:\{1,\ldots,n\}\to\{\sigma,D_\sigma\} | \#\gamma_1^{-1}(\sigma)=k_1\}} \gamma_1(1)\ldots\gamma_1(n)(a_1)\cdot D_\sigma^{k_1}(a_2a_3) \right) =$$

$$= \sum_{k_1=0}^{n} \left(\sum_{\{\gamma_1:\{1,\ldots,n\}\to\{\sigma,D_\sigma\} | \#\gamma_1^{-1}(\sigma)=k_1\}} \gamma_1(1)\ldots\gamma_1(n)(a_1)\cdot \right.$$

$$\left. \cdot \sum_{k_2=0}^{k_1} \left(\sum_{\{\gamma_2:\{1,\ldots,k_1\}\to\{\sigma,D_\sigma\} | \#\gamma_2^{-1}(\sigma)=k_2\}} \gamma_2(1)\ldots\gamma_2(k_1)(a_2)D_\sigma^{k_2}(a_3) \right) \right) =$$

$$= \sum_{n\geq k_1\geq k_2\geq 0} \left(\sum_{\substack{\{\gamma_1:\{1,\ldots,n\}\to\{\sigma,D_\sigma\} | \#\gamma_1^{-1}(\sigma)=k_1\}\times \\ \times\{\gamma_2:\{1,\ldots,k_1\}\to\{\sigma,D_\sigma\} | \#\gamma_2^{-1}(\sigma)=k_2\}}} \gamma_1(1)\ldots\gamma_1(n)(a_1)\gamma_2(1)\ldots\gamma_2(k_1)(a_2)D_\sigma^{k_2}(a_3) \right)$$

Noting the pattern we state and prove the general formula.

Theorem 5 (Generalized Leibniz formula for σ-derivations)**.** *Suppose A is associative, and let D_σ be a σ-derivation in A. Then for all integers $n \geq 1$ and $m \geq 1$, and for all m-tuples $(a_1, \dots, a_m) \in A^m$ we have*

$$D_\sigma^n(a_1 a_2 \dots a_m) = \\ = \sum_{\substack{n = k_0 \geq k_1 \geq \cdots \\ \cdots \geq k_j \geq k_{j+1} \geq \cdots \\ \cdots \geq k_{m-1} \geq k_m = 0}} \left(\sum_{\substack{\{\gamma_1 : \{1,\dots,n\} \to \{\sigma, D_\sigma\} | \#\gamma_1^{-1}(\sigma) = k_1\} \times \dots \\ \dots \times \{\gamma_m : \{1,\dots,k_{m-1}\} \to \{\sigma, D_\sigma\} | \#\gamma_m^{-1}(\sigma) = k_m\}}} \gamma_1(1)\dots\gamma_1(n)(a_1)\dots\gamma_j(1)\dots\gamma_j(k_{j-1})(a_j)\dots \right. \\ \left. \dots \gamma_{m-1}(1)\dots\gamma_{m-1}(k_{m-2})(a_{m-1}) \cdot \gamma_m(1)\dots\gamma_m(k_{m-1})(a_m) \right) \quad (29)$$

Remark 4. The last factor $\gamma_m(1)\dots\gamma_m(k_{m-1})(a_m)$ in the sum is equal to $D_\sigma^{k_{m-1}}(a_m)$, since $k_m = 0$.

Proof. We prove (29) by induction on m. For $m = 1$ the right hand side of (29) becomes

$$\sum_{n=k_0 \geq k_1 = 0} \left(\sum_{\{\gamma_1 : \{1,\dots,n\} \to \{\sigma, D_\sigma\} | \#\gamma_1^{-1}(\sigma) = k_1\}} \gamma_1(1)\dots\gamma_1(n)(a_1) \right) = D_\sigma \dots D_\sigma(a_1) = D_\sigma^n(a_1)$$

Thus (29) holds for all integers $n \geq 1$ when $m = 1$. It also holds for all $n \geq 1$ and $m = 2$ since it becomes (28). Now suppose it holds for $m = p$, where $p \geq 1$ is some integer. Then

$$D_\sigma^n(a_1 a_2 \dots a_{m+1}) = \\ = \sum_{\substack{n = k_0 \geq k_1 \geq \cdots \\ \cdots \geq k_j \geq k_{j+1} \geq \cdots \\ \cdots \geq k_{m-1} \geq k_m = 0}} \left(\sum_{\substack{\{\gamma_1 : \{1,\dots,n\} \to \{\sigma, D_\sigma\} | \#\gamma_1^{-1}(\sigma) = k_1\} \times \dots \\ \dots \times \{\gamma_m : \{1,\dots,k_{m-1}\} \to \{\sigma, D_\sigma\} | \#\gamma_m^{-1}(\sigma) = k_m\}}} \gamma_1(1)\dots\gamma_1(n)(a_1)\dots\gamma_j(1)\dots\gamma_j(k_{j-1})(a_j)\dots \right. \\ \left. \dots \gamma_{m-1}(1)\dots\gamma_{m-1}(k_{m-2})(a_{m-1}) \cdot \gamma_m(1)\dots\gamma_m(k_{m-1})(a_m a_{m+1}) \right) \quad (30)$$

We use (28) on the last factor

$$\gamma_m(1)\dots\gamma_m(k_{m-1})(a_m a_{m+1}) = D_\sigma^{k_{m-1}}(a_m a_{m+1})$$

of the sum (30). Thus (30) equals

$$\sum_{\substack{n=k_0\geq k_1\geq\cdots\\ \cdots\geq k_j\geq k_{j+1}\geq\cdots\\ \cdots\geq k_{m-1}\geq k_m=0}} \Bigg(\sum_{\substack{\{\gamma_1:\{1,\dots,n\}\to\{\sigma,D_\sigma\}|\#\gamma_1^{-1}(\sigma)=k_1\}\times\dots\\ \dots\times\{\gamma_{m-1}:\{1,\dots,k_{m-2}\}\to\{\sigma,D_\sigma\}|\#\gamma_{m-1}^{-1}(\sigma)=k_{m-1}\}}} \gamma_1(1)\dots\gamma_1(n)(a_1)\dots\gamma_j(1)\dots\gamma_j(k_{j-1})(a_j)\dots$$

$$\dots\gamma_{m-1}(1)\dots\gamma_{m-1}(k_{m-2})(a_{m-1})\cdot$$

$$\cdot\sum_{k_m=0}^{k_{m-1}}\Bigg(\sum_{\{\gamma_m:\{1,\dots,k_{m-1}\}\to\{\sigma,D_\sigma\}|\#\gamma_m^{-1}(\sigma)=k_m\}}\gamma_m(1)\dots\gamma_m(k_{m-1})(a_m)D_\sigma^{k_m}(a_{m+1})\Bigg)\Bigg)=$$

$$=\sum_{\substack{n=k_0\geq k_1\geq\cdots\\ \cdots\geq k_j\geq k_{j+1}\geq\cdots\\ \cdots\geq k_m\geq k_{m+1}=0}} \Bigg(\sum_{\substack{\{\gamma_1:\{1,\dots,n\}\to\{\sigma,D_\sigma\}|\#\gamma_1^{-1}(\sigma)=k_1\}\times\dots\\ \dots\times\{\gamma_m:\{1,\dots,k_{m-1}\}\to\{\sigma,D_\sigma\}|\#\gamma_m^{-1}(\sigma)=k_m\}}} \gamma_1(1)\dots\gamma_1(n)(a_1)\dots\gamma_j(1)\dots\gamma_j(k_{j-1})(a_j)\dots$$

$$\dots\gamma_m(1)\dots\gamma_m(k_{m-1})(a_m)\cdot D_\sigma^{k_m}(a_{m+1})\Bigg)=$$

$$=\sum_{\substack{n=k_0\geq k_1\geq\cdots\\ \cdots\geq k_j\geq k_{j+1}\geq\cdots\\ \cdots\geq k_m\geq k_{m+1}=0}} \Bigg(\sum_{\substack{\{\gamma_1:\{1,\dots,n\}\to\{\sigma,D_\sigma\}|\#\gamma_1^{-1}(\sigma)=k_1\}\times\dots\\ \dots\times\{\gamma_{m+1}:\{1,\dots,k_m\}\to\{\sigma,D_\sigma\}|\#\gamma_{m+1}^{-1}(\sigma)=k_{m+1}\}}} \gamma_1(1)\dots\gamma_1(n)(a_1)\dots\gamma_j(1)\dots\gamma_j(k_{j-1})(a_j)\dots$$

$$\dots\gamma_m(1)\dots\gamma_m(k_{m-1})(a_m)\cdot\gamma_{m+1}(1)\dots\gamma_{m+1}(k_m)(a_{m+1})\Bigg)$$

Hence by the principle of mathematical induction the formula (29) holds for all integers $n\geq 1$ and $m\geq 1$. □

The following corollary is obtained by taking $n=1$ in the formula (29).

Corollary 6. *If A is associative, and D_σ a σ-derivation. Then*

$$\begin{aligned}D_\sigma(a_1a_2\dots a_m)=D_\sigma(a_1)a_2\dots a_m&+\\ &+\sum_{j=2}^{m-1}\sigma(a_1)\dots\sigma(a_{j-1})D_\sigma(a_j)a_{j+1}\dots a_m\\ &+\sigma(a_1)\sigma(a_2)\dots\sigma(a_{m-1})D_\sigma(a_m)\end{aligned}\tag{31}$$

for all $a_1,\dots,a_m\in A$.

Remark 5. Corollary 6 has an important consequence; if A is generated by some set X, then any σ-derivation D_σ in A is uniquely determined by the elements $D_\sigma(x)$ and $\sigma(x)$ for $x\in X$. Note that it is not required that σ is a homomorphism for this to be true.

4.3 Formulas for (σ,τ)-derivations

In this final subsection we find generalizations of the previously proved formulas to (σ,τ)-derivations. In the following propositions it will be assumed that A is an algebra, and that σ and τ are linear mappings $A \to A$. We also find it convenient to introduce a "flipping" function

$$\theta : \{D,\sigma\} \to \{D,\tau\}$$

defined by

$$\theta(D) = \tau \qquad \theta(\sigma) = D.$$

Proposition 7 (Leibniz formula for (σ,τ)-derivations)**.** *Let D be a (σ,τ)-derivation on A and $a,b \in A$. Then for integers $n \geq 1$,*

$$D^n(ab) = \sum_{\{\gamma:\{1,\ldots,n\}\to\{D,\sigma\}\}} \Big(\gamma(1)\ldots\gamma(n)\Big)(a)\cdot\Big((\theta\gamma)(1)\ldots(\theta\gamma)(n)\Big)(b), \tag{32}$$

where $(\theta\gamma)(k) = \theta\big(\gamma(k)\big)$.

Proof. We use induction on n. For $n = 1$, we have

$$\sum_{\{\gamma:\{1\}\to\{D,\sigma\}\}} \gamma(1)(a)\cdot(\theta\gamma)(1)(b) = D(a)\tau(b) + \sigma(a)D(b) = D^1(ab)$$

Assume the formula (32) holds for $n = k$, where $k \geq 1$. Then

$$\begin{aligned}
&D^{k+1}(ab) = D(D^k(ab)) = \\
&= D\left(\sum_{\{\gamma:\{1,\ldots,k\}\to\{D,\sigma\}\}} \Big(\gamma(1)\ldots\gamma(k)\Big)(a)\cdot\Big((\theta\gamma)(1)\ldots(\theta\gamma)(k)\Big)(b)\right) = \\
&= \sum_{\{\gamma:\{1,\ldots,k\}\to\{D,\sigma\}\}} \Bigg[\Big(\sigma\cdot\gamma(1)\ldots\gamma(k)\Big)(a)\cdot\Big(D\cdot(\theta\gamma)(1)\ldots(\theta\gamma)(k)\Big)(b) + \\
&\quad + \Big(D\cdot\gamma(1)\ldots\gamma(k)\Big)(a)\cdot\Big(\tau\cdot(\theta\gamma)(1)\ldots(\theta\gamma)(k)\Big)(b)\Bigg] = \\
&= \sum_{\{\gamma:\{0,\ldots,k\}\to\{D,\sigma\}\}} \Big(\gamma(0)\ldots\gamma(k)\Big)(a)\cdot\Big((\theta\gamma)(0)\ldots(\theta\gamma)(k)\Big)(b) = \\
&= \sum_{\{\gamma:\{1,\ldots,k+1\}\to\{D,\sigma\}\}} \Big(\gamma(1)\ldots\gamma(k+1)\Big)(a)\cdot\Big((\theta\gamma)(1)\ldots(\theta\gamma)(k+1)\Big)(b)
\end{aligned}$$

This shows that (32) holds for $n = k+1$. Thus by the principle of mathematical induction, the formula (32) holds for every integer $n \geq 1$, which was to be proved. $\square$

Lemma 8. *Let D be a (σ,τ)-derivation on A. Then for integers $n \geq 1$ and elements $a, b \in A$,*

$$D^n(ab) = \sum_{i=0}^{n} \sum_{G_i} \big(\gamma(1)...\gamma(n)\big)(a) \cdot \big((\theta\gamma)(1)...(\theta\gamma)(n)\big)(b) \tag{33}$$

where

$$G_i = \{\gamma : \{1,...,n\} \to \{D,\sigma\} \Big| \gamma(j) = \sigma \text{ if } j > n-i, \text{ and } \gamma(n-i) = D \text{ if } i \neq n\}.$$

Proof. It is enough to show that the sets G_i, $i = 0,...,n$ form a partition of the set $G = \{\gamma : \{1,2,...,n\} \to \{D,\sigma\}\}$ of all functions $\{1,2,...,n\} \to \{D,\sigma\}$ because then the result will follow from Proposition 7. Since each $G_i \subseteq G$, clearly $\cup_{i=0}^{n} G_i \subseteq G$. Also, we claim that $G_p \cap G_q = \emptyset$ when $p \neq q$. Otherwise, let $\gamma \in G_p \cap G_q$ for $p \neq q$ and choose notation so that $q > p$. Then $p \neq n$ so $\gamma(n-p) = D$ since $\gamma \in G_p$, but $n - p > n - q$ so that $\gamma(n-p) = \sigma$, since $\gamma \in G_q$. This contradiction shows that $G_p \cap G_q = \emptyset$. It remains to show that $G \subseteq \cup_{i=0}^{n} G_i$. Let $\gamma \in G$. If $\gamma(j) \neq D$ for $j = 1,...,n$ then $\gamma \in G_n \subseteq \cup_{i=0}^{n} G_i$. Otherwise, there exists a smallest integer p such that $\gamma(n-p) = D$. Then $0 \leq p \leq n-1$ and we claim that $\gamma \in G_p$. This is true, since $\gamma(n-p) = D$ and if there were a $j > n-p$ such that $\gamma(j) = D$, the integer $n - j$ would satisfy $\gamma(n-(n-j)) = D$ and $n - j < p$ which would contradict the minimality of p. Hence we have shown that G is the disjoint union of G_i, $i = 0,...,n$. □

We introduce some notation which will be used in the general formula below. Fix two integers $n \geq 1$ and $m \geq 2$, and let there be given a nonincreasing finite sequence of nonnegative integers

$$n = k_0 \geq k_1 \geq ... \geq k_{m-1} \geq k_m = 0 \tag{34}$$

and associate to this sequence the following collection of sets:

$$\Gamma_i = \{\gamma_{i+1} : \{1,...,k_i\} \to \{D,\sigma\} \Big| \gamma_{i+1}(j) = \sigma \text{ if } j > k_i - k_{i+1}, \\ \text{and } \gamma_{i+1}(k_i - k_{i+1}) = D \text{ if } k_{i+1} \neq k_i\}$$

for $0 \leq i \leq m-1$.

Remark 6. For each i, the set Γ_i depends only on k_i and k_{i+1} from the sequence.

Remark 7. The sets Γ_i are connected to the sets G_i from Lemma 8 in the following sence. If we take $m = 2$, then a sequence (34) reduces to $n = k_0 \geq k_1 \geq k_2 = 0$, in other words a single integer k_1 between 0 and n. Then $\Gamma_0 = G_{k_1}$.

Proposition 9 (Generalized Leibniz formula for (σ,τ)-derivations). *Suppose A is associative, and let D be a (σ,τ)-derivation on A. Then for integers $n \geq 1$, $m \geq 2$, and elements $a_1, a_2, ..., a_m \in A$ the following formula holds:*

$$
\begin{aligned}
D^n(a_1 a_2 ... a_m) = \sum_{\substack{n=k_0\geq k_1 \geq ...\\ \geq k_{m-1}\geq k_m = 0}} \sum_{\Gamma_0\times\Gamma_1\times...\times\Gamma_{m-2}} & \\
\big(\gamma_1(1)...\gamma_1(k_0)\big)(a_1)\left(\prod_{j_1=1}^{k_0-k_1}(\theta\gamma_1)(j_1)\right)\Bigg(_1 & \\
\big(\gamma_2(1)...\gamma_2(k_1)\big)(a_2)\left(\prod_{j_2=1}^{k_1-k_2}(\theta\gamma_2)(j_2)\right)\Bigg(_2 \dots & \\
\dots\big(\gamma_{m-1}(1)...\gamma_{m-1}(k_{m-2})\big)(a_{m-1})\left(\prod_{j_{m-1}=1}^{k_{m-2}-k_{m-1}}(\theta\gamma_{m-1})(j_{m-1})\right)\Bigg(_{m-1} & \\
D^{k_{m-1}}(a_m)\Bigg)_{m-1}\Bigg)_{m-2}\dots\Bigg)_1 &
\end{aligned}
$$

where the parenthesis are indexed so that $(_i$ and $)_i$ is a matching pair. Put more concisely,

$$
D^n(a_1 a_2 ... a_m) = \sum_{\substack{n=k_0\geq k_1 \geq ...\\ \geq k_{m-1}\geq k_m = 0}} \sum_{\Gamma_0\times\Gamma_1\times...\times\Gamma_{m-2}} L_1 D_1 L_2 D_2 ... L_{m-1} D_{m-1} D^{k_{m-1}}(a_m) \tag{35}
$$

where $L_j \in \mathcal{L}(A)$ is the operator of left multiplication by $\big(\gamma_j(1)...\gamma_j(k_{j-1})\big)(a_j)$ and $D_j = \prod_{i=1}^{k_{j-1}-k_j}(\theta\gamma_j)(i) \in \mathcal{L}(A)$.

Remark 8. Note that since τ only is an arbitrary linear operator, we cannot simplify expressions of the form $\tau(ab)$. Thus the formula is the best we can do, in the sence that each product of operators

$$
D_j = \prod_{i=1}^{k_{j-1}-k_j}(\theta\gamma_j)(i)
$$

is either the identity operator (when $k_{j-1} = k_j$) or ends with τ. This can be seen by considering the definition of the set Γ_{j-1}, which contains the element γ_j. Namely, γ_j must satisfy

$$
\gamma_j(k_{j-1} - k_j) = D,
$$

when $k_{j-1} \neq k_j$, which become τ after application of θ.

Proof. We use induction over m. For m=2, the formula (35) becomes

$$D^n(a_1a_2) = \sum_{n=k_0\geq k_1\geq k_2=0} \sum_{\Gamma_0} L_1 D_1 D^{k_1}(a_2)$$

where $L_1 \in \mathcal{L}(A)$ is the operator of left multiplication by $\big(\gamma_1(1)...\gamma_1(k_0)\big)(a_1)$ and $D_1 = \prod_{i=1}^{k_0-k_1}(\theta\gamma_1)(i)$. Substituting this we get

$$D^n(a_1a_2) = \sum_{k_1=0}^{n} \sum_{\Gamma_0} \big(\gamma_1(1)...\gamma_1(n)\big)(a_1)\Bigg(\prod_{i=1}^{n-k_1}(\theta\gamma_1)(i)\Bigg)D^{k_1}(a_2) \tag{36}$$

Now the functions γ_1 satisfy $\gamma_1(j) = \sigma$ if $j > k_0 - k_1 = n - k_1$, and therefore $\theta\gamma_1(j) = D_\sigma$ if $j > n - k_1$. Thus we can rewrite (36) to

$$D^n(a_1a_2) = \sum_{k_1=0}^{n} \sum_{\Gamma_0} \big(\gamma_1(1)...\gamma_1(n)\big)(a_1)\big((\theta\gamma_1)(1)...(\theta\gamma_1)(n)\big)(a_2)$$

Recalling the equality of sets Γ_0 and G_{k_1}, and using Lemma 8, we conclude that the formula (35) is valid for $m = 2$ and every integer $n \geq 1$. Suppose now that (35) holds for every integer $n \geq 1$, and $m \leq p$, where $p \geq 2$. Let $a_1, a_2, ..., a_{p+1} \in A$. Then, using (35) for $m = p$ we have

$$\begin{aligned} D^n(a_1a_2...a_pa_{p+1}) &= D^n(a_1a_2...a_{p-1}(a_pa_{p+1})) = \\ &= \sum_{\substack{n=k_0\geq k_1\geq ... \\ \geq k_{p-1}\geq k_p=0}} \sum_{\Gamma_0\times\Gamma_1\times...\times\Gamma_{p-2}} L_1D_1L_2D_2...L_{p-1}D_{p-1}D^{k_{p-1}}(a_pa_{p+1}) \end{aligned} \tag{37}$$

Now, using (35) for $m = 2$ we know that

$$D^{k_{p-1}}(a_pa_{p+1}) = \sum_{k_p=0}^{k_{p-1}} \sum_{\Gamma_{p-1}} L_pD_pD^{k_p}(a_{p+1}) \tag{38}$$

If we substitute (38) into (37) we get

$$D^n(a_1a_2...a_pa_{p+1}) = \sum_{\substack{n=k_0\geq k_1\geq ... \\ \geq k_p\geq k_{p+1}=0}} \sum_{\Gamma_0\times\Gamma_1\times...\times\Gamma_{p-1}} L_1D_1L_2D_2...L_pD_pD^{k_p}(a_{p+1})$$

This proves the induction step and thus by the principle of mathematical induction, (35) holds for every integers $n \geq 1$ and $m \geq 2$ which was to be proved. □

The next proposition could be obtained as a corollary of Proposition 9 but we provide an alternative proof.

Proposition 10. *Suppose A is associative, and let D be a (σ, τ)-derivation on A. Then for integers $m \geq 2$,*

$$\begin{aligned} D(a_1 a_2 ... a_m) = D(a_1)\tau(a_2 a_3 ... a_m) & \\ + \sum_{j=2}^{m-1} \sigma(a_1)\sigma(a_2)...\sigma(a_{j-1})D(a_j)\tau(a_{j+1}a_{j+2}...a_m) & \\ + \sigma(a_1)\sigma(a_2)...\sigma(a_{m-1})D(a_m) & \quad (39) \end{aligned}$$

and also

$$\begin{aligned} D(a_1 a_2 ... a_m) = D(a_1)\tau(a_2)\tau(a_3)...\tau(a_m) & \\ + \sum_{j=2}^{m-1} \sigma(a_1 a_2 ... a_{j-1})D(a_j)\tau(a_{j+1})\tau(a_{j+2})...\tau(a_m) & \\ + \sigma(a_1 a_2 ... a_{m-1})D(a_m) & \quad (40) \end{aligned}$$

where $a_i \in A$.

Proof. We use induction on m. For $m = 2$ the identities hold, since they are reduced to the formula (3) which is true by definition of a (σ, τ)-derivation. Assume that (39) holds for $m = k$, where $k \geq 2$. Then

$$\begin{aligned} D(a_1 a_2 ... a_{k+1}) &= D(a_1)\tau(a_2 ... a_{k+1}) + \sigma(a_1)D(a_2 ... a_{k+1}) = \\ &= D(a_1)\tau(a_2 a_3 ... a_{k+1}) + \sigma(a_1)\Big(D(a_2)\tau(a_3 a_4 ... a_{k+1}) \\ &\quad + \sum_{j=3}^{k} \sigma(a_2)\sigma(a_3)...\sigma(a_{j-1})D(a_j)\tau(a_{j+1}a_{j+2}...a_{k+1}) \\ &\quad + \sigma(a_2)\sigma(a_3)...\sigma(a_k)D(a_{k+1})\Big) = \\ &= D(a_1)\tau(a_2 a_3 ... a_{k+1}) \\ &\quad + \sum_{j=2}^{k} \sigma(a_1)\sigma(a_2)\sigma(a_3)...\sigma(a_{j-1})D(a_j)\tau(a_{j+1}a_{j+2}...a_{k+1}) \\ &\quad + \sigma(a_1)\sigma(a_2)\sigma(a_3)...\sigma(a_k)D(a_{k+1}) \end{aligned}$$

This shows that (39) holds for $m = k + 1$. By induction it holds for all integers $m \geq 2$.

Now assume that (40) holds for $m = k$, where $k \geq 2$. Then

$$\begin{aligned}
D(a_1a_2...a_{k+1}) &= D(a_1a_2...a_k)\tau(a_{k+1}) + \sigma(a_1a_2...a_k)D(a_{k+1}) = \\
&= \Big(D(a_1)\tau(a_2)\tau(a_3)...\tau(a_k) \\
&\quad + \sum_{j=2}^{k-1} \sigma(a_1a_2...a_{j-1})D(a_j)\tau(a_{j+1})\tau(a_{j+2})...\tau(a_k) \\
&\quad + \sigma(a_1a_2...a_{k-1})D(a_k)\Big)\tau(a_{k+1}) + \sigma(a_1a_2...a_k)D(a_{k+1}) = \\
&= D(a_1)\tau(a_2)\tau(a_3)...\tau(a_k)\tau(a_{k+1}) \\
&\quad + \sum_{j=2}^{k} \sigma(a_1a_2...a_{j-1})D(a_j)\tau(a_{j+1})\tau(a_{j+2})...\tau(a_k)\tau(a_{k+1}) \\
&\quad + \sigma(a_1a_2...a_k)D(a_{k+1})
\end{aligned}$$

This shows that (40) holds for $m = k+1$. By induction it holds for all integers $m \geq 2$. The proof is finished. □

5 The general structure of $\mathfrak{D}_{(\sigma,\tau)}(A)$

5.1 Associativity conditions

We describe a necessary condition for (σ, τ)-derivations on associative algebras.

Proposition 11. *If A is an associative algebra and D is a (σ, τ)-derivation in A, then*

$$\big(\sigma(ab) - \sigma(a)\sigma(b)\big)D(c) = D(a)\big(\tau(bc) - \tau(b)\tau(c)\big), \tag{41}$$

or equivalently

$$\sigma(ab)D(c) + D(a)\tau(b)\tau(c) = \sigma(a)\sigma(b)D(c) + D(a)\tau(bc). \tag{42}$$

Proof. Since A is associative, we have

$$\begin{aligned}
0 &= D(a(bc) - (ab)c) = D(a(bc)) - D((ab)c) = \\
&= D(a)\tau(bc) + \sigma(a)D(bc) - \big(D(ab)\tau(c) + \sigma(ab)D(c)\big) = \\
&= D(a)\tau(bc) + \sigma(a)\big(D(b)\tau(c) + \sigma(b)D(c)\big) \\
&\quad - \big(D(a)\tau(b) + \sigma(a)D(b)\big)\tau(c) - \sigma(ab)D(c) = \\
&= D(a)\big(\tau(bc) - \tau(b)\tau(c)\big) + \big(\sigma(a)\sigma(b) - \sigma(ab)\big)D(c)
\end{aligned}$$

Thus equation (41) holds. Equation (42) is obtained from (41) by expanding the parenthesis and adding the expression $\sigma(a)\sigma(b)D(c) + D(a)\tau(b)\tau(c)$ to both sides. □

Corollary 12. *If A is an associative algebra with no zero divizors, and D is a nonzero (σ, τ)-derivation on A, then σ is an algebra endomorphism if and only if τ is an algebra endomorphism.*

Corollary 13. *If A is an associative algebra and $D \in \mathfrak{D}_\sigma(A)$, then*

$$\big(\sigma(ab) - \sigma(a)\sigma(b)\big)D(c) = 0 \tag{43}$$

for all $a, b, c \in A$.

Proof. Take $\tau = \mathrm{id}_A$ in (41). □

The following corollary is important, and follows directly from Corollary 13.

Corollary 14. *Let A be an associative algebra with no zero divizors, and let σ be a linear operator on A. If σ is not an algebra endomorphism on A, then $\mathfrak{D}_\sigma(A)$=\{ 0 \}, that is, the only possible σ-derivation on A is the zero operator.*

Corollary 15. *If A is an associative algebra, $D \in \mathfrak{D}_\sigma(A)$ and the set*

$$D(A) = \{D(c) | c \in A\}$$

contains at least one element which is nonzero and not a zero-divizor in A, then

$$\sigma(ab) = \sigma(a)\sigma(b)$$

for all $a, b \in A$, i.e. σ is a homomorphism.

5.2 Necessary conditions for σ to be a σ-derivation

Proposition 16. *If A is an associative algebra and $\sigma \in \mathfrak{D}_\sigma(A)$, then*

$$\sigma(a_1)a_2\sigma(a_3) = 0 \tag{44}$$

for all $a_1, a_2, a_3 \in A$, and in particular

$$\sigma(a)^3 = 0 \tag{45}$$

for all $a \in A$.

Proof. Using that σ is a σ-derivation we get

$$\sigma(a_1a_2) = \sigma(a_1)a_2 + \sigma(a_1)\sigma(a_2), \tag{46}$$

which implies

$$\sigma(a_1a_2) - \sigma(a_1)\sigma(a_2) = \sigma(a_1)a_2. \tag{47}$$

By Corollary 13,

$$(\sigma(a_1a_2) - \sigma(a_1)\sigma(a_2))\sigma(a_3) = 0. \tag{48}$$

Substituting (47) in (48) we deduce

$$\sigma(a_1)a_2\sigma(a_3) = 0, \tag{49}$$

which was to be proved. The equation (45) is obtained by taking $a_1 = a$, $a_2 = \sigma(a)$, and $a_3 = a$. □

Corollary 17. *If A is an associative algebra with unit e and $\sigma \in \mathfrak{D}_\sigma(A)$, then*

$$\sigma(a_1)\sigma(a_2) = 0 \tag{50}$$

for all $a_1, a_2 \in A$, and in particular

$$\sigma(a)^2 = 0 \tag{51}$$

$$\sigma(a)\sigma(e) = 0 \tag{52}$$

for all $a \in A$.

Proof. Set $a_2 = e$ in (44) to get (50). The equality (51) is obtained from (50) when $a_1 = a_2 = a$, and (52) is obtained when $a_1 = a$ and $a_2 = e$. □

5.3 $\mathfrak{D}_{(\sigma,\tau)}(A)$ as a bimodule over subalgebras of A.

In this section, A is assumed to be an associative algebra. (The construction also works for Lie algebras, but we shall not consider this case here.) By definition that $\mathfrak{D}_{(\sigma,\tau)}(A)$ is a subset of $\mathcal{L}(A)$. Now, if D and E are two (σ, τ)-derivations on A then

$$(\lambda D + \mu E)(a_1a_2) = \lambda D(a_1a_2) + \mu E(a_1a_2) = \tag{53}$$
$$= \lambda\big(D(a_1)\tau(a_2) + \sigma(a_1)D(a_2)\big)+ \tag{54}$$
$$+ \mu\big(E(a_1)\tau(a_2) + \sigma(a_1)E(a_2)\big) = \tag{55}$$
$$= (\lambda D + \mu E)(a_1)\tau(a_2) + \sigma(a_1)(\lambda D + \mu E)(a_2) \tag{56}$$

for all $\lambda, \mu \in \mathbb{C}$ and $a_1, a_2 \in A$. So for every pair of linear operators $\sigma, \tau \in \mathcal{L}(A)$, $\mathfrak{D}_{(\sigma,\tau)}(A)$ is a linear subspace of $\mathcal{L}(A)$.

Remark 9. In general, these subspaces can have nonzero intersection. In other words there can exist linear operators $\sigma_1, \sigma_2, \tau_1, \tau_2$ on A such that $\sigma_1 \neq \sigma_2$ or $\tau_1 \neq \tau_2$ and still $\mathfrak{D}_{\sigma_1,\tau_1}(A) \cap \mathfrak{D}_{\sigma_2,\tau_2}(A) \neq 0$. Let for instance σ be a nonzero homomorphism on some algebra A, as in Example 1, page 7. Then $\sigma \in \mathfrak{D}_{\frac{1}{2}\sigma,\frac{1}{2}\sigma}(A) \cap \mathfrak{D}_{\frac{1}{3}\sigma,\frac{2}{3}\sigma}(A)$, but $\frac{1}{2}\sigma \neq \frac{1}{3}\sigma$.

Consider the algebra $\mathcal{L}(A)$ of linear transformations of the associative algebra A. This algebra $\mathcal{L}(A)$ is in particular a complex linear space and it is possible to define on this linear space the structure of a (A,A)-bimodule in the following natural way. Define maps

$$\lambda : A \times \mathcal{L}(A) \to \mathcal{L}(A) \tag{57}$$

$$\rho : \mathcal{L}(A) \times A \to \mathcal{L}(A) \tag{58}$$

by

$$\lambda(x,F) = \mathtt{L}_x \cdot F \tag{59}$$

and

$$\rho(F,y) = \mathtt{R}_y \cdot F, \tag{60}$$

where $\mathtt{R}_a$ and $\mathtt{L}_a$ denote the operators of multiplication by an element $a \in A$ on the right and left, respectively. The functions λ and ρ are bilinear and

$$\lambda(x_1x_2, D) = \mathtt{L}_{x_1x_2}D = \mathtt{L}_{x_1}\mathtt{L}_{x_2}D = \lambda(x_1, \lambda(x_2, D)) \tag{61}$$

$$\rho(D, y_1y_2) = \mathtt{R}_{y_1y_2}D = \mathtt{R}_{y_2}\mathtt{R}_{y_1}D = \rho(\rho(D,y_1),y_2) \tag{62}$$

$$\begin{aligned}\lambda(x_1, \rho(D,y_1)) = \lambda(x_1, \mathtt{R}_{y_1}D) = \mathtt{L}_{x_1}\mathtt{R}_{y_1}D = \\ = \mathtt{R}_{y_1}\mathtt{L}_{x_1}D = \rho(\mathtt{L}_{x_1}D, y_1) = \rho(\lambda(x_1,D),y_1)\end{aligned} \tag{63}$$

In equation (63) we used that A is associative, that is, left and right multiplication operators commute. The identities (61)-(63) together with bilinearity show that λ and ρ are (A,A)-bimodule multiplication maps on $\mathcal{L}(A)$.

Given linear operators σ and τ on A, form the following two subsets of A:

$$X_{(\sigma,\tau)} = \{c \in A \mid \lambda(c,D) \in \mathfrak{D}_{(\sigma,\tau)}(A) \quad \text{for all } D \in \mathfrak{D}_{(\sigma,\tau)}(A)\} \tag{64}$$

$$Y_{(\sigma,\tau)} = \{c \in A \mid \rho(D,c) \in \mathfrak{D}_{(\sigma,\tau)}(A) \quad \text{for all } D \in \mathfrak{D}_{(\sigma,\tau)}(A)\} \tag{65}$$

Lemma 18. *Let σ and τ be linear operators on A. Then*

$$X_{(\sigma,\tau)} = \{c \in A \mid [c,\sigma(a)]D(b) = 0 \quad \textit{for all } a,b \in A, D \in \mathfrak{D}_{(\sigma,\tau)}(A)\} \tag{66}$$

$$Y_{(\sigma,\tau)} = \{c \in A \mid D(b)[c,\tau(a)] = 0 \quad \textit{for all } a,b \in A, D \in \mathfrak{D}_{(\sigma,\tau)}(A)\} \tag{67}$$

Proof. Let $D \in \mathfrak{D}_{(\sigma,\tau)}(A)$. By Definition 3 of (σ,τ)-derivations, $\lambda(c,D) = \mathtt{L}_cD$ is a (σ,τ)-derivation if and only if

$$\begin{aligned}0 = \mathtt{L}_cD(ab) - \big((\mathtt{L}_cD(a))\tau(b) + \sigma(a)\mathtt{L}_cD(b)\big)\\ = cD(a)\tau(b) + c\sigma(a)D(b) - cD(a)\tau(b) - \sigma(a)cD(b)\\ = (c\sigma(a) - \sigma(a)c)D(b) = [c,\sigma(a)]D(b)\end{aligned} \tag{68}$$

for all $a, b \in A$. Since $D \in \mathfrak{D}_{(\sigma,\tau)}(A)$ was arbitrary, we have proved (66). Similarly, if $D \in \mathfrak{D}_{(\sigma,\tau)}(A)$, then $\rho(D, c) = \mathtt{R}_c D$ is a (σ, τ)-derivation if and only if

$$\begin{aligned} 0 = \mathtt{R}_c D(ab) - \big((\mathtt{R}_c D(a))\tau(b) + \sigma(a)\mathtt{R}_c D(b)\big) \\ = D(a)\tau(b)c + \sigma(a)D(b)c - D(a)c\tau(b) - \sigma(a)D(b)c \\ = D(a)(\tau(b)c - c\tau(b)) = D(a)[\tau(b), c] \end{aligned} \tag{69}$$

for all $a, b \in A$, which proves (67). □

Theorem 19. *$X_{(\sigma,\tau)}$ and $Y_{(\sigma,\tau)}$ are subalgebras of A.*

Proof. We will use the equalities (66) and (67) to prove the statement. First we show that $X_{(\sigma,\tau)}$ and $Y_{(\sigma,\tau)}$ are linear subspaces of A. Let $\lambda, \mu \in \mathbb{C}$, $a, b \in A$ and $D \in \mathfrak{D}_{(\sigma,\tau)}(A)$. Then for $c, d \in X_{(\sigma,\tau)}$,

$$[\lambda c + \mu d, \sigma(a)]D(b) = \lambda[c, \sigma(a)]D(b) + \mu[d, \sigma(a)]D(b) = 0$$

and for $c, d \in Y_{(\sigma,\tau)}$,

$$D(b)[\lambda c + \mu d, \tau(a)] = \lambda D(b)[c, \tau(a)] + \mu D(b)[d, \tau(a)] = 0$$

This shows that $X_{(\sigma,\tau)}$ and $Y_{(\sigma,\tau)}$ are linear subspaces of A. Next, we show that $X_{(\sigma,\tau)}$ and $Y_{(\sigma,\tau)}$ are closed under the product in A. Let $c, d \in X_{(\sigma,\tau)}$. Then for any $a, b \in A$ and $D \in \mathfrak{D}_{(\sigma,\tau)}(A)$, we have

$$\begin{aligned} [cd, \sigma(a)]D(b) =& \big(cd\sigma(a) - \sigma(a)cd\big)D(b) = \\ =& \big(cd\sigma(a) - c\sigma(a)d + c\sigma(a)d - \sigma(a)cd\big)D(b) = \\ =& c[d, \sigma(a)]D(b) + [c, \sigma(a)]dD(b) = \\ =& c \cdot 0 + [c, \sigma(a)] \cdot (\mathtt{L}_d D)(b) = 0. \end{aligned}$$

The last equality follows from that $c \in X_{(\sigma,\tau)}$ and that $\mathtt{L}_d D \in \mathfrak{D}_{(\sigma,\tau)}(A)$ since $d \in X_{(\sigma,\tau)}$ and $D \in \mathfrak{D}_{(\sigma,\tau)}(A)$. Hence we have shown that $X_{(\sigma,\tau)}$ is closed under multiplication in A. Similarly for $Y_{(\sigma,\tau)}$, when $c, d \in Y_{(\sigma,\tau)}$, we have for any $a, b \in A$ and $D \in \mathfrak{D}_{(\sigma,\tau)}(A)$,

$$\begin{aligned} D(b)[cd, \tau(a)] =& D(b)\big(cd\tau(a) - \tau(a)cd\big) = \\ =& D(b)\big(cd\tau(a) - c\tau(a)d + c\tau(a)d - \tau(a)cd\big) = \\ =& D(b)c[d, \tau(a)] + D(b)[c, \tau(a)]d = \\ =& (\mathtt{R}_c D)(b)[d, \tau(a)] + 0 \cdot d = 0. \end{aligned}$$

Therefore $Y_{(\sigma,\tau)}$ is closed under multiplication in A. Thus $X_{(\sigma,\tau)}$ and $Y_{(\sigma,\tau)}$ are subalgebras of A. □

Corollary 20. *$\mathfrak{D}_{(\sigma,\tau)}(A)$ is an $(X_{(\sigma,\tau)}, Y_{(\sigma,\tau)})$-bimodule under the restriction of the maps λ and ρ to $X_{(\sigma,\tau)} \times \mathfrak{D}_{(\sigma,\tau)}(A)$ and $\mathfrak{D}_{(\sigma,\tau)}(A) \times Y_{(\sigma,\tau)}$ respectively.*

For a subset $S \subseteq A$ of an algebra A, the *centralizer of S in A* is the subalgebra $\mathrm{Cen}_A(S) \subseteq A$ of A defined by

$$\mathrm{Cen}_A(S) = \{c \in A \bigm| cx = xc \quad \forall\, x \in S\}$$

Proposition 21. *Let A be an associative algebra, and let σ and τ be linear operators on A. Then*

$$\mathrm{Cen}_A(\sigma(A)) \subseteq X_{(\sigma,\tau)} \tag{70}$$

and

$$\mathrm{Cen}_A(\tau(A)) \subseteq Y_{(\sigma,\tau)} \tag{71}$$

Furthermore, if there is some $D \in \mathfrak{D}_{(\sigma,\tau)}(A)$ such that its image contains some right cancellable element, then equality holds in (70). Similarly, if there is some $D \in \mathfrak{D}_{(\sigma,\tau)}(A)$ such that its image contains some left cancellable element, then equality holds in (71). In particular, equality holds in both (70) and (71) if A has no zero-divizors and $\mathfrak{D}_{(\sigma,\tau)}(A) \neq 0$.

Proof. If $c \in \mathrm{Cen}_A(\sigma(A))$, then $[c, \sigma(a)] = 0$ for any $a \in A$. Thus, using (66), we have $c \in X_{(\sigma,\tau)}$ so (70) holds. The proof of (71) is similar. Suppose now that the image of some $D \in \mathfrak{D}_{(\sigma,\tau)}(A)$ contains a right cancellable element $D(b)$. Let $c \in X_{(\sigma,\tau)}$. Then, using (66), we have in particular that $[c, \sigma(a)]D(b) = 0$ for any $a \in A$, which imply $c \in \mathrm{Cen}_A(\sigma(A))$, since $D(b)$ was right canellable. Analogously, if $D(b)$ is left cancellable for some $D \in \mathfrak{D}_{(\sigma,\tau)}(A)$, $b \in A$, then any $c \in Y_{(\sigma,\tau)}$ satisfies $D(b)[c, \tau(a)] = 0$ for all $a \in A$, which shows $c \in \mathrm{Cen}_A(\tau(A))$. □

Remark 10. It would be interesting to find an example where at least one of the inclusions (70),(71) is strict.

6 Generalized derivations on unique factorization domains

In Section 6.1 we consider the algebra $\mathbb{C}[x]$ of complex polynomials in one variable and describe for any linear operator σ on $\mathbb{C}[x]$ all possible σ-derivations in $\mathbb{C}[x]$. In Section 6.2 we determine which of these σ-derivations that are homogenous as linear operators $\mathbb{C}[x] \to \mathbb{C}[x]$ with respect to the natural grading (76). In Section 6.3 we state and prove a formula for calculating with σ-derivations on $\mathbb{C}[x_1, \ldots, x_n]$. Finally, in Section 6.4 we study (σ, τ)-derivations on algebras which, viewed as rings, are unique factorization domains. In particular we obtain a classification of (σ, τ)-derivations on $\mathbb{C}[x_1, \ldots, x_n]$ when σ and τ are algebra endomorphisms.

6.1 σ-Derivations on $\mathbb{C}[x]$

The algebra $\mathbb{C}[x]$ has no zero-divizors, so it follows from Corollary 14 that if σ is a linear operator on $\mathbb{C}[x]$ such that there exists some nonzero σ-derivation D_σ, then σ must be a homomorphism. Therefore either $\sigma(1) = 0$ or $\sigma(1) = 1$. If $\sigma(1) = 0$ then $\sigma = 0$ and the σ-derivations are precisely the left multiplication operators. If $\sigma(1) = 1$ then σ is uniquely defined by the value on the generator x and we have the following result.

Proposition 22. *Let $\sigma \neq 0$ be a nonzero algebra endomorphism of $\mathbb{C}[x]$. Then a linear operator D_σ in $\mathbb{C}[x]$ is a σ-derivation if and only if the following two conditions are satisfied:*

i. $D_\sigma(1) = 0$, and

ii. For every integer $k \geq 1$, the following equation holds:

$$D_\sigma(x^k) = \left(x^{k-1} + x^{k-2}\sigma(x) + \ldots + x\sigma(x)^{k-2} + \sigma(x)^{k-1}\right) D_\sigma(x). \tag{72}$$

If $\sigma = 0$, then $D_\sigma(p) = D_\sigma(1)p$ for all $p \in \mathbb{C}[x]$, where $D_\sigma(1)$ can be any element of $\mathbb{C}[x]$.

Proof. Suppose first that D_σ is a σ-derivation. Then

$$D_\sigma(1) = D_\sigma(1 \cdot 1) = D_\sigma(1) \cdot 1 + \sigma(1) D_\sigma(1) = 2D_\sigma(1)$$

so that $D_\sigma(1) = 0$. For the second property, we use the formula in Corollary 6 with $m = k$ and $a_1 = a_2 = \ldots a_k = x$ and get

$$\begin{aligned} D_\sigma(x^k) &= D_\sigma(x) \cdot x^{k-1} + \sum_{j=2}^{k-1} \left(\sigma(x)^{j-1} \cdot D_\sigma(x) \cdot x^{k-j}\right) + \sigma(x)^{k-1} \cdot D_\sigma(x) = \\ &= \left(x^{k-1} + x^{k-2}\sigma(x) + \ldots + x\sigma(x)^{k-2} + \sigma(x)^{k-1}\right) D_\sigma(x). \end{aligned} \tag{73}$$

Conversely, suppose we are given a linear operator D_σ on $\mathbb{C}[x]$ satisfying $D_\sigma(1) = 0$ and formula (72). We must prove that D_σ is a σ-derivation. By linearity it suffices to check that $D_\sigma(x^k x^l) = D_\sigma(x^k)x^l + \sigma(x^k)D_\sigma(x^l)$ for all non-negative integers k and l. If at least one of k and l is zero, say k, then we have

$$D_\sigma(x^0 x^l) = D_\sigma(x^l) = 0 \cdot x^l + 1 \cdot D_\sigma(x^l) = D_\sigma(x^0)x^l + \sigma(x^0)D_\sigma(x^l),$$

which is the required equality for a σ-derivation. If both k and l are positive integers, consider the following calculation:

$$\begin{aligned}
D_\sigma(x^k \cdot x^l) = D_\sigma(x^{k+l}) = \\
&= \big(x^{k+l-1} + x^{k+l-2}\sigma(x) + \ldots + x^l\sigma(x)^{k-1} + \\
&\quad + x^{l-1}\sigma(x)^k + x^{l-2}\sigma(x)^{k+1} + \ldots + x\sigma(x)^{k+l-2} + \sigma(x)^{k+l-1}\big)D_\sigma(x) = \\
&= \big(x^{k-1} + x^{k-2}\sigma(x) + \ldots + x\sigma(x)^{k-2} + \sigma(x)^{k-1}\big)D_\sigma(x)x^l + \\
&\quad + \sigma(x)^k\big(x^{l-1} + x^{l-2}\sigma(x) + \ldots + x\sigma(x)^{l-2} + \sigma(x)^{l-1}\big)D_\sigma(x) = \\
&= D_\sigma(x^k)x^l + \sigma(x^k)D_\sigma(x^l)
\end{aligned}$$

We have shown that $D_\sigma(x^k x^l) = D_\sigma(x^k)x^l + \sigma(x^k)D_\sigma(x^l)$ for all non-negative integers k and l. This concludes the proof. □

Remark 11. If D_σ is a σ-derivation in $\mathbb{C}[x]$, then $D_\sigma(x)$ is a polynomial $\in \mathbb{C}[x]$. Conversely, by Proposition 22, given any polynomial $p(x) \in \mathbb{C}[x]$, there is a unique σ-derivation such that $D_\sigma(x) = p(x)$. Thus, for fixed σ, there is a bijective correspondance $\mathfrak{D}_\sigma(\mathbb{C}[x]) \to \mathbb{C}[x]$ defined by $D_\sigma \mapsto D_\sigma(x)$. Also, this correspondance takes linear combinations into linear combinations, so there is a isomorphism of linear spaces $\mathfrak{D}_\sigma(\mathbb{C}[x]) \simeq \mathbb{C}[x]$. This isomorphism induces a direct sum decomposition of $\mathfrak{D}_\sigma(\mathbb{C}[x])$ from the natural $\mathbb{Z}_{\geq 0}$-grading on $\mathbb{C}[x]$:

$$\mathfrak{D}_\sigma(\mathbb{C}[x]) = \bigoplus_{k \in \mathbb{Z}_{\geq 0}} \mathfrak{D}_{\sigma,k} \tag{74}$$

where $\mathfrak{D}_{\sigma,k} = \{D_\sigma \in \mathfrak{D}_\sigma(\mathbb{C}[x]) \mid \exists \lambda \in \mathbb{C} \text{ such that } D_\sigma(x) = \lambda x^k\}$. As we shall soon see, when σ is operator of multiplication by scalar, the homogenous elements in the direct sum (74) is precisely the homogenous σ-derivations, in the sence of Definition 7.

We can use Proposition 22 and linearity to derive a formula which describes the action of an arbitrary σ-derivation on an arbitrary polynomial in $\mathbb{C}[x]$.

Corollary 23. *Let $\sigma \neq 0$ be a nonzero algebra endomorphism of $\mathbb{C}[x]$, $p(x) = \sum_{i=0}^n c_i x^i \in \mathbb{C}[x]$ a polynomial, and D_σ a σ-derivation in $\mathbb{C}[x]$. Then*

$$D_\sigma(p(x)) = \sum_{j=0}^{n-1} d_j x^j D_\sigma(x)$$

where

$$d_j = \sum_{i=j}^{n-1} c_{i+1}\sigma(x)^{i-j}$$

Proof. Consider the following computations:

$$\begin{aligned}
D_\sigma(p(x)) &= D_\sigma(\sum_{i=0}^{n} c_i x^i) = \sum_{i=0}^{n} c_i D_\sigma(x^i) \\
&= \sum_{i=1}^{n} c_i \big(x^{i-1} + x^{i-2}\sigma(x) + \ldots + x\sigma(x)^{i-2} + \sigma(x)^{i-1}\big) D_\sigma(x) \\
&= \sum_{i=1}^{n} \sum_{j=0}^{i-1} \Big(c_i x^j \sigma(x)^{i-1-j}\Big) D_\sigma(x) \\
&= \sum_{j=0}^{n-1} \sum_{i=j+1}^{n} \Big(c_i x^j \sigma(x)^{i-1-j}\Big) D_\sigma(x) \\
&= \sum_{j=0}^{n-1} \Big(\sum_{i=j+1}^{n} c_i \sigma(x)^{i-1-j}\Big) x^j D_\sigma(x) \\
&= \sum_{j=0}^{n-1} \Big(\sum_{i=j}^{n-1} c_{i+1} \sigma(x)^{i-j}\Big) x^j D_\sigma(x) \qquad (75)
\end{aligned}$$

The proof is finished. □

6.2 Homogenous σ-derivations on $\mathbb{C}[x]$

The algebra $\mathbb{C}[x]$ has a natural $\mathbb{Z}_{\geq 0}$-grading:

$$\mathbb{C}[x] = \bigoplus_{k \in \mathbb{Z}_{\geq 0}} \mathbb{C}x^k \qquad (76)$$

We state and prove a proposition which describes all σ-derivations in $\mathbb{C}[x]$ which are homogenous as linear operators with respect to the grading (76), in the sence of Definition 7.

Proposition 24. *Let $\sigma \neq 0$ be a nonzero algebra endomorphism of $\mathbb{C}[x]$ and D_σ be a σ-derivation in $\mathbb{C}[x]$. Then D_σ is homogenous if and only if the following two conditions are satisfied:*

1. *$D_\sigma(x)$ is a homogenous element of $\mathbb{C}[x]$, that is, $D_\sigma(x) = \mu x^r$ for some complex number $\mu \in \mathbb{C}$ and some non-negative integer $r \in \mathbb{Z}_{\geq 0}$.*

2. *$\sigma(x) = \lambda x$ for some complex number λ.*

In this case,

$$D_\sigma(x^k) = \begin{cases} \frac{\lambda^k - 1}{\lambda - 1} x^{k-1} D_\sigma(x), & \text{if } \lambda \neq 1 \\ k x^{k-1} D_\sigma(x), & \text{if } \lambda = 1 \end{cases}$$

Proof. Suppose first that D_σ is a homogenous σ-derivation. Then the first condition is obviously satisfied. Using formula (72) we have for all integers $k \geq 2$,

$$D_\sigma(x^k) = \big(x^{k-1} + x^{k-2}\sigma(x) + \ldots + x\sigma(x)^{k-2} + \sigma(x)^{k-1}\big) D_\sigma(x). \tag{77}$$

By assumption $D_\sigma(x)$ and $D_\sigma(x^k)$ are homogenous elements of $\mathbb{C}[x]$ since x and x^k are homogenous. Furthermore, whenever $q(x)$ and $p(x)q(x)$ are homogenous polynomials, $p(x)$ must also be homogenous. Using this we deduce that the left factor $\big(x^{k-1} + x^{k-2}\sigma(x) + \ldots + x\sigma(x)^{k-2} + \sigma(x)^{k-1}\big)$ of the right hand side of (77) must also be homogenous for all integers $k \geq 2$. In other words we have

$$x^{k-1} + x^{k-2}\sigma(x) + \ldots + x\sigma(x)^{k-2} + \sigma(x)^{k-1} = c_k x^{p_k} \tag{78}$$

for all integers $k \geq 2$ and some $c_k \in \mathbb{C}$ and $p_k \in \mathbb{Z}_{\geq 0}$. Multiply both sides of (78) by $x - \sigma(x)$ to get

$$x^k - \sigma(x)^k = c_k x^{p_k+1} - c_k x^{p_k}\sigma(x). \tag{79}$$

If $\sigma(x) = 0$ then $\lambda = 0$ satisfies the required condition 2. If $\sigma(x) \neq 0$, let m denote the degree $\deg \sigma(x)$ of the polynomial $\sigma(x)$. Since D_σ is homogenous, it follows from (72) that $\sigma(x)$ cannot be a constant. Hence $m \geq 1$ so that $\deg(\sigma(x)^k) \geq mk \geq k$ and $\deg(-c_k x^{p_k}\sigma(x)) = p_k + m \geq p_k + 1$. Equating highest degree on both sides of equation (79) gives us

$$mk = p_k + m \Longrightarrow p_k = m(k-1) \tag{80}$$

For $k = 2$ we have $p_2 = m \cdot (2-1) = m$ and equation (78) becomes

$$x + \sigma(x) = c_2 x^m \Longrightarrow \sigma(x) = c_2 x^m - x \tag{81}$$

For $k = 3$ we have $p_3 = m \cdot (3-1) = 2m$ and equation (78) becomes

$$x^2 + \sigma(x)x + \sigma(x)^2 = c_3 x^{2m} \tag{82}$$

Substitute the expression for $\sigma(x)$ obtained in (81) into equation (82):

$$x^2 + c_2 x^{m+1} - x^2 + c_2^2 x^{2m} - 2c_2 x^{m+1} + x^2 = c_3 x^{2m},$$

or, after simplification,

$$x^2 = (c_3 - c_2^2)x^{2m} + c_2 x^{m+1} \tag{83}$$

From (83) it is clear that at least one of $2m$ and $m+1$ has to be equal to 2. In any case, $m = 1$, and substituting this into (81) we have $\sigma(x) = c_2 x - x = (c_2 - 1)x$, so if we take $\lambda = c_2 - 1$, the second condition is satisfied.

Conversely, if $D_\sigma(x)$ is homogenous and $\sigma(x) = \lambda x$, where $\lambda \in \mathbb{C}$, equation (72) yields

$$D_\sigma(x^k) = \big(x^{k-1} + x^{k-2}\lambda x + \ldots + x(\lambda x)^{k-2} + (\lambda x)^{k-1}\big) D_\sigma(x) =$$

$$= \Big(\sum_{j=0}^{k-1} \lambda^j\Big) x^{k-1} D_\sigma(x) = \begin{cases} \frac{\lambda^k - 1}{\lambda - 1} x^{k-1} D_\sigma(x), & \text{if } \lambda \neq 1 \\ k x^{k-1} D_\sigma(x), & \text{if } \lambda = 1 \end{cases}$$

It follows that indeed D_σ is a homogenous linear operator, which was to be proved. $\square$

6.3 A formula for σ-derivations on $\mathbb{C}[x_1, \ldots, x_n]$

For notational purposes, we denote the algebra by $\mathbb{C}[X]$, where X is some finite set. $\mathbb{C}[X]$ can be thought of as a free commutative algebra with unit on the finite set X.

Proposition 25. *Let $\sigma \neq 0$ be a nonzero algebra endomorphism of $\mathbb{C}[X]$. Then a linear operator D_σ on $\mathbb{C}[X]$ is a σ-derivation if and only if*

i. $D_\sigma(1) = 0$, and

ii. for all $2n$-tuples, $(k_1, k_2, ..., k_n, x_1, x_2, ..., x_n)$, $n \geq 1$ of positive integers $k_i \in \mathbb{Z}_{>0}$ and generators $x_i \in X$, the following equation holds:

$$D_\sigma(x_1^{k_1} x_2^{k_2} \ldots x_n^{k_n}) = \\ = \sum_{j=1}^{n} \left(\left(\prod_{i=1}^{j-1} \sigma(x_i)^{k_i} \right) \left(s^{k_j - 1}(x_j, \sigma(x_j)) D_\sigma(x_j) \right) \left(\prod_{i=j+1}^{n} x_i^{k_i} \right) \right) \quad (84)$$

where we denote by $s^k(x, \sigma(x)) = \sum_{i=0}^{k} x^i \sigma(x)^{k-i}$, when $k \in \mathbb{Z}_{\geq 0}$.

Proof. Suppose that D_σ is a σ-derivation. Then

$$D_\sigma(1) = D_\sigma(1 \cdot 1) = D_\sigma(1) \cdot 1 + \sigma(1) D_\sigma(1) = 2D_\sigma(1),$$

so that $D_\sigma(1) = 0$. For the second property, use Corollary 6 to deduce

$$D_\sigma(x_1^{k_1} \cdot \ldots \cdot x_n^{k_n}) = D_\sigma(x_1^{k_1}) x_2^{k_2} \cdot \ldots \cdot x_n^{k_n} \\ + \sigma(x_1)^{k_1} D_\sigma(x_2^{k_2}) x_3^{k_3} \cdot \ldots \cdot x_n^{k_n} + \ldots + \sigma(x_1)^{k_1} \sigma(x_2)^{k_2} \cdot \ldots \cdot \sigma(x_n)^{k_n} \quad (85)$$

For each i we can use the same calculation as in (73) and obtain

$$D_\sigma(x_i^{k_i}) = s^{k_i - 1}(x_i, \sigma(x_i)) D_\sigma(x_i)$$

Substituting this into (85) we get the desired formula.

Conversely, suppose the two conditions are satisfied, then we must check that D_σ is indeed a σ-derivation. Since D_σ is linear, it suffices to check that

$$D_\sigma(x_1^{k_1} \ldots x_n^{k_n} \cdot y_1^{l_1} \ldots y_m^{l_n}) = \\ = D_\sigma(x_1^{k_1} \ldots x_n^{k_n}) y_1^{l_1} \ldots y_m^{l_n} + \sigma(x_1^{k_1} \ldots x_n^{k_n}) D_\sigma(y_1^{l_1} \ldots y_m^{l_n}) \quad (86)$$

holds for any set of $n+m$ generators $x_1, \ldots, x_n, y_1, \ldots y_m$, and any non-negative integers k_i, l_j, for $i = 1, \ldots, n$ and $j = 1, \ldots m$, where n and m are positive integers. We proceed by induction on $n+m$. Suppose first that $n+m=2$, that is, $n=m=1$. If at least one of k_1 and l_1, say k_1, is equal to zero, we have

$$D_\sigma(x_1^0 y_1^{l_1}) = 0\cdot y_1^{l_1} + 1\cdot D_\sigma(y_1^{l_1}) = D_\sigma(x_1^0)y_1^{l_1} + \sigma(x_1^0)D_\sigma(y_1^{l_1})$$

where we used the first condition that $D_\sigma(x_1^0) = D_\sigma(1) = 0$. Now if both $k_1 \geq 1$ and $l_1 \geq 1$ then using (84) on the $2\cdot 2$-tuple (k_1,k_2,x_1,x_2) where we have set $k_2 = l_1$ and $x_2 = y_1$ for notational purposes, we have

$$\begin{aligned}
D_\sigma(x_1^{k_1}y_1^{l_1}) &= D_\sigma(x_1^{k_1}x_2^{k_2}) = \\
&= \sum_{j=1}^{2}\left(\Big(\prod_{i=1}^{j-1}\sigma(x_i)^{k_i}\Big)\Big(s^{k_j-1}(x_j,\sigma(x_j))D_\sigma(x_j)\Big)\Big(\prod_{i=j+1}^{2}x_i^{k_i}\Big)\right) = \\
&= \left(\Big(\prod_{i=1}^{1-1}\sigma(x_i)^{k_i}\Big)\Big(s^{k_1-1}(x_1,\sigma(x_1))D_\sigma(x_1)\Big)\Big(\prod_{i=1+1}^{2}x_i^{k_i}\Big)\right) + \\
&\quad + \left(\Big(\prod_{i=1}^{2-1}\sigma(x_i)^{k_i}\Big)\Big(s^{k_2-1}(x_2,\sigma(x_2))D_\sigma(x_2)\Big)\Big(\prod_{i=2+1}^{2}x_i^{k_i}\Big)\right) = \\
&= \Big(D_\sigma(x_1^{k_1})\Big)x_2^{k_2} + \Big(\sigma(x_1)^{k_1}\Big)\Big(D_\sigma(x_2^{k_2})\Big) = \\
&= D_\sigma(x_1^{k_1})y_1^{l_1} + \sigma(x_1^{k_1})D_\sigma(y_1^{l_1})
\end{aligned}$$

Thus (86) holds for $n+m=2$ generators x_1,y_1 and any non-negative integers k_1,l_1. Suppose now that (86) holds for any set of $p+q$ generators, with corresponding non-negative exponents, whenever $p+q<n+m$. Let $x_1,\ldots,x_n,y_1,\ldots,y_m$ be any $n+m$ generators and let $k_1,\ldots,k_n,l_1,\ldots,l_m$ be $n+m$ non-negative integers. We wish to prove (86). If there is some integer i such that $1\leq i\leq n$ and $k_i=0$, we have

$$\begin{aligned}
&D_\sigma(x_1^{k_1}\ldots x_n^{k_n}\cdot y_1^{l_1}\ldots y_m^{l_m}) = D_\sigma(x_1^{k_1}\ldots x_{i-1}^{k_{i-1}}x_{i+1}^{k_{i+1}}\ldots x_n^{k_n}\cdot y_1^{l_1}\ldots y_m^{l_m}) = \\
&D_\sigma(x_1^{k_1}\ldots x_{i-1}^{k_{i-1}}x_{i+1}^{k_{i+1}}\ldots x_n^{k_n})y_1^{l_1}\ldots y_m^{l_n} + \\
&\qquad\qquad + \sigma(x_1^{k_1}\ldots x_{i-1}^{k_{i-1}}x_{i+1}^{k_{i+1}}\ldots x_n^{k_n})D_\sigma(y_1^{l_1}\ldots y_m^{l_n}) = \\
&\qquad\qquad = D_\sigma(x_1^{k_1}\ldots x_n^{k_n})y_1^{l_1}\ldots y_m^{l_n} + \sigma(x_1^{k_1}\ldots x_n^{k_n})D_\sigma(y_1^{l_1}\ldots y_m^{l_n})
\end{aligned}$$

where we used the induction hypothesis in the second equality. Therefore, we can assume that $k_i\neq 0$ for $i=1,\ldots,n$, and similarly we can assume $l_j\neq 0$ for $j=1,\ldots,m$. Now set $x_{n+1}=y_1,\ldots,x_{n+m}=y_m$ and $k_{n+1}=l_1,\ldots,k_{n+m}=l_m$ and use (84) on the

$2(n+m)$-tuple $(k_1, \dots, k_{n+m}, x_1, \dots, x_{n+m})$:

$$\begin{aligned}
&D_\sigma(x_1^{k_1} \dots x_n^{k_n} \cdot y_1^{l_1} \dots y_m^{l_m}) = D_\sigma(x_1^{k_1} x_2^{k_2} \dots x_{n+m}^{k_{n+m}}) = \\
&\qquad \sum_{j=1}^{n+m} \left(\Big(\prod_{i=1}^{j-1} \sigma(x_i)^{k_i} \Big) \Big(s^{k_j - 1}(x_j, \sigma(x_j)) D_\sigma(x_j) \Big) \Big(\prod_{i=j+1}^{n+m} x_i^{k_i} \Big) \right) = \\
&= \sum_{j=1}^{n} \left(\Big(\prod_{i=1}^{j-1} \sigma(x_i)^{k_i} \Big) \Big(s^{k_j - 1}(x_j, \sigma(x_j)) D_\sigma(x_j) \Big) \Big(\prod_{i=j+1}^{n+m} x_i^{k_i} \Big) \right) + \\
&\qquad + \sum_{j=n+1}^{n+m} \left(\Big(\prod_{i=1}^{j-1} \sigma(x_i)^{k_i} \Big) \Big(s^{k_j - 1}(x_j, \sigma(x_j)) D_\sigma(x_j) \Big) \Big(\prod_{i=j+1}^{n+m} x_i^{k_i} \Big) \right) = \\
&= \sum_{j=1}^{n} \left(\Big(\prod_{i=1}^{j-1} \sigma(x_i)^{k_i} \Big) \Big(s^{k_j - 1}(x_j, \sigma(x_j)) D_\sigma(x_j) \Big) \Big(\prod_{i=j+1}^{n} x_i^{k_i} \Big) \right) \prod_{i=n+1}^{n+m} x_i^{k_i} + \\
&\quad + \Big(\prod_{i=1}^{n} \sigma(x_i)^{k_i} \Big) \sum_{j=n+1}^{n+m} \left(\Big(\prod_{i=n+1}^{j-1} \sigma(x_i)^{k_i} \Big) \Big(s^{k_j - 1}(x_j, \sigma(x_j)) D_\sigma(x_j) \Big) \Big(\prod_{i=j+1}^{n+m} x_i^{k_i} \Big) \right) = \\
&= D_\sigma(x_1^{k_1} \dots x_n^{k_n}) x_{n+1}^{k_{n+1}} \dots x_{n+m}^{k_{n+m}} + \sigma(x_1^{k_1} \dots x_n^{k_n}) D_\sigma(x_{n+1}^{k_{n+1}} \dots x_{n+m}^{k_{n+m}}) = \\
&\qquad\qquad = D_\sigma(x_1^{k_1} \dots x_n^{k_n}) y_1^{l_1} \dots y_m^{l_m} + \sigma(x_1^{k_1} \dots x_n^{k_n}) D_\sigma(y_1^{l_1} \dots y_m^{l_m})
\end{aligned}$$

Thus (86) holds for $n+m$ generators, with non-negative exponents. By the principle of mathematical induction, (86) holds for any set of $n+m$ generators, where $n \geq 1$ and $m \geq 1$ are two positive integers, and all non-negative integers $k_1, \dots, k_n, l_1, \dots, l_m$. This completes the proof. □

6.4 (σ, τ)-Derivations on UFD:s and on $\mathbb{C}[x_1, \dots, x_n]$

When we speak of unique factorization domains we shall always mean a commutative associative algebra over $\mathbb{C}$ with unity 1 and with no zero-divizors, such that any element can be written in a unique way (up to a multiple of an invertible element) as a product of irreducible elements (i.e. elements which cannot be written as a product of two non-invertible elements). Examples of unique factorization domains include $\mathbb{C}[x_1, \dots, x_n]$, and the algebra $\mathbb{C}[t, t^{-1}]$ of Laurent polynomials.

If A is a unique factorization domain, it is in particular commutative, and therefore $\mathfrak{D}_{(\sigma,\tau)}(A)$ carries a natural A-module structure by pointwise multiplication, as defined in Section 5.3. If $a, b \in A$ we shall write $a \big| b$ if there is an element $c \in A$ such that $ac = b$. If $S \subseteq A$ is a subset of A, the greatest common divizor $GCD(S)$ of S is defined as an element of A satisfying

$$GCD(S) \big| a \quad \text{for } a \in S, \tag{87}$$

and

$$b\big|a \quad \text{for } a \in S \Longrightarrow b\big|GCD(S). \tag{88}$$

Using that A is a unique factorization domain one can show that $GCD(S)$ exists for any nonempty subset S of A and is unique up to a multiple of an invertible element in A. It follows directly from the definition of GCD that

$$S \subseteq T \subseteq A \quad \Longrightarrow \quad GCD(T)\big|GCD(S). \tag{89}$$

Remark 12. When we have $A = \mathbb{C}[x_1,\ldots,x_n]$, we will order the set of monomials lexicographically, and always choose the unique GCD having leading coefficient 1 with respect to this order.

Lemma 26. *Let A be a commutative algebra with no zero-divizors. Let σ and τ be different algebra endomorphisms on A, and let D be a (σ,τ)-derivation on A. Then*

$$\ker(\tau-\sigma) \subseteq \ker D. \tag{90}$$

Remark 13. Under some extra conditions we have equality in (90) for any nonzero D (see Corollary 28).

Proof. Let $y \in A$ be such that $\tau(y) \neq \sigma(y)$, and let $x \in \ker(\tau-\sigma)$. Then

$$\begin{aligned} 0 = D(xy-yx) &= D(x)\tau(y) + \sigma(x)D(y) - D(y)\tau(x) - \sigma(y)D(x) = \\ &= D(x)(\tau(y)-\sigma(y)) - D(y)(\tau(x)-\sigma(x)) = D(x)(\tau(y)-\sigma(y)) \end{aligned}$$

which imply $D(x)=0$, since we assumed A had no zero-divizors. □

We now prove the main result in this section.

Theorem 27. *Let σ and τ be different algebra endomorphisms on a unique factorization domain A (for example $\mathbb{C}[x_1,\ldots,x_n]$). Then $\mathfrak{D}_{(\sigma,\tau)}(A)$ is free of rank one as an A-module with generator*

$$\frac{\tau-\sigma}{g} : x \mapsto \frac{(\tau-\sigma)(x)}{g}, \tag{91}$$

where $g = GCD\big((\tau-\sigma)(A)\big)$.

Proof. We note first that $(\tau-\sigma)/g$ is a (σ,τ)-derivation on A:

$$\begin{aligned} \frac{(\tau-\sigma)(xy)}{g} &= \frac{\tau(x)\tau(y)-\sigma(x)\sigma(y)}{g} = \\ &= \frac{\big(\tau(x)-\sigma(x)\big)\tau(y) + \sigma(x)\big(\tau(y)-\sigma(y)\big)}{g} = \\ &= \frac{(\tau-\sigma)(x)}{g}\cdot\tau(y) + \sigma(x)\cdot\frac{(\tau-\sigma)(y)}{g}, \end{aligned}$$

for $x, y \in A$. Next we show that $(\tau - \sigma)/g$ generates a free A-module of rank one. So suppose that

$$x \cdot \frac{\tau - \sigma}{g} = 0, \tag{92}$$

for some $x \in A$. Since $\tau \neq \sigma$, there is an $y \in A$ such that $(\tau - \sigma)(y) \neq 0$. Application of both sides in (92) to this y yields

$$x \cdot \frac{(\tau - \sigma)(y)}{g} = 0.$$

Since A has no zero-divizors, it then follows that $x = 0$. Thus

$$A \cdot \frac{\tau - \sigma}{g}$$

is a free A-module of rank one.

It remains to show that $\mathfrak{D}_{(\sigma,\tau)}(A) \subseteq A \cdot \frac{\tau-\sigma}{g}$. Let D be a (σ, τ)-derivation on A. We want to find $a_D \in A$ such that

$$D(x) = a_D \cdot \frac{(\tau - \sigma)(x)}{g} \tag{93}$$

for $x \in A$. We will define

$$a_D = \frac{D(x) \cdot g}{(\tau - \sigma)(x)} \tag{94}$$

for some x such that $(\tau - \sigma)(x) \neq 0$. For this to be possible, we must show two things. First of all, that

$$(\tau - \sigma)(x) \mid D(x) \cdot g \quad \text{for any } x \text{ with } (\tau - \sigma)(x) \neq 0 \tag{95}$$

and secondly, that

$$\frac{D(x) \cdot g}{(\tau - \sigma)(x)} = \frac{D(y) \cdot g}{(\tau - \sigma)(y)} \quad \text{for any two } x, y \text{ with } (\tau - \sigma)(x) \neq 0 \neq (\tau - \sigma)(y). \tag{96}$$

Suppose for a moment that (95) and (96) were true. Then it is clear that if we define a_D by (94), the formula (93) holds for any $x \in A$ satisfying $(\tau - \sigma)(x) \neq 0$. But (93) also holds when $x \in A$ is such that $(\tau - \sigma)(x) = 0$, because then $D(x) = 0$ also, by Lemma 26.

We first prove (95). Let $x, y \in A$ be such that $(\tau - \sigma)(x) \neq 0 \neq (\tau - \sigma)(y)$. Then we have

$$\begin{aligned} 0 = D(xy - yx) &= D(x)\tau(y) + \sigma(x)D(y) - D(y)\tau(x) - \sigma(y)D(x) = \\ &= D(x)(\tau(y) - \sigma(y)) - D(y)(\tau(x) - \sigma(x)), \end{aligned}$$

so that

$$D(x)(\tau(y)-\sigma(y)) = D(y)(\tau(x)-\sigma(x)). \tag{97}$$

Now define a function $h : A \times A \to A$ by setting

$$h(z,w) = GCD(\tau(z)-\sigma(z), \tau(w)-\sigma(w)) \qquad \text{for } z,w \in A.$$

By the choice of x and y, we have $h(x,y) \neq 0$. Divide both sides of (97) by $h(x,y)$:

$$D(x)\frac{\tau(y)-\sigma(y)}{h(x,y)} = D(y)\frac{\tau(x)-\sigma(x)}{h(x,y)}. \tag{98}$$

Then it is true that

$$GCD\left(\frac{\tau(y)-\sigma(y)}{h(x,y)}, \frac{\tau(x)-\sigma(x)}{h(x,y)}\right) = 1.$$

Therefore we deduce from (98) that

$$\frac{\tau(x)-\sigma(x)}{h(x,y)} \Bigm| D(x),$$

i.e. that

$$(\tau-\sigma)(x) \bigm| D(x)\cdot h(x,y) \tag{99}$$

for any $x,y \in A$ with $(\tau-\sigma)(x) \neq 0 \neq (\tau-\sigma)(y)$. Let $S = A\setminus\ker(\tau-\sigma)$. Then from (99) and property (88) of the GCD we get

$$(\tau-\sigma)(x) \bigm| D(x)\cdot GCD(h(x,S)) \tag{100}$$

for all $x \in A$ with $(\tau-\sigma)(x) \neq 0$. But

$$\begin{aligned} GCD(h(x,S)) &= GCD\Big(\big\{GCD\big((\tau-\sigma)(x),(\tau-\sigma)(s)\big) \;:\; s\in S\big\}\Big) = \\ &= GCD\big((\tau-\sigma)(S)\cup\{(\tau-\sigma)(x)\}\big) = \\ &= GCD\big((\tau-\sigma)(A)\cup\{(\tau-\sigma)(x)\}\big) = \\ &= g. \end{aligned}$$

Thus (100) is equivalent to (95) which was to be proved.

Finally, we prove (96). Let $x,y \in A$ be such that $(\tau-\sigma)(x) \neq 0 \neq (\tau-\sigma)(y)$. Then

$$\begin{aligned} 0 &= D(xy-yx) = D(x)\tau(y)+\sigma(x)D(y)-D(y)\tau(x)-\sigma(y)D(x) = \\ &= D(x)(\tau(y)-\sigma(y))-D(y)(\tau(x)-\sigma(x)), \end{aligned}$$

which, after multiplication by g and division by $(\tau-\sigma)(x)\cdot(\tau-\sigma)(y)$ proves (96). This finishes the proof of the existence of a_D, and the proof of the theorem is complete. $\square$

Corollary 28. *Let σ and τ be different algebra endomorphisms on a unique factorization domain A (for example $\mathbb{C}[x_1, \dots, x_n]$). Then*

$$\ker(\tau - \sigma) = \ker D$$

for any nonzero (σ, τ)-derivation D on A.

Proof. Let D be a nonzero (σ, τ)-derivation. By uniqueness of a_D in Theorem 27, we must have $a_D \neq 0$. Using and the absence of zero-divizors, we see that $a_D \cdot \frac{(\tau-\sigma)(x)}{g} = 0$ if and only if $(\tau - \sigma)(x) = 0$. □

Corollary 29. *Let σ and τ be different algebra endomorphisms on a unique factorization domain (for example $\mathbb{C}[x_1, \dots, x_n]$). Then the following two statements are equivalent.*

1. *Any (σ, τ)-derivation of A is inner,*
2. *$g = GCD\big((\tau - \sigma)(A)\big)$ is an invertible element of A.*

Proof. Suppose any (σ, τ)-derivation is inner. By Theorem 27, the map

$$\frac{\tau - \sigma}{g}$$

is a (σ, τ)-derivation. Since this is inner by assumption, there is some element $r \in A$ such that for all $x \in A$,

$$\frac{(\tau - \sigma)(x)}{g} = r\tau(x) - \sigma(x)r = r(\tau(x) - \sigma(x)) = rg\frac{(\tau - \sigma)(x)}{g}.$$

From the uniqueness of a_D, we must have $rg = 1$, so g is invertible.

Conversely, if g is invertible, then by Theorem 27 we have for any (σ, τ)-derivation D,

$$D(x) = a_D \cdot \frac{(\tau - \sigma)(x)}{g} = a_D g^{-1} \cdot \tau(x) - \sigma(x) \cdot a_D g^{-1}$$

so $D = \delta_{a_D g^{-1}}$, with the notation from Definition 4. Thus any (σ, τ)-derivation is inner and the proof is finished. □

The following proposition is useful for explicit calculations.

Proposition 30. *Let σ and τ be different nonzero algebra endomorphism on a unique factorization domain A. Let $X \subseteq A$ be a set of generators of A. Then*

$$GCD\big((\tau - \sigma)(A)\big) = GCD\big((\tau - \sigma)(X)\big). \tag{101}$$

Proof. We show that each side of (101) divides the other. Using (89) it follows that the left hand side divides the right hand side. We will now show that

$$GCD\big((\tau-\sigma)(X)\big)\Big|(\tau-\sigma)(a) \quad \text{for any } a \in A, \tag{102}$$

from which the conclusion will follow, by the property (88) of the greatest common divizor. Since $\tau-\sigma$ is a linear operator, it is enough to prove that (102) holds for any element a of the form

$$a = x_1 \cdot \ldots \cdot x_n \tag{103}$$

where $x_i \in X$ for $i = 1,\dots,n$. We will use induction over n. When $n = 0$, we have $a = 1$ as a product over an empty set, and (102) holds, since $\sigma(1) = 1 = \tau(1)$ for nonzero endomorphisms σ and τ. When $n = 1$, we have $a \in X$, and (102) holds by definition. Assume now that (102) holds for any a of the form (103) with $n \le k$, where $k \ge 1$. Then, if a is a product of $k+1$ generators, we can factorize it into two elements b and c, both of which are of the form (103) with $1 \le n \le k$. Then we get, since $\tau-\sigma$ is a (σ,τ)-derivation on A,

$$(\tau-\sigma)(a) = (\tau-\sigma)(bc) = (\tau-\sigma)(b)\sigma(c) + \tau(b)(\tau-\sigma)(c),$$

which is divisible by $GCD\big((\tau-\sigma)(X)\big)$ by the induction hypothesis. Consequently (102) holds for any a which is a product of $k+1$ generators. This which finishes the induction step and the proof of the proposition. □

Remark 14. Note that the Jackson q-differentiation $D_{x,q}$, $q \in \mathbb{C}$, considered in Section 3.3 acting on $\mathbb{C}[x]$ is precisely the (σ,τ)-derivation $(\tau-\sigma)/g$ from Theorem 27, with $\sigma(p(x)) = p(qx)$ and $\tau = \mathrm{id}$, because by Proposition 30, $g = (\mathrm{id}-\sigma)(x) = x - qx$.

Proposition 31. *Let σ be a nonzero algebra endomorphism on $\mathbb{C}[x_1,\dots,x_n]$. Then there are unique (σ,σ)-derivations $\frac{\partial_\sigma}{\partial x_i}$, $i = 1,\dots,n$, satisfying*

$$\frac{\partial_\sigma}{\partial x_i}(x_j) = \delta_{i,j} \quad \textit{for all } i,j = 1,\dots,n. \tag{104}$$

Furthermore, $\mathfrak{D}_{(\sigma,\sigma)}(\mathbb{C}[x_1,\dots,x_n])$ is a free $\mathbb{C}[x_1,\dots,x_n]$-module of rank n, and the (σ,σ)-derivations (104) form a basis.

Proof. Uniqueness of the operators $\frac{\partial_\sigma}{\partial x_i}$ is clear, because they are uniquely determined by the value on the generators x_i of the algebra. To show existence, let $\frac{\partial_\sigma}{\partial x_i}$ be the unique linear operator on $\mathbb{C}[x_1,\dots,x_n]$ satisfying

$$\frac{\partial_\sigma}{\partial x_i}(x_1^{k_1}\dots x_n^{k_n}) = k_i\sigma(x_1^{k_1}\dots x_{i-1}^{k_{i-1}}x_i^{k_i-1}x_{i+1}^{k_{i+1}}\dots x_n^{k_n}) \tag{105}$$

for all integers $k_j \geq 0$, $j = 1, \dots, n$, where we interpret the right hand side as zero, if $k_i = 0$. Then it is easy to see that (104) is satisfied. We must show that (105) indeed defines (σ, σ)-derivations. It is enough to show that $\frac{\partial_\sigma}{\partial x_i}(ab) = \frac{\partial_\sigma}{\partial x_i}(a)\sigma(b) + \sigma(a)\frac{\partial_\sigma}{\partial x_i}(b)$ for monomials a and b, because then the identity will hold for arbitrary polynomials as well, by bilinearity.

$$\begin{aligned}
\frac{\partial_\sigma}{\partial x_i}(x_1^{k_1}\dots x_n^{k_n}\cdot x_1^{l_1}\dots x_n^{l_n}) = \frac{\partial_\sigma}{\partial x_i}(x_1^{k_1+l_1}\dots x_n^{k_n+l_n}) = \\
&= (k_i+l_i)\sigma(x_1^{k_1+l_1}\dots x_{i-1}^{k_{i-1}+l_{i-1}}x_i^{k_i+l_i-1}x_{i+1}^{k_{i+1}+l_{i+1}}\dots x_n^{k_n+l_n}) = \\
&= k_i\sigma(x_1^{k_1}\dots x_{i-1}^{k_{i-1}}x_i^{k_i-1}x_{i+1}^{k_{i+1}}\dots x_n^{k_n})\cdot\sigma(x_1^{l_1}\dots x_n^{l_n}) + \\
&\quad + \sigma(x_1^{k_1}\dots x_n^{k_n})\cdot l_i\sigma(x_1^{l_1}\dots x_{i-1}^{l_{i-1}}x_i^{l_i-1}x_{i+1}^{l_{i+1}}\dots x_n^{l_n}) = \\
&= \frac{\partial_\sigma}{\partial x_i}(x_1^{k_1}\dots x_n^{k_n})\cdot\sigma(x_1^{l_1}\dots x_n^{l_n}) + \\
&\quad + \sigma(x_1^{k_1}\dots x_n^{k_n})\cdot\frac{\partial_\sigma}{\partial x_i}(x_1^{l_1}\dots x_n^{l_n})
\end{aligned}$$

We now show that $X = \{\frac{\partial_\sigma}{\partial x_1}, \dots, \frac{\partial_\sigma}{\partial x_n}\}$ form a basis for $\mathfrak{D}_{(\sigma,\sigma)}(\mathbb{C}[x_1, \dots, x_n])$. X is a linearly independent set, because if

$$\sum_{i=1}^{n} p_i(x_1, \dots, x_n)\frac{\partial_\sigma}{\partial x_i} = 0$$

for some polynomials $p_i \in \mathbb{C}[x_1, \dots, x_n]$, we apply both sides to x_j to obtain $p_j = 0$ for $j = 1, \dots, n$. Also, $\mathfrak{D}_{(\sigma,\sigma)}(\mathbb{C}[x_1, \dots, x_n])$ is generated by X, since if D is any (σ, σ)-derivation on $\mathbb{C}[x_1, \dots, x_n]$, the (σ, σ)-derivation

$$\sum_{i=1}^{n} D(x_i)\frac{\partial_\sigma}{\partial x_i}$$

and D coincide on every generator x_j and hence must be equal. The proof is finished. □

Theorem 27 and Proposition 31, together with the trivial fact that the zero map 0 is the only $(0, 0)$-derivation (this holds in any algebra A such that $A = A \cdot A$, in particular when A has a unit), gives a complete classification of all possible (σ, τ)-derivations on the algebra of complex polynomials in n indeterminates, in the case when σ and τ are algebra endomorphisms.

7 Equations for σ-derivations on the quantum plane

In this section we shall consider σ-derivations on the quantum plane A which is the algebra with unit I and generators A,B satisfying the defining commutation relation

$$AB - qBA = 0,$$

where $q \in \mathbb{C}$. In Section 7.2 we will assume that σ is affine, and we find matrix equations which must be satisfied for affine σ-derivations. We begin however with a general theorem.

7.1 The general case

Proposition 32. *Let σ be a linear operator on the quantum plane, and let D_σ be a σ-derivation. Write*

$$D_\sigma(A) = \sum_{\substack{0 \le i,j \le n \\ i+j \le n}} \alpha_{ij} B^i A^j, \qquad D_\sigma(B) = \sum_{\substack{0 \le i,j \le n \\ i+j \le n}} \beta_{ij} B^i A^j,$$

$$\sigma(A) = \sum_{\substack{0 \le i,j \le n \\ i+j \le n}} \gamma_{ij} B^i A^j, \qquad \sigma(B) = \sum_{\substack{0 \le i,j \le n \\ i+j \le n}} \delta_{ij} B^i A^j,$$

where $n \ge 0$ is an integer and $\alpha_{ij}, \beta_{ij}, \gamma_{ij}, \delta_{ij}$ are complex numbers. Then the following five equations are true.

$$\gamma_{00}\beta_{00} - \delta_{00}\alpha_{00}q = 0, \tag{106}$$

$$\alpha_{\lambda-1,0} + \sum_{\substack{0 \le i,k \le n \\ i+k \le \lambda}} (\gamma_{i0}\beta_{k0} - \delta_{i0}\alpha_{k0}q) = 0 \quad \textit{when} \quad 1 \le \lambda \le n, \tag{107}$$

$$\beta_{0,\mu-1}q + \sum_{\substack{0 \le j,l \le n \\ j+l \le \mu}} (\gamma_{0j}\beta_{0l} - \delta_{0j}\alpha_{0l}q) = 0 \quad \textit{when} \quad 1 \le \mu \le n, \tag{108}$$

$$\alpha_{\lambda-1,\mu}q^\mu - \beta_{\lambda,\mu-1}q + \sum_{\substack{0 \le i,j,k,l \le n \\ i+j,k+l \le n \\ i+k=\lambda, j+l=\mu}} (\gamma_{ij}\beta_{kl}q^{jk} - \delta_{ij}\alpha_{kl}q^{jk+1}) = 0 \quad \textit{when} \quad 2 \le \lambda+\mu \le n+1, \tag{109}$$

$$\sum_{\substack{0 \le i,j,k,l \le n \\ i+j,k+l \le n \\ i+k=\lambda, j+l=\mu}} (\gamma_{ij}\beta_{kl}q^{jk} - \delta_{ij}\alpha_{kl}q^{jk+1}) = 0 \quad \textit{when} \quad 0 \le \lambda,\mu \le 2n \textit{ and } n+1 < \lambda+\mu \le 2n. \tag{110}$$

Proof. We have

$$\begin{aligned}
0 &= D_\sigma(AB - qBA) = D_\sigma(A)B - qD_\sigma(B)A + \sigma(A)D_\sigma(B) - q\sigma(B)D_\sigma(A) = \\
&= \sum_{\substack{0 \le i,j \le n \\ i+j \le n}} (\alpha_{ij} \underbrace{B^i A^j B}_{=q^j B^{i+1} A^j} - q\beta_{ij} B^i A^j A) + \\
&\quad + \sum_{\substack{0 \le i,j \le n \\ i+j \le n}} \sum_{\substack{0 \le k,l \le n \\ k+l \le n}} (\gamma_{ij} B^i A^j \beta_{kl} B^k A^l - q\delta_{ij} B^i A^j \alpha_{kl} B^k A^l) = \\
&= \sum_{i=1}^{n+1} \alpha_{i-1,0} q^0 B^i A^0 - \sum_{j=1}^{n+1} \beta_{0,j-1} q B^0 A^j + \\
&\quad + \sum_{\substack{1 \le i,j \le n+1 \\ i+j \le n+1}} (\alpha_{i-1,j} q^j - \beta_{i,j-1} q) B^i A^j + \\
&\quad + \sum_{\substack{0 \le i,j \le n \\ i+j \le n}} \sum_{\substack{0 \le k,l \le n \\ k+l \le n}} (\gamma_{ij}\beta_{kl} q^{jk} - \delta_{ij}\alpha_{kl} q^{jk+1}) B^{i+k} A^{j+l} = \\
&= \sum_{i=1}^{n+1} (\alpha_{i-1,0} B^i - \beta_{0,i-1} q A^i) + \\
&\quad + \sum_{\substack{1 \le i,j \le n+1 \\ i+j \le n+1}} (\alpha_{i-1,j} q^j - \beta_{i,j-1} q) B^i A^j + \\
&\quad + \sum_{\substack{0 \le \lambda,\mu \le 2n \\ \lambda+\mu \le 2n}} \Big(\sum_{\substack{0 \le i,j,k,l \le n \\ i+j,k+l \le n \\ i+k=\lambda, j+l=\mu}} (\gamma_{ij}\beta_{kl} q^{jk} - \delta_{ij}\alpha_{kl} q^{jk+1}) \Big) B^\lambda A^\mu
\end{aligned} \tag{111}$$

For shorter notation, let

$$\Theta_{\lambda\mu} = \sum_{\substack{0 \le i,j,k,l \le n \\ i+j,k+l \le n \\ i+k=\lambda, j+l=\mu}} (\gamma_{ij}\beta_{kl} q^{jk} - \delta_{ij}\alpha_{kl} q^{jk+1})$$

Then the equation can be written

$$\begin{aligned}
0 = {} & \Theta_{00} I + \sum_{\lambda=1}^{n+1} (\alpha_{\lambda-1,0} + \Theta_{\lambda,0}) B^{\lambda} + \sum_{\mu=1}^{n+1} (-\beta_{0,\mu-1} q + \Theta_{0,\mu}) A^{\mu} + \\
& + \sum_{\substack{1 \le \lambda, \mu \le n+1 \\ \lambda+\mu \le n+1}} (\alpha_{\lambda-1,\mu} q^{\mu} - \beta_{\lambda,\mu-1} q + \Theta_{\lambda\mu}) B^{\lambda} A^{\mu} + \\
& + \sum_{\substack{0 \le \lambda, \mu \le 2n \\ n+1 < \lambda+\mu \le 2n}} \Theta_{\lambda\mu} B^{\lambda} A^{\mu} \qquad (112)
\end{aligned}$$

Now each coefficient has to be zero so we get the following equations:

1. $\Theta_{00} = 0$.
2. $\alpha_{\lambda-1,0} + \Theta_{\lambda 0} = 0$ when $1 \le \lambda \le n+1$.
3. $\beta_{0,\mu-1} q + \Theta_{0\mu} = 0$ when $1 \le \mu \le n+1$.
4. $\alpha_{\lambda-1,\mu} q^{\mu} - \beta_{\lambda,\mu-1} q + \Theta_{\lambda\mu} = 0$ when $1 \le \lambda, \mu \le n+1$ and $\lambda + \mu \le n+1$.
5. $\Theta_{\lambda\mu} = 0$ when $0 \le \lambda, \mu \le 2n$ and $n+1 < \lambda + \mu \le 2n$.

The three first of these equations can be simplified by substituting the expression for $\Theta_{\lambda\mu}$.

$$0 = \Theta_{00} = \sum_{\substack{0 \le i,j,k,l \le n \\ i+j, k+l \le n \\ i+k=0, j+l=0}} (\gamma_{ij} \beta_{kl} q^{jk} - \delta_{ij} \alpha_{kl} q^{jk+1}) = \gamma_{00} \beta_{00} - \delta_{00} \alpha_{00} q$$

In the same way, the second equation can also be written more simple.

$$\begin{aligned}
0 = \alpha_{\lambda-1,0} + \Theta_{\lambda,0} = \alpha_{\lambda-1,0} + & \sum_{\substack{0 \le i,j,k,l \le n \\ i+j, k+l \le n \\ i+k=\lambda \\ j+l=0 \Rightarrow j=l=0}} (\gamma_{ij} \beta_{kl} q^{jk} - \delta_{ij} \alpha_{kl} q^{jk+1}) = \\
= \alpha_{\lambda-1,0} + & \sum_{\substack{0 \le i,k \le n \\ i+k \le \lambda}} (\gamma_{i0} \beta_{k0} - \delta_{i0} \alpha_{k0} q)
\end{aligned}$$

when $1 \le \lambda \le n$. Similarly the third equation may be simplified to the following.

$$\beta_{0,\mu-1} q + \sum_{\substack{0 \le j,l \le n \\ j+l \le \mu}} (\gamma_{0j} \beta_{0l} - \delta_{0j} \alpha_{0l} q) = 0$$

when $1 \le \mu \le n$. The proof is finished. $\square$

7.2 Matrix equations in the affine case

Let us study the conditions (1-5) above in more detail when $n = 1$. The first equation does not depend of n, so it is the same.

$$\gamma_{00}\beta_{00} - q\delta_{00}\alpha_{00} = 0 \tag{113}$$

For the second condition, note that if an integer λ fulfills $1 \leq \lambda \leq 2$ then either $\lambda = 1$ or $\lambda = 2$. Thus we get two equations. First, when $\lambda = 1$,

$$\alpha_{0,0} + \underbrace{\gamma_{10}\beta_{00}q^0 - \delta_{10}\alpha_{00}q^1}_{i=1,k=0} + \underbrace{\gamma_{00}\beta_{10}q^0 - \delta_{00}\alpha_{10}q^1}_{i=0,k=1} = 0, \tag{114}$$

and when $\lambda = 2$,

$$\alpha_{1,0} + \underbrace{\gamma_{10}\beta_{10}q^0 - \delta_{10}\alpha_{10}q^1}_{i=1,k=1} = 0. \tag{115}$$

The third condition is analogous to the second. We get two equations again. First, when $\mu = 1$,

$$\beta_{0,0}q + \underbrace{\gamma_{01}\beta_{00}q^0 - \delta_{01}\alpha_{00}q^1}_{j=1,l=0} + \underbrace{\gamma_{00}\beta_{01}q^0 - \delta_{00}\alpha_{01}q^1}_{j=0,l=1} = 0, \tag{116}$$

and when $\mu = 2$,

$$\beta_{0,1}q + \underbrace{\gamma_{01}\beta_{01}q^0 - \delta_{01}\alpha_{01}q^1}_{j=1,l=1} = 0. \tag{117}$$

For the fourth equation, observe that $1 \leq \lambda, \mu \leq 2$ and $\lambda + \mu \leq 2$ imply that $\lambda = \mu = 1$. So we get only one equation. To write it down, we need to investigate the sum Θ_{11}. The conditions on the indices are

$$\begin{gathered} 0 \leq i, j, k, l \leq 1, \\ i + j, k + l \leq 1, \\ i + k = j + l = 1. \end{gathered}$$

This gives two terms, one when $i = l = 1$, $j = k = 0$, and one when $i = l = 0$, $j = k = 1$. Explicitly the equation is

$$\alpha_{01}q - \beta_{10}q + \underbrace{\gamma_{10}\beta_{01}q^0 - \delta_{10}\alpha_{01}q^1}_{i=l=1,j=k=0} + \underbrace{\gamma_{01}\beta_{10}q^1 - \delta_{01}\alpha_{10}q^2}_{i=l=0,j=k=1} = 0. \tag{118}$$

The fifth and last condition will not occur in the case $n = 1$ since there are no numbers x satisfying $2 < x \leq 2$. Thus, in matrix notation, we have the following equation.

$$\left[\begin{array}{ccc|ccc} -\delta_{00}q & 0 & 0 & \gamma_{00} & 0 & 0 \\ 1-\delta_{10}q & -\delta_{00}q & 0 & \gamma_{10} & \gamma_{00} & 0 \\ -\delta_{01}q & 0 & -\delta_{00}q & -q+\gamma_{01} & 0 & \gamma_{00} \\ 0 & 1-\delta_{10}q & 0 & 0 & \gamma_{10} & 0 \\ 0 & -\delta_{01}q^2 & q-\delta_{10}q & 0 & -q+\gamma_{01}q & \gamma_{10} \\ 0 & 0 & -\delta_{01}q & 0 & 0 & -q+\gamma_{01} \end{array}\right] \begin{bmatrix} \alpha_{00} \\ \alpha_{10} \\ \alpha_{01} \\ \beta_{00} \\ \beta_{10} \\ \beta_{01} \end{bmatrix} = 0 \tag{119}$$

The rows corresponds to equations (113), (114), (116), (115), (118), and (117), respectively.

So far we have not used the fact that the quantum plane has no zero-divizors. Using this fact and Corollary 14 we can say more. Namely we know that if D_σ is nonzero, then σ must be a homomorphism of algebras. The following proposition gives a necessary condition for affine endomorphisms of the quantum plane when $q \neq 0, \pm 1$.

Proposition 33. *Let $\mathcal{A}$ be the quantum plane with generators A and B and the relation $AB = qBA$, where $q \in \mathbb{C}$, and suppose $q \neq 0, \pm 1$. Let $\sigma : \mathcal{A} \to \mathcal{A}$ be an algebra endomorphism such that*

$$\sigma(A) = \gamma_{00} I + \gamma_{10} B + \gamma_{01} A$$

$$\sigma(B) = \delta_{00} I + \delta_{10} B + \delta_{01} A$$

Then exactly one of the following conditions is satisfied.

1. *$\sigma(A) = 0$ or $\sigma(B) = 0$.*

2. *$\sigma(A) = \gamma_{01} A$ and $\sigma(B) = \delta_{10} B$, and $\gamma_{01}, \delta_{10} \neq 0$.*

Proof. Suppose that we are given an endomorphism σ of $\mathcal{A}$ of the form in the proposition. It is clear that the two conditions cannot be satisfied simultaneously. Using that $AB = qBA$ and that σ is an algebra endomorphism we can do the following calculation.

$$\begin{aligned}
0 &= \sigma(AB - qBA) = \sigma(A)\sigma(B) - q\sigma(B)\sigma(A) = \\
&= (\gamma_{00} I + \gamma_{10} B + \gamma_{01} A)(\delta_{00} I + \delta_{10} B + \delta_{01} A) \\
&\quad -q(\delta_{00} I + \delta_{10} B + \delta_{01} A)(\gamma_{00} I + \gamma_{10} B + \gamma_{01} A) = \\
&= (\gamma_{00}\delta_{00} - q\delta_{00}\gamma_{00}) I \\
&\quad +\big(\gamma_{00}\delta_{10} + \gamma_{10}\delta_{00} - q(\delta_{00}\gamma_{10} + \delta_{10}\gamma_{00})\big) B \\
&\quad +\big(\gamma_{00}\delta_{01} + \gamma_{01}\delta_{00} - q(\delta_{00}\gamma_{01} + \delta_{01}\gamma_{00})\big) A \\
&\quad +(\gamma_{10}\delta_{10} - q\delta_{10}\gamma_{10}) B^2 \\
&\quad +\big(\gamma_{10}\delta_{01} + q\gamma_{01}\delta_{10} - q(\delta_{10}\gamma_{01} + q\delta_{01}\gamma_{10})\big) BA \\
&\quad +(\gamma_{01}\delta_{01} - q\delta_{01}\gamma_{01}) A^2 = \\
&= (1-q)\gamma_{00}\delta_{00} I \\
&\quad +(1-q)(\gamma_{00}\delta_{10} + \gamma_{10}\delta_{00}) B \\
&\quad +(1-q)(\gamma_{00}\delta_{01} + \gamma_{01}\delta_{00}) A \\
&\quad +(1-q)\gamma_{10}\delta_{10} B^2 \\
&\quad +(1-q^2)\gamma_{10}\delta_{01} BA \\
&\quad +(1-q)\gamma_{01}\delta_{01} A^2
\end{aligned} \tag{120}$$

Since the set $\{B^iA^j \mid i,j \geq 0\}$ is linear independant, each coefficient must be zero and $q \neq 0, \pm 1$, we have the following equations.

$$\gamma_{00}\delta_{00} = 0 \tag{121}$$
$$\gamma_{00}\delta_{10} + \gamma_{10}\delta_{00} = 0 \tag{122}$$
$$\gamma_{00}\delta_{01} + \gamma_{01}\delta_{00} = 0 \tag{123}$$
$$\gamma_{10}\delta_{10} = 0 \tag{124}$$
$$\gamma_{10}\delta_{01} = 0 \tag{125}$$
$$\gamma_{01}\delta_{01} = 0 \tag{126}$$

To proceed, we now suppose that condition 1 of Proposition 33 is not satisfied, that is, we suppose that at least one of $\gamma_{00}, \gamma_{10}, \gamma_{01}$ is nonzero, and that at least one of $\delta_{00}, \delta_{10}, \delta_{01}$ is nonzero. Then we show that condition 2 of Proposition 33 is satisfied. By equation (121), at least one of γ_{00} and δ_{00} is zero. Assume first that $\gamma_{00} \neq 0$, and $\delta_{00} = 0$. Then, equations (122) and (123) imply that $\delta_{10} = \delta_{01} = 0$, which contradicts the assumption about $\sigma(B)$ being nonzero. Similarly, if $\gamma_{00} = 0$ and $\delta_{00} \neq 0$, then equations (122) and (123) imply that $\gamma_{10} = \gamma_{01} = 0$, which contradicts the assumption about $\sigma(A)$ being nonzero. Hence in fact we must have $\gamma_{00} = \delta_{00} = 0$. Now if $\gamma_{10} \neq 0$, equations (124) and (125) imply that $\delta_{10} = \delta_{01} = 0$, which again is a contradiction. Similarly if $\delta_{01} \neq 0$, then equations (125) and (126) imply that $\gamma_{10} = \gamma_{01} = 0$, which is a contradiction. Thus $\gamma_{10} = \delta_{01} = 0$, so that the only coefficients which are nonzero are γ_{01} and δ_{10}, as desired. □

Corollary 34. *Let $\sigma : A \to A$ is a linear operator on the quantum plane with $q \neq 0, \pm 1$, and let D_σ be a σ-derivation. Write*

$$\sigma(A) = \gamma_{00}I + \gamma_{10}B + \gamma_{01}A,$$

$$\sigma(B) = \delta_{00}I + \delta_{10}B + \delta_{01}A,$$

$$D_\sigma(A) = \alpha_{00}I + \alpha_{10}B + \alpha_{01}A,$$

$$D_\sigma(B) = \beta_{00}I + \beta_{10}B + \beta_{01}A,$$

where $\alpha_{ij}, \beta_{ij}, \gamma_{ij}, \delta_{ij} \in \mathbb{C}$. Then one of the following equations is satisfied.

$$\begin{bmatrix} -\delta_{00}q & 0 & 0 \\ 1-\delta_{10}q & -\delta_{00}q & 0 \\ -\delta_{01}q & 0 & -\delta_{00}q \\ 0 & 1-\delta_{10}q & 0 \\ 0 & -\delta_{01}q^2 & q-\delta_{10}q \\ 0 & 0 & -\delta_{01}q \end{bmatrix} \begin{bmatrix} \alpha_{00} \\ \alpha_{10} \\ \alpha_{01} \end{bmatrix} = 0 \tag{127}$$

$$\begin{bmatrix} \gamma_{00} & 0 & 0 \\ \gamma_{10} & \gamma_{00} & 0 \\ -q+\gamma_{01} & 0 & \gamma_{00} \\ 0 & \gamma_{10} & 0 \\ 0 & -q+\gamma_{01}q & \gamma_{10} \\ 0 & 0 & -q+\gamma_{01} \end{bmatrix} \begin{bmatrix} \beta_{00} \\ \beta_{10} \\ \beta_{01} \end{bmatrix} = 0 \tag{128}$$

$$\left[\begin{array}{ccc|ccc} 1-\delta_{10}q & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -q+\gamma_{01} & 0 & 0 \\ 0 & 1-\delta_{10}q & 0 & 0 & 0 & 0 \\ 0 & 0 & q-\delta_{10}q & 0 & -q+\gamma_{01}q & 0 \\ 0 & 0 & 0 & 0 & 0 & -q+\gamma_{01} \end{array}\right] \begin{bmatrix} \alpha_{00} \\ \alpha_{10} \\ \alpha_{01} \\ \beta_{00} \\ \beta_{10} \\ \beta_{01} \end{bmatrix} = 0 \tag{129}$$

Proof. Since the quantum plane has no zero divizors, by Corollary 14, it is enough to consider the case when σ is an algebra endomorphism on A, since otherwise we must have $\alpha_{ij} = \beta_{ij} = 0$ for all i, j so that these equations are trivially satisfied. Then, by Proposition 33, either $\sigma(A) = 0$ or $\sigma(B) = 0$, or $\sigma(A) = \gamma_{01}A$ and $\sigma(B) = \delta_{10}B$. In the first case, substituting $\gamma_{00} = \gamma_{10} = \gamma_{01} = 0$ into the matrix equation (119) and simplifying, we come to the first equation, (127). Similarly, if $\sigma(B) = 0$, substitution of $\delta_{00} = \delta_{10} = \delta_{01} = 0$ into (119) gives the second matrix equation in the corollary, equation (128). Finally, if $\sigma(A) = \gamma_{01}A$ and $\sigma(B) = \delta_{10}B$, then $\gamma_{00} = \gamma_{10} = \delta_{00} = \delta_{01} = 0$ and substituting this in (119) and removing the first row since it is zero, we get equation (129). $\square$

8 Homogenous (σ, τ)-derivations on graded algebras

8.1 Necessary conditions on the grading semigroup

In this section we consider (σ, τ)-derivations on semigroup-graded associative algebras with homogenous σ and τ, and state some necessary conditions which the semigroup must satisfy if a given (σ, τ)-derivation is homogenous.

Proposition 35. *Let S be s semigroup, $A = \oplus_{s\in S} A_s$ an S-graded associative algebra. Let σ, τ be homogenous linear operators on A, $D \in \mathfrak{D}_{(\sigma,\tau)}(A)$ be a (σ, τ)-derivation on A, and suppose D is homogenous. Then for any $s \in S$ and $t \in S$ there is an element $u \in S$ such that*

$$D(A_s)\tau(A_t) \subseteq A_u \quad \text{and} \quad D(A_sA_t) \subseteq A_u \quad \text{and} \quad \sigma(A_s)D(A_t) \subseteq A_u. \tag{130}$$

The element u can be chosen arbitrary if and only if

$$D(A_s)\tau(A_t) = D_\sigma(A_sA_t) = \sigma(A_s)D(A_t) = 0. \tag{131}$$

Otherwise, when (131) is not true, u must be unique.

Proof. Since σ, τ, and D are homogenous, and since the product of homogenous elements of A is homogenous, there are some $u_1, u_2, u_3 \in S$ such that $D(A_sA_t) \subseteq A_{u_1}$, $D(A_s)\tau(A_t) \subseteq A_{u_2}$, and $\sigma(A_s)D(A_t) \subseteq A_{u_3}$. We show that u_1, u_2, u_3 can be chosen to be equal. For $a_s \in A_s$ and $a_t \in A_t$ we have

$$D(a_sa_t) = D(a_s)\tau(a_t) + \sigma(a_s)D(a_t) \tag{132}$$

since D is a (σ,τ)-derivation. If $D(A_sA_t) = 0$, then u_1 can be chosen arbitrary, and from (132) follows that $D(A_s)\tau(A_t) = -\sigma(A_s)D(A_t)$ and therefore we can take $u_3 = u_2$. Hence $u = u_2$ satisfies (130), and is unique, unless $D(A_s)\tau(A_t) = -\sigma(A_s)D(A_t) = 0$. Suppose now that $D(A_sA_t) \neq 0$. Then u_1 is unique, and we take $u = u_1$. Using (132) and that each element of A is uniquely expressed as a finite sum $\sum x_g$ of homogenous elements $x_g \in A_g$, we have $D(A_s)\tau(A_t) \subseteq A_u$ and $\sigma(A_s)D(A_t) \subseteq A_u$. Hence the condition (130) holds. □

Proposition 36. *Let S be a semigroup, $A = \oplus_{s\in S}A_s$ an S-graded associative algebra, τ a homogenous linear operator of right degree $m \in S$, σ a homogenous linear operator of right degree $n \in S$, and D a (σ,τ)-derivation, which is also homogenous of right degree $k \in S$. Then for all $t \in S$ such that there is some left cancellable $s \in S$ such that $D(A_sA_t) \neq 0$, it is true that*

1. *$D(A_s)\tau(A_t) \neq 0 \Longrightarrow tk = ktm$.*
2. *$\sigma(A_s)D(A_t) \neq 0 \Longrightarrow tk = ntk$.*

Proof. By Proposition 35 we find a u such that (130) holds. But we also have

$$\begin{aligned} D(A_sA_t) &\subseteq D(A_{st}) \subseteq A_{stk} \\ D(A_s)\tau(A_t) &\subseteq A_{sk}A_{tm} \subseteq A_{sktm} \\ \sigma(A_s)D(A_t) &\subseteq A_{sn}A_{tk} \subseteq A_{sntk} \end{aligned}$$

If $D(A_sA_t) \neq 0$ and $D(A_s)\tau(A_t) \neq 0$, we have that $A_{stk} \cap A_u \neq 0$ and $A_{sktm} \cap A_u \neq 0$, and thus $A_{stk} = A_u = A_{sktm}$. Hence $stk = sktm$, and since s is left cancellable, $tk = ktm$. Similarly, if $D(A_sA_t) \neq 0$ and $\sigma(A_s)D(A_t) \neq 0$, we have that $A_{stk} \cap A_u \neq 0$ and $A_{sntk} \cap A_u \neq 0$, and thus $A_{stk} = A_u = A_{sntk}$. Hence $stk = sntk$, and since s is left cancellable, $tk = ntk$. □

When the grading semigroup is a group, we can say more, as expressed in the following corollary.

Corollary 37. *Let Γ be a group with its identity denoted 1, and $A = \oplus_{g\in\Gamma} A_g$ a Γ-graded algebra. Suppose τ is homogenous of right degree $m \in \Gamma$, σ is homogenous of right degree $n \in \Gamma$, and D is a (σ, τ)-derivation, homogenous of right degree $k \in \Gamma$. Suppose $D(A_1 A_1) \neq 0$. Then*

1. *If $D(A_1)\tau(A_1) \neq 0$ then $m = 1$*
2. *If $\sigma(A_1)D(A_1) \neq 0$ then $n = 1$*

Proof. Taking $t = s = 1$ in Proposition 36, we deduce that if $D(A_1)\tau(A_1) \neq 0$ then $1k = k1m$ which implies $m = 1$. And if $\sigma(A_1)D(A_1) \neq 0$ then $1k = n1k$ which implies $n = 1$. □

8.2 A projection formula

Let Γ be a group, and $A = \oplus_{g\in\Gamma} A_g$ a Γ-graded algebra. For elements $a \in A$, $g \in \Gamma$, denote by $a_g \in A_g$ the g:th homogenous components of a. Equivalently, a_g is the element obtained by projecting a onto the subspace A_g. We have

$$a = \sum_{g\in\Gamma} a_g \tag{133}$$

where $a_g \in A_g$ for $g \in \Gamma$ and there are only finitely many elements $g \in \Gamma$ such that a_g is non nonzero. The following proposition can

Proposition 38. *Let $A = \bigoplus_{g\in\Gamma} A_g$ be a Γ-graded associative algebra, where Γ is a group. Let σ and τ be linear operators on A, not necessarily homogenous, and D a (σ, τ)-derivation in A. Suppose $a, b \in A$ are such that $\sigma(a)$ and $\tau(b)$ are homogenous elements of A. Let $s \in \Gamma$ and $t \in \Gamma$ be such that $\sigma(a) \in A_s$ and $\tau(b) \in A_t$. Then we have the following formula*

$$(D(ab))_g = (D(a))_{gt^{-1}}\tau(b) + \sigma(a)(D(b))_{s^{-1}g} \tag{134}$$

Proof. Using that D is a (σ, τ)-derivation we have $D(ab) = D(a)\tau(b) + \sigma(a)D(b)$. If we use (133) we have

$$D(ab) = \sum_{g\in\Gamma} (D(a))_g \tau(b) + \sigma(a) \sum_{g\in\Gamma} (D(b))_g \tag{135}$$

Since the mappings $g \mapsto gt^{-1}$ and $g \mapsto s^{-1}g$ are bijections of Γ we can rewrite (135) as

$$\begin{aligned} D(ab) &= \sum_{g\in\Gamma} (D(a))_{gt^{-1}}\tau(b) + \sum_{g\in\Gamma} \sigma(a)(D(b))_{s^{-1}g} = \\ &= \sum_{g\in\Gamma} \left((D(a))_{gt^{-1}}\tau(b) + \sigma(a)(D(b))_{s^{-1}g}\right) \end{aligned} \tag{136}$$

Now since t was chosen so that $\tau(b) \in A_t$, we have

$$(D(a))_{gt^{-1}}\tau(b) \in A_{gt^{-1}}A_t \subseteq A_{gt^{-1}t} = A_g$$

and similarly, s was chosen so that $\sigma(a) \in A_s$, so

$$\sigma(a)(D(b))_{s^{-1}g} \in A_sA_{s^{-1}g} \subseteq A_{ss^{-1}g} = A_g$$

Thus $(D(a))_{gt^{-1}}\tau(b)+\sigma(a)(D(b))_{s^{-1}g} \in A_g$ is homogenous of degree g. This fact together with formula (136) imply the desired relation (134). □

Corollary 39. *Let $A = \bigoplus_{g\in\Gamma} A_g$ be a Γ-graded associative algebra, where Γ is a group. Let σ and τ be homogenous linear operators on A, and D a (σ, τ)-derivation in A, and $a, b \in A$ be homogenous. Let $t \in \Gamma$ be such that $\tau(b) \in A_t$ and $s \in \Gamma$ be such that $\sigma(a) \in A_s$. Then $D(ab)$ is a homogenous element of A of degree $h \in \Gamma$, i.e. $D(ab) \in A_h$, if and only if*

$$(D(a))_{gt^{-1}} \cdot \tau(b) + \sigma(a) \cdot (D(b))_{s^{-1}g} \neq 0 \Longrightarrow g = h \tag{137}$$

for all $g \in \Gamma$.

9 Generalized products and Jacobi type identities

In this section we consider some general conditions which are necessary and sufficient for σ and (σ, τ)-derivations to be closed under certain types of products. Throughout, A will denote an associative algebra.

First we derive conditions for $\mathfrak{D}_\sigma(A)$ to be closed under the multiplication in $\mathcal{L}(A)$, that is, function composition. Let $D_\sigma, E_\sigma \in \mathfrak{D}_\sigma(A)$. Then

$$D_\sigma E_\sigma(ab) = D_\sigma(E_\sigma(ab)) = D_\sigma(E_\sigma(a)b + \sigma(a)E_\sigma(b)) =$$

$$= (D_\sigma E_\sigma)(a)b + (\sigma E_\sigma)(a)D_\sigma(b) + (D_\sigma\sigma)(a)E_\sigma(b) + \sigma^2(a)(D_\sigma E_\sigma)(b) \tag{138}$$

Comparing (138) with Definition 2 of a σ-derivation we get the following result:

Proposition 40. *Let D_σ and $E_\sigma \in \mathfrak{D}_\sigma(A)$. Then $D_\sigma E_\sigma \in \mathfrak{D}_\sigma(A)$ if and only if*

$$(\sigma E_\sigma)(a)D_\sigma(b) + (D_\sigma\sigma)(a)E_\sigma(b) + \sigma^2(a)(D_\sigma E_\sigma)(b) = \sigma(a)(D_\sigma E_\sigma)(b) \tag{139}$$

for all $a, b \in A$.

Corollary 41. *If σ is an idempotent, i.e. $\sigma^2 = \sigma$, and if*

$$(\sigma E_\sigma)(a)D_\sigma(b) + (D_\sigma\sigma)(a)E_\sigma(b) = 0 \tag{140}$$

for all $D_\sigma, E_\sigma \in \mathfrak{D}_\sigma(A)$ and $a, b \in A$, then $\mathfrak{D}_\sigma(A)$ is a subalgebra of $\mathcal{L}(A)$.

9.1 Lie algebra structure

One can deduce the following corollary by exchanging D_σ and E_σ in (138) and subtracting the obtained equality from (138).

Corollary 42. *Let D_σ and E_σ be σ-derivations on A. If $[\sigma, D_\sigma](a)E_\sigma(b) = [\sigma, E_\sigma](a)D_\sigma(b)$ for all $a, b \in A$, then $D_\sigma E_\sigma - E_\sigma D_\sigma$ is a σ^2-derivation.*

Corollary 43. *If $\sigma D_\sigma = D_\sigma \sigma$ and $\sigma E_\sigma = E_\sigma \sigma$ then $D_\sigma E_\sigma - E_\sigma D_\sigma$ is a σ^2-derivation.*

Corollaries 42 and 43 also follow from the following more general proposition.

Proposition 44. *Let $k, l \in \mathbb{Z}_{\geq 0}$, and let $\sigma \in \mathcal{L}(A)$ be a linear operator. Suppose D_{σ^k} is a σ^k-derivation and E_{σ^l} a σ^l-derivation. Then $[D_{\sigma^k}, E_{\sigma^l}]$ is a σ^{k+l}-derivation if and only if*

$$[\sigma^l, D_{\sigma^k}](a)E_{\sigma^l}(b) = [\sigma^k, E_{\sigma^l}](a)D_{\sigma^k}(b)$$

for all $a, b \in A$.

Proof. Consider the following calculation:

$$\begin{aligned}
&\big(D_{\sigma^k}E_{\sigma^l} - E_{\sigma^l}D_{\sigma^k}\big)(ab) = \\
&\qquad = D_{\sigma^k}\big(E_{\sigma^l}(a)b + \sigma^l(a)E_{\sigma^l}(b)\big) - E_{\sigma^l}\big(D_{\sigma^k}(a)b + \sigma^k(a)D_{\sigma^k}(b)\big) = \\
&= D_{\sigma^k}(E_{\sigma^l}(a))b + \sigma^k(E_{\sigma^l}(a))D_{\sigma^k}(b) + D_{\sigma^k}(\sigma^l(a))E_{\sigma^l}(b) + \sigma^k(\sigma^l(a))D_{\sigma^k}(E_{\sigma^l}(b)) - \\
&- E_{\sigma^l}(D_{\sigma^k}(a))b - \sigma^l(D_{\sigma^k}(a))E_{\sigma^l}(b) - E_{\sigma^l}(\sigma^k(a))D_{\sigma^k}(b) - \sigma^l(\sigma^k(a))E_{\sigma^l}(D_{\sigma^k}(b)) = \\
&\qquad = (D_{\sigma^k}E_{\sigma^l} - E_{\sigma^l}D_{\sigma^k})(a)b + \sigma^{k+l}(a)(D_{\sigma^k}E_{\sigma^l} - E_{\sigma^l}D_{\sigma^k})(b) + \\
&\qquad\qquad + (\sigma^k E_{\sigma^l} - E_{\sigma^l}\sigma^k)(a)D_{\sigma^k}(b) - (\sigma^l D_{\sigma^k} - D_{\sigma^k}\sigma^l)(a)E_{\sigma^l}(b) =
\end{aligned}$$

This proves the claim. □

Corollary 45. *Let $k, l \in \mathbb{Z}_{\geq 0}$, $\sigma \in \mathcal{L}(A)$, and $D \in \mathfrak{D}_{\sigma^k}(A), E \in \mathfrak{D}_{\sigma^l}(A)$. If σ^l commutes with D and σ^k commutes with E, then $[D, E] \in \mathfrak{D}_{\sigma^{k+l}}(A)$*

Corollary 46. *Let $k, l \in \mathbb{Z}_{\geq 0}$, $\sigma \in \mathcal{L}(A)$ and $D \in \mathfrak{D}_{\sigma^k}(A)$, $E \in \mathfrak{D}_{\sigma^l}(A)$. If σ commutes with D and E then $[D, E] \in \mathfrak{D}_{\sigma^{k+l}}(A)$.*

For a linear operator $\sigma \in \mathcal{L}(A)$, denote by $|\sigma|$ the order of σ, that is the smallest positive number k such that $\sigma^k = 1$, the identity of $\mathcal{L}(A)$. If no such number exists, we write $|\sigma| = \infty$.

Corollary 47. *Let A be an associative algebra, and let $\sigma \in \mathcal{L}(A)$ be a linear operator on A such that $[\sigma, \mathfrak{D}_{\sigma^k}(A)] = 0$ for every $k \in \mathbb{Z}_{\geq 0}$. Then*

1. *If $|\sigma| < \infty$, then the linear space*

$$\mathfrak{D}_{(\sigma)}(A) = \bigoplus_{k \in \mathbb{Z}_{|\sigma|}} \mathfrak{D}_{\sigma^k}(A) \tag{141}$$

   is a $\mathbb{Z}_{|\sigma|}$-graded Lie algebra under the multiplication $[D_{\sigma^k}, E_{\sigma^l}] = D_{\sigma^k}E_{\sigma^l} - E_{\sigma^l}D_{\sigma^k}$.

2. *If $|\sigma| = \infty$ then the linear space*

$$\mathfrak{D}_{(\sigma)}(A) = \bigoplus_{k \in \mathbb{Z}_{\geq 0}} \mathfrak{D}_{\sigma^k}(A) \tag{142}$$

   is an $\mathbb{Z}_{\geq 0}$-graded Lie algebra under the multiplication $[D_{\sigma^k}, E_{\sigma^l}] = D_{\sigma^k}E_{\sigma^l} - E_{\sigma^l}D_{\sigma^k}$.

Note that the subspace $\mathfrak{D}_{\sigma^0}(A)$ of $\mathfrak{D}_{(\sigma)}(A)$ consisting of degree zero elements is a Lie subalgebra, and it is precisely the standard Lie algebra of derivations in the algebra A.

Define subspaces

$$\mathfrak{E}_k = \{D \in \mathfrak{D}_{\sigma^k}(A) \mid [\sigma, D] = 0\} = \mathfrak{D}_{\sigma^k}(A) \bigcap \ker(\operatorname{ad}\sigma)$$

where $\operatorname{ad}\sigma : \mathcal{L}(A) \to \mathcal{L}(A)$ is defined by $(\operatorname{ad}\sigma)(B) = \sigma B - B\sigma$ for any $B \in \mathcal{L}(A)$. If $D \in \mathfrak{E}_k$ and $E \in \mathfrak{E}_l$ we use the Jacobi identity to establish

$$[\sigma, [D, E]] = -[D, [E, \sigma]] - [E, [\sigma, D]] = 0$$

which implies that $[D, E] \in \mathfrak{E}_{k+l}$.

Corollary 48. *Let A be associative and σ a linear operator A. Then*

1. *If $|\sigma| < \infty$ there is a $\mathbb{Z}_{|\sigma|}$-graded Lie algebra*

$$\mathfrak{E}_{(\sigma)}(A) = \bigoplus_{k \in \mathbb{Z}_{|\sigma|}} \mathfrak{E}_k \tag{143}$$

2. *If $|\sigma| = \infty$ there is an $\mathbb{Z}_{\geq 0}$-graded Lie algebra*

$$\mathfrak{E}_{(\sigma)}(A) = \bigoplus_{k \in \mathbb{Z}_{\geq 0}} \mathfrak{E}_k \tag{144}$$

where $\mathfrak{E}_k = \mathfrak{D}_{\sigma^k}(A) \bigcap \ker(\operatorname{ad}\,\sigma)$.

9.2 The f bracket

Fix two linear operators σ and τ on A. Let $f : \mathfrak{D}_{(\sigma,\tau)}(A) \times \mathfrak{D}_{(\sigma,\tau)}(A) \times A \to A$ be a function which is linear in the third variable, and denote its value on (D, E, a) by $f_{D,E}(a)$. In other words, assign for each pair (D, E) of (σ, τ)-derivations a linear operator $f_{D,E}$ on A. Now define the f-bracket

$$[\cdot, \cdot]_f : \mathfrak{D}_{(\sigma,\tau)}(A) \times \mathfrak{D}_{(\sigma,\tau)}(A) \to \mathcal{L}(A)$$

$$[D, E]_f = DE - f_{D,E}ED$$

We want to find conditions on the function f which ensure that for given $D \in \mathfrak{D}_{(\sigma,\tau)}(A)$ and $E \in \mathfrak{D}_{(\sigma,\tau)}(A)$ the bracket product $[D, E]_f$ is again in the linear space $\mathfrak{D}_{(\sigma,\tau)}(A)$ of (σ, τ)-derivations.

Proposition 49. *Let $D \in \mathfrak{D}_{(\sigma,\tau)}(A)$ and $E \in \mathfrak{D}_{(\sigma,\tau)}(A)$. Then $[D, E]_f \in \mathfrak{D}_{(\sigma,\tau)}(A)$ if and only if*

$$\begin{aligned} D(E(a))\tau(\tau(b)) + \sigma(E(a))D(\tau(b)) + D(\sigma(a))\tau(E(b)) + \sigma(\sigma(a))D(E(b)) \\ -f_{D,E}E(D(a))\tau(\tau(b)) - f_{D,E}\sigma(D(a))E(\tau(b)) \\ -f_{D,E}E(\sigma(a))\tau(D(b)) - f_{D,E}\sigma(\sigma(a))E(D(b)) \\ -DE(a)\tau(b) + f_{D,E}ED(a)\tau(b) - \sigma(a)DE(b) + \sigma(a)f_{D,E}ED(b) = 0 \end{aligned} \tag{145}$$

for all $a, b \in A$.

Proof. The proof is given by the following computation:

$$\begin{aligned} 0 &= [D, E]_f(ab) - \Big([D, E]_f(a)\tau(b) + \sigma(a)[D, E]_f(b)\Big) = \\ &= \Big(DE - f_{D,E}ED\Big)(ab) - \Big(DE - f_{D,E}ED\Big)(a)\tau(b) - \sigma(a)\Big(DE - f_{D,E}ED\Big)(b) = \\ &= D\Big(E(a)\tau(b) + \sigma(a)E(b)\Big) - f_{D,E}E\Big(D(a)\tau(b) + \sigma(a)D(b)\Big) \\ &\qquad - DE(a)\tau(b) + f_{D,E}ED(a)\tau(b) - \sigma(a)DE(b) + \sigma(a)f_{D,E}ED(b) = \\ &= D(E(a))\tau(\tau(b)) + \sigma(E(a))D(\tau(b)) + D(\sigma(a))\tau(E(b)) + \sigma(\sigma(a))D(E(b)) \\ &\qquad - f_{D,E}E(D(a))\tau(\tau(b)) - f_{D,E}\sigma(D(a))E(\tau(b)) \\ &\qquad - f_{D,E}E(\sigma(a))\tau(D(b)) - f_{D,E}\sigma(\sigma(a))E(D(b)) \\ &\qquad - DE(a)\tau(b) + f_{D,E}ED(a)\tau(b) - \sigma(a)DE(b) + \sigma(a)f_{D,E}ED(b) \end{aligned}$$

$\square$

We get the following corollary for σ-derivations.

Corollary 50. *Let $D \in \mathfrak{D}_\sigma(A)$ and $E \in \mathfrak{D}_\sigma(A)$. Then $[D,E]_f \in \mathfrak{D}_\sigma(A)$ if and only if*

$$\begin{aligned}\sigma(E(a))D(b) + D(\sigma(a))E(b) + \sigma(\sigma(a))D(E(b))\\ - f_{D,E}E(D(a))b - f_{D,E}\sigma(D(a))E(b)\\ - f_{D,E}E(\sigma(a))D(b) - f_{D,E}\sigma(\sigma(a))E(D(b))\\ f_{D,E}ED(a)b - \sigma(a)DE(b) + \sigma(a)f_{D,E}ED(b) = 0\end{aligned} \tag{146}$$

for all $a,b \in A$.

Corollary 50 may be used to state some sufficient conditions which guarantee that for given σ-derivations D and E the f-bracket $[D,E]_f$ is again a σ-derivation.

Proposition 51. *Let $D \in \mathfrak{D}_\sigma(A)$ and $E \in \mathfrak{D}_\sigma(A)$. If*

1. *$f_{D,E}(ab) = f_{D,E}(a)b$ for all $a,b \in A$,*
2. *$\sigma E - f_{D,E}E\sigma = 0$,*
3. *$D\sigma - f_{D,E}\sigma D = 0$,*
4. *$\sigma(\sigma(a)) = \sigma(a)$ for all $a \in A$,*
5. *$f_{D,E}(\sigma(\sigma(a)))E(D(b)) = \sigma(a)f_{D,E}(E(D(b)))$ for all $a,b \in A$,*

then $[D,E]_f = DE - f_{D,E}ED \in \mathfrak{D}_\sigma(A)$.

Proof. It is enough to check that condition (146) in Corollary 50 is satisfied. This follows from the following calculation:

$$\begin{aligned}&\sigma(E(a))D(b) + D(\sigma(a))E(b) + \sigma(\sigma(a))D(E(b))\\ &\quad - f_{D,E}(E(D(a))b) - f_{D,E}(\sigma(D(a))E(b))\\ &\quad - f_{D,E}(E(\sigma(a))D(b)) - f_{D,E}(\sigma(\sigma(a))E(D(b)))\\ &\quad + f_{D,E}(E(D(a)))b - \sigma(a)D(E(b)) + \sigma(a)f_{D,E}(E(D(b)))\\ =&\sigma(E(a))D(b) - f_{D,E}(E(\sigma(a))D(b)) + D(\sigma(a))E(b) - f_{D,E}(\sigma(D(a))E(b))\\ &\quad + \sigma(\sigma(a))D(E(b)) - \sigma(a)D(E(b))\\ &\quad - f_{D,E}(E(D(a))b) + f_{D,E}(E(D(a)))b\\ &\quad - f_{D,E}(\sigma(\sigma(a))E(D(b))) + \sigma(a)f_{D,E}(E(D(b)))\\ =&(\sigma(E(a)) - f_{D,E}(E(\sigma(a))))D(b) + (D(\sigma(a)) - f_{D,E}(\sigma(D(a))))E(b)\\ &\quad + (\sigma(\sigma(a)) - \sigma(a))D(E(b))\\ &\quad - f_{D,E}(E(D(a)))b + f_{D,E}(E(D(a)))b\\ &\quad - f_{D,E}(\sigma(\sigma(a)))E(D(b)) + \sigma(a)f_{D,E}(E(D(b)))\\ =&0\end{aligned}$$

by conditions 1-5. ∎

Proposition 52. *If*

1. $f_{D,E}(ab) = f_{D,E}(a)b = af_{D,E}(b)$ *for all* $a, b \in A$
2. $\sigma E - f_{D,E}E\sigma = 0$
3. $D\sigma - f_{D,E}\sigma D = 0$
4. $\sigma(\sigma(a)) = \sigma(a)$ *for all* $a \in A$

Then $[D,E]_f = DE - f_{D,E}ED$ *gives an algebra multiplication on the linear space of* σ*-derivations.*

Proof. This follows directly from Proposition 51. The only property we have to check is the fifth:

$$\begin{aligned} f_{D,E}(\sigma(\sigma(a)))E(D(b)) &= f_{D,E}(\sigma(\sigma(a))E(D(b))) \\ &= \sigma(\sigma(a))f_{E,D}(E(D(b))) = \sigma(a)f_{D,E}(E(D(b))) \end{aligned}$$

for all $a, b \in A$ because $f_{D,E}(ab) = f_{D,E}(a)b = af_{D,E}(b)$ for all $a, b \in A$ by assumption. □

9.3 Generalization of ϵ-derivations

Proposition 53. *Let* $\sigma_D, \sigma_E \in \mathcal{L}(A)$ *be linear operators on* A*, and let* D *be a* σ_D*-derivation of* A *and* E *be a* σ_E*-derivation of* A*. Then* $[D,E]_f$ *is a* $\sigma_D\sigma_E$*-derivation of* A *if and only if*

$$\begin{aligned} (\sigma_D E - f_{D,E}E\sigma_D)(a)D(b) + (D\sigma_E - f_{D,E}\sigma_E D)(a)E(b) + \\ \sigma_D\sigma_E(a)(f_{D,E}ED)(b) - (f_{D,E}\sigma_E\sigma_D)(a)(ED)(b) = 0 \end{aligned} \quad (147)$$

for all $a, b \in A$.

Proof. $[D,E]_f \in \mathfrak{D}_{\sigma_D\sigma_E}(A)$ if and only if

$$\begin{aligned} 0 =&(DE - f_{D,E}ED)(ab) - \\ &\Big((DE - f_{D,E}ED)(a)b + \sigma_D\sigma_E(a)(DE - f_{D,E}ED)(b)\Big) = \\ =&D\Big(E(a)b + \sigma_E(a)E(b)\Big) - f_{D,E}E\Big(D(a)b + \sigma_D(a)D(b)\Big) - \\ &(DE)(a)b + (f_{D,E}ED)(a)b - \sigma_D\sigma_E(a)(DE)(b) + \sigma_D\sigma_E(a)(f_{D,E}ED)(b) = \\ =&(DE)(a)b + (\sigma_D E)(a)D(b) + (D\sigma_E)(a)E(b) + (\sigma_D\sigma_E)(a)(DE)(b) - \\ &(f_{D,E}ED)(a)b - (f_{D,E}\sigma_E D)(a)E(b) - \\ &(f_{D,E}E\sigma_D)(a)D(b) - (f_{D,E}\sigma_E\sigma_D)(a)(ED)(b) - \\ &(DE)(a)b + (f_{D,E}ED)(a)b - \sigma_D\sigma_E(a)(DE)(b) + \sigma_D\sigma_E(a)(f_{D,E}ED)(b) = \\ =&(\sigma_D E - f_{D,E}E\sigma_D)(a)D(b) + (D\sigma_E - f_{D,E}\sigma_E D)(a)E(b) + \\ &\sigma_D\sigma_E(a)(f_{D,E}ED)(b) - (f_{D,E}\sigma_E\sigma_D)(a)(ED)(b) \end{aligned}$$

for all $a, b \in A$. $\square$

Corollary 54. *If D is a σ_D-derivation, E is a σ_E-derivation, and*

1. $(f_{D,E}\sigma_E\sigma_D)(a)(ED)(b) = (\sigma_D\sigma_E)(a)(f_{D,E}ED)(b)$
2. $\big(\sigma_D E - f_{D,E}E\sigma_D\big)(a)D(b) = 0$
3. $\big(D\sigma_E - f_{D,E}\sigma_E D\big)(a)E(b) = 0$

for all $a, b \in A$, then $[D, E]_f$ is a $\sigma_D\sigma_E$-derivation.

Corollary 55. *If D is a σ_D-derivation, E is a σ_E-derivation, and*

1. $f_{D,E}(a)b = af_{D,E}(b)$
2. $\sigma_D\sigma_E = \sigma_E\sigma_D$
3. $\sigma_D E - f_{D,E}E\sigma_D = 0$
4. $D\sigma_E - f_{D,E}\sigma_E D = 0$

for all $a, b \in A$, then $[D, E]_f$ is a $\sigma_D\sigma_E$-derivation.

Example 3. Let $A = \oplus_{\gamma\in\Gamma}A_\gamma$ be a Γ-graded associative algebra, where Γ is an abelian group, and let ϵ be a commutation factor on Γ as defined in Section 3.2 by the relations (15) through (18). Let $\gamma, \delta \in \Gamma$ and consider the two elements $C \in D(A, \epsilon)_\gamma$ and $D \in D(A, \epsilon)_\delta$. Define two linear operators σ_C and σ_D on A as follows. For a homogenous element $a \in A_\alpha$, set

$$\sigma_C(a) = \epsilon(\gamma, \alpha)a$$

$$\sigma_D(a) = \epsilon(\delta, \alpha)a$$

Then relation (24) shows that C is a σ_C-derivation, and D is a σ_D-derivation. Now define a linear operator $f_{C,D} : A \to A$ by

$$f_{C,D}(a) = \epsilon(\gamma, \delta)a$$

for homogenous $a \in A$. Then for all $a, b \in A$,

$$f_{C,D}(a)b = (\epsilon(\gamma,\delta)a)b = a(\epsilon(\gamma,\delta)b) = af_{C,D}(b)$$

$$\begin{aligned}(\sigma_C\sigma_D)(a) &= \sigma_C(\sigma_D(a)) = \sigma_C(\epsilon(\delta,\alpha)a) = \epsilon(\gamma,\alpha)\epsilon(\delta,\alpha)a = \\ &= \epsilon(\delta,\alpha)\epsilon(\gamma,\alpha)a = \sigma_D(\sigma_C(a)) = (\sigma_D\sigma_C)(a)\end{aligned}$$

$$\begin{aligned}(\sigma_C D - f_{C,D}D\sigma_C)(a) &= \sigma_C(D(a)) - f_{C,D}(D(\sigma_C(a))) = \\ &= \epsilon(\gamma,\delta+\alpha)D(a) - f_{C,D}(D(\epsilon(\gamma,\alpha)a)) = \\ &= \epsilon(\gamma,\delta)\epsilon(\gamma,\alpha)D(a) - \epsilon(\gamma,\delta)\epsilon(\gamma,\alpha)D(a) = 0\end{aligned}$$

$$\begin{aligned}(C\sigma_D - f_{C,D}\sigma_D C)(a) &= C(\sigma_D(a)) - f_{C,D}(\sigma_D(C(a))) = \\ &= C(\epsilon(\delta,\alpha)a) - f_{C,D}(\epsilon(\delta,\gamma+\alpha)C(a)) = \\ &= \epsilon(\delta,\alpha)C(a) - \epsilon(\gamma,\delta)\epsilon(\delta,\gamma)\epsilon(\delta,\alpha)D(a) = 0\end{aligned}$$

By Corollary 55, we conclude that $[C,D]_f = CD - f_{C,D}DC = CD - \epsilon(\gamma,\delta)DC$ is a $\sigma_C\sigma_D$-derivation, that is,

$$\begin{aligned}[C,D]_f(ab) &= [C,D]_f(a)b + \sigma_C\sigma_D(a)[C,D]_f(b) = \\ &= [C,D]_f(a)b + \epsilon(\gamma,\alpha)\epsilon(\delta,\alpha)a[C,D]_f(b) = \\ &= [C,D]_f(a)b + \epsilon(\gamma+\delta,\alpha)a[C,D]_f(b)\end{aligned}$$

Thus $[C,D]_f \in D(A,\epsilon)_{\gamma+\delta}$ and therefore we have shown that $D(A,\epsilon) = \oplus_{\gamma\in\Gamma} D(A,\epsilon)_\gamma$ is a graded algebra, as claimed in Section 3.2.

9.4 The (f,g)-bracket

In this subsection we consider another type of bracket product. Suppose we have associated for any two (σ,τ)-derivations D, E on some associative algebra A two linear operators $f_{D,E}$ and $g_{D,E}$, and define

$$[D,E]_{f,g} = Df_{D,E}E - Eg_{D,E}D,$$

for $D, E \in \mathfrak{D}_{(\sigma,\tau)}(A)$.

Proposition 56. *Let $D, E \in \mathfrak{D}_{(\sigma,\tau)}(A)$. Then $[D,E]_{f,g} \in \mathfrak{D}_{(\sigma,\tau)}(A)$ if and only if*

$$\begin{aligned}&Df_{D,E}(E(a)\tau(b)) + Df_{D,E}(\sigma(a)E(b)) \\ &\qquad - Eg_{D,E}(D(a)\tau(b)) + Eg_{D,E}(\sigma(a)D(b)) \\ &\qquad - Df_{D,E}E(a)\tau(b) + Eg_{D,E}D(a)\tau(b) \\ &\qquad\qquad - \sigma(a)Df_{D,E}E(b) + \sigma(a)Eg_{D,E}D(b) = 0\end{aligned}$$

for all $a, b \in A$.

Proof.

$$\begin{aligned}
0 &= Df_{D,E}E(ab) - Eg_{D,E}D(ab) \\
&\qquad - \Big((Df_{D,E}E - Eg_{D,E}D)(a)\tau(b) + \sigma(a)(Df_{D,E}E - Eg_{D,E}D)(b)\Big) = \\
&= Df_{D,E}\Big(E(a)\tau(b) + \sigma(a)E(b)\Big) - Eg_{D,E}\Big(D(a)\tau(b) + \sigma(a)D(b)\Big) \\
&\qquad - Df_{D,E}E(a)\tau(b) + Eg_{D,E}D(a)\tau(b) \\
&\qquad - \sigma(a)Df_{D,E}E(b) + \sigma(a)Eg_{D,E}D(b) = \\
&= Df_{D,E}(E(a)\tau(b)) + Df_{D,E}(\sigma(a)E(b)) \\
&\qquad - Eg_{D,E}(D(a)\tau(b)) + Eg_{D,E}(\sigma(a)D(b)) \\
&\qquad - Df_{D,E}E(a)\tau(b) + Eg_{D,E}D(a)\tau(b) \\
&\qquad - \sigma(a)Df_{D,E}E(b) + \sigma(a)Eg_{D,E}D(b)
\end{aligned}$$

□

Corollary 57. *Let $D, E \in \mathfrak{D}_{(\sigma,\tau)}(A)$. If*

1. $Df_{D,E}\big(E(a)\cdot\tau(b)\big) = Df_{D,E}E(a)\cdot\tau(b)$
2. $Df_{D,E}\big(\sigma(a)\cdot E(b)\big) = \sigma(a)Df_{D,E}E(b)$
3. $Eg_{D,E}\big(D(a)\cdot\tau(b)\big) = Eg_{D,E}D(a)\cdot\tau(b)$
4. $Eg_{D,E}\big(\sigma(a)\cdot D(b)\big) = \sigma(a)Eg_{D,E}D(b)$

for all $a, b \in A$, then $[D, E]_{f,g} \in \mathfrak{D}_{(\sigma,\tau)}(A)$.

Corollary 58. *Let $D, E \in \mathfrak{D}_{(\sigma,\tau)}(A)$. If*

1. $Df_{D,E}\mathtt{R}_{\tau(a)}E = \mathtt{R}_{\tau(a)}Df_{D,E}E$
2. $Df_{D,E}\mathtt{L}_{\sigma(a)}E = \mathtt{L}_{\sigma(a)}Df_{D,E}E$
3. $Eg_{D,E}\mathtt{R}_{\tau(a)}D = \mathtt{R}_{\tau(a)}Eg_{D,E}D$
4. $Eg_{D,E}\mathtt{L}_{\sigma(a)}D = \mathtt{L}_{\sigma(a)}Eg_{D,E}D$

for all $a \in A$, then $[D, E]_{f,g} \in \mathfrak{D}_{(\sigma,\tau)}(A)$.

Corollary 59. *Let $D, E \in \mathfrak{D}_{(\sigma,\tau)}(A)$. If*

1. $Df_{D,E}(ab) = Df_{D,E}(a)\cdot b = a\cdot Df_{D,E}(b)$
2. $Eg_{D,E}(ab) = Eg_{D,E}(a)\cdot b = a\cdot Eg_{D,E}(b)$

for all $a, b \in A$, then $[D, E]_{f,g} \in \mathfrak{D}_{(\sigma,\tau)}(A)$.

9.5 Jacobi identity for the f-bracket $[\cdot,\cdot]_f$.

In this section we find a Jacobi-type identity for the f-bracket previously introduced.

Theorem 60. *Let $\mathcal{L}$ be any associative algebra, suppose we are given $\mathcal{B} \subseteq \mathcal{L}$ a subset and $f_{(-,-)} : \mathcal{B} \times \mathcal{B} \to \mathcal{L}$ a function such that*

$$[C, D]_f \equiv CD - f_{C,D}DC \in \mathcal{B} \tag{148}$$

whenever $C, D \in \mathcal{B}$. Suppose further that there is a function

$$\Phi_{(-,-,-)} : \mathcal{B} \times \mathcal{B} \times \mathcal{B} \to \mathcal{L}$$

satisfying the following two conditions:

1. *For every triple (C, D, E) of elements of $\mathcal{B}$,*

$$\Phi_{C,D,E} = \Phi_{D,E,C} \cdot f_{[D,E]_f,C} \tag{149}$$

2. *For every triple (C', D', E') of elements of $\mathcal{B}$, either*

$$\Phi_{C,D,E} \cdot f_{C,D}DCE = \Phi_{D,E,C} \cdot Df_{E,C}CE \tag{150}$$

for any cyclic permutation (C, D, E) of (C', D', E'), or

$$\Phi_{C,D,E} \cdot f_{C,D}DCE = \Phi_{C,D,E} \cdot Cf_{D,E}ED \tag{151}$$

for any cyclic permutation (C, D, E) of (C', D', E'), or

$$\Phi_{C,D,E} \cdot f_{C,D}DCE = \Phi_{E,C,D} \cdot Ef_{C,D}DC \tag{152}$$

for any cyclic permutation (C, D, E) of (C', D', E').

Then

$$\Phi_{C,D,E}[[C, D]_f, E]_f + \Phi_{E,C,D}[[E, C]_f, D]_f + \Phi_{D,E,C}[[D, E]_f, C]_f = 0 \tag{153}$$

for all $C, D, E \in \mathcal{B}$.

Proof.

$$
\begin{aligned}
&\Phi_{C,D,E}[[C,D]_f,E]_f+\Phi_{E,C,D}[[E,C]_f,D]_f+\Phi_{D,E,C}[[D,E]_f,C]_f\\
&=\Phi_{C,D,E}[CD-f_{C,D}DC,E]_f+\Phi_{E,C,D}[EC-f_{E,C}CE,D]_f+\Phi_{D,E,C}[DE-f_{D,E}ED,C]_f\\
&\qquad=\Phi_{C,D,E}\Big((CD-f_{C,D}DC)E-f_{[C,D]_f,E}E(CD-f_{C,D}DC)\Big)\\
&\qquad+\Phi_{E,C,D}\Big((EC-f_{E,C}CE)D-f_{[E,C]_f,D}D(EC-f_{E,C}CE)\Big)\\
&\qquad+\Phi_{D,E,C}\Big((DE-f_{D,E}ED)C-f_{[D,E]_f,C}C(DE-f_{D,E}ED)\Big)\\
&=\Phi_{C,D,E}\Big(CDE-f_{C,D}DCE\Big)-\Phi_{C,D,E}\Big(f_{[C,D]_f,E}(ECD-Ef_{C,D}DC)\Big)\\
&+\Phi_{E,C,D}\Big(ECD-f_{E,C}CED\Big)-\Phi_{E,C,D}\Big(f_{[E,C]_f,D}(DEC-Df_{E,C}CE)\Big)\\
&+\Phi_{D,E,C}\Big(DEC-f_{D,E}EDC\Big)-\Phi_{D,E,C}\Big(f_{[D,E]_f,C}(CDE-Cf_{D,E}ED)\Big)
\end{aligned}
\tag{154}
$$

If we use the condition (149), we see that (154) equals

$$
\begin{aligned}
&\Phi_{C,D,E}\Big(CDE-f_{C,D}DCE\Big)-\Phi_{E,C,D}\Big(ECD-Ef_{C,D}DC\Big)\\
&\qquad+\Phi_{E,C,D}\Big(ECD-f_{E,C}CED\Big)-\Phi_{D,E,C}\Big(DEC-Df_{E,C}CE\Big)\\
&\qquad\qquad+\Phi_{D,E,C}\Big(DEC-f_{D,E}EDC\Big)-\Phi_{C,D,E}\Big(CDE-Cf_{D,E}ED\Big)
\end{aligned}
$$

$$
\begin{aligned}
&=-\Phi_{C,D,E}f_{C,D}DCE+\Phi_{E,C,D}Ef_{C,D}DC\\
&\qquad-\Phi_{E,C,D}f_{E,C}CED+\Phi_{D,E,C}Df_{E,C}CE\\
&\qquad\qquad-\Phi_{D,E,C}f_{D,E}EDC+\Phi_{C,D,E}Cf_{D,E}ED=0
\end{aligned}
\tag{155}
$$

In the final step we used the second condition. The proof is finished. □

Using equation (149) repeatedly one obtains:

$$
\begin{aligned}
\Phi_{C,D,E}&=\Phi_{D,E,C}f_{[D,E]_f,C}\\
&=\Phi_{E,C,D}f_{[E,C]_f,D}f_{[D,E]_f,C}\\
&=\Phi_{C,D,E}f_{[C,D]_f,E}f_{[E,C]_f,D}f_{[D,E]_f,C}
\end{aligned}
\tag{156}
$$

Thus, for instance, when $\mathcal{L}$ has a unit 1, and $\Phi_{C,D,E}$ is known to be left cancellable, the cyclic product $f_{[C,D]_f,E}f_{[E,C]_f,D}f_{[D,E]_f,C}$ must be 1.

Corollary 61 (Jacobi identity)**.** *Let $\mathcal{L}$ be an associative algebra with unit $1_{\mathcal{L}}$, $\mathcal{B}\subseteq\mathcal{L}$ a subset closed under $[\cdot,\cdot]_f$. Suppose the function f satisfies two conditions:*

$$
f_{[C,D]_f,E}f_{[E,C]_f,D}f_{[D,E]_f,C}=1_{\mathcal{L}}
\tag{157}
$$

and

$$f_{[D,E]_f,C}f_{C,D}D = Df_{E,C} \tag{158}$$

for all $C, D, E \in \mathcal{B}$. Then

$$[[C,D]_f,E]_f + f_{[C,D]_f,E}[[E,C]_f,D]_f + f_{[C,D]_f,E}f_{[E,C]_f,D}[[D,E]_f,C]_f = 0 \tag{159}$$

for all $C, D, E \in \mathcal{B}$.

Proof. Define two triples (C, D, E), (C', D', E') of elements of $\mathcal{B}$ to be equivalent if one can be permuted into the other cyclically. We will define Φ on $\mathcal{B}^{\times 3}$ by defining it on each element of each equivalence class. So given a class $\overline{(C,D,E)}$, define

$$\Phi_{C,D,E} = 1, \quad \Phi_{E,C,D} = f_{[C,D]_f,E}, \quad \Phi_{D,E,C} = f_{[C,D]_f,E}f_{[E,C]_f,D}$$

This definition obviously depends on the choice of representative (C, D, E) for the equivalence class. However, it will be shown that this will not matter. This procedure defines Φ on the whole of $\mathcal{B}^{\times 3}$. We now show that the conditions in Theorem 60 are satisfied.

For the first condition, let (C, D, E) be any triple of elements of $\mathcal{B}$, and consider

$$\Phi_{D,E,C} \cdot f_{[D,E]_f,C} \tag{160}$$

If we defined $\Phi_{D,E,C} = 1$ or $\Phi_{D,E,C} = f_{[E,C]_f,D}$ then clearly (160) equals $\Phi_{C,D,E}$ so that the condition holds. And if we defined $\Phi_{D,E,C} = f_{[C,D]_f,E}f_{[E,C]_f,D}$ then using (157) we get

$$\Phi_{D,E,C} \cdot f_{[D,E]_f,C} = f_{[C,D]_f,E}f_{[E,C]_f,D}f_{[D,E]_f,C} = 1 = \Phi_{C,D,E} \tag{161}$$

Hence the first condition holds.

For the second condition, we will show that the equality (150) is satisfied for any triple (C, D, E) of elements of $\mathcal{B}$. Multiply both sides of equation (158) by CE from the right, and by $\Phi_{D,E,C}$ from the left to deduce that

$$\Phi_{D,E,C}f_{[D,E]_f,C}f_{C,D}DCE = \Phi_{D,E,C}Df_{E,C}CE \tag{162}$$

is satisfied for all $C, D, E \in \mathcal{B}$. But we have just shown that $\Phi_{D,E,C}f_{[D,E]_f,C} = \Phi_{C,D,E}$ and substituting this into (162) we see that equation (150) indeed is satisfied for all $C, D, E \in \mathcal{B}$.

Thus by Theorem 60,

$$\Phi_{C,D,E}[[C,D]_f,E]_f + \Phi_{E,C,D}[[E,C]_f,D]_f + \Phi_{D,E,C}[[D,E]_f,C]_f = 0 \tag{163}$$

for all $C, D, E \in \mathcal{B}$.

Now given arbitrary $C, D, E \in \mathcal{B}$ we show that (159) holds. If $\Phi_{C,D,E}$ was chosen to be 1, then direct substitution of the values of Φ into (163) will give (159). Otherwise we have one of the two equations:

$$f_{[D,E]_f,C}[[C,D]_f,E]_f + f_{[D,E]_f,C}f_{[C,D]_f,E}[[E,C]_f,D]_f + [[D,E]_f,C]_f = 0 \tag{164}$$

if $\Phi_{C,D,E} = f_{[D,E]_f,C}$, or

$$f_{[E,C]_f,D} f_{[D,E]_f,C} [[C,D]_f, E]_f + [[E,C]_f, D]_f + f_{[E,C]_f,D} [[D,E]_f, C]_f = 0 \tag{165}$$

if $\Phi_{C,D,E} = f_{[E,C]_f,D} f_{[D,E]_f,C}$. But if we multiply equation (164) by $f_{[C,D]_f,E} f_{[E,C]_f,D}$ and equation (165) by $f_{[C,D]_f,E}$ and then use (157) both of these become

$$[[C,D]_f, E]_f + f_{[C,D]_f,E} [[E,C]_f, D]_f + f_{[C,D]_f,E} f_{[E,C]_f,D} [[D,E]_f, C]_f = 0$$

which is exactly equation (159). We have shown that this equation holds for arbitrary $C, D, E \in \mathcal{B}$ which was to be proved. □

Example 4. Take $\mathcal{L} = \mathcal{L}(A)$, the algebra of linear transformations of some associative algebra A, and take $\mathcal{B} = \mathfrak{D}(A)$ the linear space of ordinary derivations in A. Define $f_{C,D} = 1_{\mathcal{L}(A)}$, the identity transformation for all $C, D \in \mathfrak{D}(A)$. Then $\mathfrak{D}(A)$ is closed under the bracket product $[\cdot,\cdot]_f$ which now is reduced to the usual Lie bracket

$$[\cdot,\cdot] : (D,E) \mapsto [D,E] = DE - ED.$$

The conditions of Corollary 61 are satisfied and the conclusion (159) assumes the form

$$[[C,D],E] + [[E,C],D] + [[D,E],C] = 0$$

which is the familiar Jacobi identity. Thus (159) is a generalization of the ordinary Jacobi identity for derivations of an associative algebra.

Example 5. Let Γ be an abelian group, ϵ a commutation factor on Γ, and $A = \bigoplus_{\gamma\in\Gamma} A_\gamma$ be a Γ-graded ϵ Lie algebra, as defined in Section 3.2. Take

$$\mathcal{L} = \mathrm{Lgr}(A, \epsilon),$$

and

$$\mathcal{B} = \cup_{\gamma\in\Gamma} D(A,\epsilon)_\gamma,$$

and define

$$f_{D,E}(a) = \epsilon(\delta,\eta)a$$

for homogenous $D \in D(A,\epsilon)_\delta$, $E \in D(A,\epsilon)_\eta$ and $a \in A$. In Example 3 on page 58, we showed that $[D,E]_f = DE - f_{D,E}ED \in D(A,\epsilon)_{\delta+\eta}$ for all homogenous $D \in D(A,\epsilon)_\delta$, $E \in D(A,\epsilon)_\eta$. Thus $\mathcal{B}$ is closed under $[\cdot,\cdot]_f$. Now for all $a \in A$,

$$\begin{aligned} f_{[C,D]_f,E} f_{[E,C]_f,D} f_{[D,E]_f,C}(a) &= \epsilon(\gamma+\delta,\eta)\epsilon(\eta+\gamma,\delta)\epsilon(\delta+\eta,\gamma)(a) = \\ &= \epsilon(\gamma,\eta)\epsilon(\delta,\eta)\epsilon(\eta,\delta)\epsilon(\gamma,\delta)\epsilon(\delta,\gamma)\epsilon(\eta,\gamma)a = a \end{aligned}$$

and

$$(f_{[D,E]_f,C}f_{C,D}D)(a) = \epsilon(\delta+\eta,\gamma)\epsilon(\gamma,\delta)D(a) = \\ = \epsilon(\delta,\gamma)\epsilon(\eta,\gamma)\epsilon(\gamma,\delta)D(a) = \epsilon(\eta,\gamma)D(a) = (Df_{E,C})(a)$$

for all homogenous $C, D, E \in \mathcal{B}$. Thus, by Corollary 61, we have

$$[[C,D]_f,E]_f + f_{[C,D]_f,E}[[E,C]_f,D]_f + f_{[C,D]_f,E}f_{[E,C]_f,D}[[D,E]_f,C]_f = 0$$

That is,

$$[[C,D]_f,E]_f + \epsilon(\gamma+\delta,\eta)[[E,C]_f,D]_f + \epsilon(\gamma+\delta,\eta)\epsilon(\eta+\gamma,\delta)[[D,E]_f,C]_f = 0$$

which after multiplication by $\epsilon(\eta,\gamma)$ becomes

$$\epsilon(\eta,\gamma)[[C,D]_f,E]_f + \epsilon(\delta,\eta)[[E,C]_f,D]_f + \epsilon(\gamma,\delta)[[D,E]_f,C]_f = 0$$

for all homogenous $C, D, E \in \mathcal{B}$, which is the ϵ Jacobi identity.

Corollary 62. *Let $\mathcal{L}$ be an associative algebra with unit $1_{\mathcal{L}}$, $\mathcal{B} \subseteq \mathcal{L}$ a subset closed under $[\cdot,\cdot]_f$. Suppose the function f satisfies two conditions:*

$$f_{[C,D]_f,E}f_{[E,C]_f,D}f_{[D,E]_f,C} = 1_{\mathcal{L}} \tag{166}$$

and

$$[C,D]_fE = C[D,E]_f \tag{167}$$

for all $C, D, E \in \mathcal{B}$. Then

$$[[C,D]_f,E]_f + f_{[C,D]_f,E}[[E,C]_f,D]_f + f_{[C,D]_f,E}f_{[E,C]_f,D}[[D,E]_f,C]_f = 0 \tag{168}$$

for all $C, D, E \in \mathcal{B}$.

Proof. The proof is the almost same as in Corollary 61. The only thing we must check is that the second condition of Theorem 60 is satisfied. We show that in fact the equality (151) is satisfied for any triple (C, D, E) of elements of $\mathcal{B}$. Expand equation (167) and cancel one term:

$$\begin{aligned} [C,D]_fE &= C[D,E]_f \\ (CD - f_{C,D}DC)E &= C(DE - f_{D,E}ED) \\ f_{C,D}DCE &= Cf_{D,E}ED \end{aligned} \tag{169}$$

After multiplication of (169) by $\Phi_{C,D,E}$ on the left, we get that equation (151) indeed is satisfied for all $C, D, E \in \mathcal{B}$. □

The following gives a weak partial converse of the above corollaries.

Proposition 63. *Let $\mathcal{L}$ be an associative algebra with unit $1_{\mathcal{L}}$, $\mathcal{B} \subseteq \mathcal{L}$ a subset closed under $[\cdot,\cdot]_f$. Let (C',D',E') be a triple of elements of $\mathcal{B}$. Suppose that for all cyclic permutations (C,D,E) of (C',D',E'),*

$$[[C,D]_f,E]_f + f_{[C,D]_f,E}[[E,C]_f,D]_f + f_{[C,D]_f,E}f_{[E,C]_f,D}[[D,E]_f,C]_f = 0. \tag{170}$$

Then

$$(f_{[D,E]_f,C}f_{[C,D]_f,E}f_{[E,C]_f,D} - 1_{\mathcal{L}})[[D,E]_f,C]_f = 0$$

for any cyclic permutation (C,D,E) of (C',D',E'). In particular, if $\mathcal{L}$ has no zero-divizors, then

$$f_{[D,E]_f,C}f_{[C,D]_f,E}f_{[E,C]_f,D} = 1_{\mathcal{L}}$$

for any cyclic permutation (C,D,E) of (C',D',E') such that $[[D,E]_f,C]_f \neq 0$.

Proof. Let (C,D,E) be any cyclic permutation of (C',D',E'). Multiply equation (170) by $f_{[D,E]_f,C}$ from the left:

$$\begin{aligned} f_{[D,E]_f,C}[[C,D]_f,E]_f + f_{[D,E]_f,C}f_{[C,D]_f,E}[[E,C]_f,D]_f \\ + f_{[D,E]_f,C}f_{[C,D]_f,E}f_{[E,C]_f,D}[[D,E]_f,C]_f = 0 \end{aligned} \tag{171}$$

Since (170) holds for any cyclic permutation of (C',D',E') we also have

$$[[D,E]_f,C]_f + f_{[D,E]_f,C}[[C,D]_f,E]_f + f_{[D,E]_f,C}f_{[C,D]_f,E}[[E,C]_f,D]_f = 0 \tag{172}$$

Finally, if we subtract (172) from (171) we have

$$(f_{[D,E]_f,C}f_{[C,D]_f,E}f_{[E,C]_f,D} - 1)[[D,E]_f,C]_f = 0 \tag{173}$$

□

9.6 Commutation operators and ω Lie algebras

In this section we generalize some of the concepts introduced in Section 3.2. Namely, we generalize commutation factors ϵ to commutation operators ω, and Γ-graded ϵ Lie algebras to Γ-graded ω Lie algebras. Throughout, Γ will denote an abelian group.

Definition 9. Let $V = \oplus_{\gamma\in\Gamma}V_\gamma$ be a Γ-graded linear space. Then a map

$$\omega : \Gamma \times \Gamma \to \mathrm{Lgr}(V)_0 \tag{174}$$

is called a *commutation operator* for V if

$$\omega(\gamma+\delta,\eta)\omega(\eta+\gamma,\delta)\omega(\delta+\eta,\gamma) = \mathrm{id}_V \tag{175}$$

for all $\gamma,\delta,\eta \in \Gamma$ and

$$\omega(\delta+\eta,\gamma)E\omega(\gamma,\delta)D = D\omega(\eta,\gamma)E \tag{176}$$

for all $\gamma,\delta,\eta \in \Gamma$ and $D \in \mathrm{Lgr}(V)_\delta$, $E \in \mathrm{Lgr}(V)_\eta$ such that $DE = ED$.

Definition 10. Let ω be a commutation operator for (the underlying linear space of) a Γ-graded algebra $A = \oplus_{\gamma\in\Gamma} A_\gamma$ in which the product is denoted by $\langle\cdot,\cdot\rangle$. Then A is called a Γ-*graded* ω *Lie algebra* if the following two identities are satisfied whenever C, D, E are homogenous elements of A of degrees γ, δ, η respectively:

$$\langle C, D\rangle = -\omega(\gamma,\delta)\Big(\langle D, C\rangle\Big) \tag{177}$$

$$\langle\langle C, D\rangle, E\rangle + \omega(\gamma+\delta,\eta)\Big(\langle\langle E, C\rangle, D\rangle\Big) + (\omega(\gamma+\delta,\eta)\omega(\eta+\gamma,\delta))\Big(\langle\langle D, E\rangle, C\rangle\Big) = 0 \tag{178}$$

Lemma 64. *Let ω be a commutation operator for a Γ-graded linear space $V = \oplus_{\gamma\in\Gamma} V_\gamma$. Then*

$$\omega(0,\gamma) = \omega(\gamma,0) = \mathrm{id}_V \quad \text{for all } \gamma\in\Gamma \tag{179}$$

$$\omega(\gamma,\delta)\omega(\delta,\gamma) = \mathrm{id}_V \quad \text{for all } \gamma,\delta\in\Gamma \tag{180}$$

Proof. If we take $\gamma = \delta = 0$ in (175) we get

$$\omega(0,\eta)\omega(\eta,0)\omega(\eta,0) = \mathrm{id}_V \qquad \text{for all } \eta\in\Gamma \tag{181}$$

Now $\mathrm{id}_V \in \mathrm{Lgr}(V)_0$ so if we use (176) with $\delta = \eta = 0$ and $D = E = \mathrm{id}_V$ we have

$$\omega(0,\gamma)\,\mathrm{id}_V\,\omega(\gamma,0)\,\mathrm{id}_V = \mathrm{id}_V\,\omega(0,\gamma)\,\mathrm{id}_V \qquad \text{for all } \gamma\in\Gamma$$

that is,

$$\omega(0,\gamma)\omega(\gamma,0) = \omega(0,\gamma) \qquad \text{for all } \gamma\in\Gamma \tag{182}$$

Thus

$$\mathrm{id}_V = \omega(0,\gamma)\omega(\gamma,0)\omega(0,\gamma) = \omega(0,\gamma)\omega(\gamma,0) = \omega(0,\gamma) \tag{183}$$

where we used (181) in the first equality, and (182) in the second and third. Another application of (182) yields, using (183),

$$\mathrm{id}_V\,\omega(\gamma,0) = \mathrm{id}_V \qquad \text{for all } \gamma\in\Gamma \tag{184}$$

Hence (179) is proved. To prove (180), use (175) with $\gamma = 0$:

$$\mathrm{id}_V = \omega(\delta,\eta)\omega(\eta,\delta)\omega(\delta+\eta,0) = \omega(\delta,\eta)\omega(\eta,\delta)\,\mathrm{id}_V \tag{185}$$

for all $\delta,\eta\in\Gamma$. The proof is finished. □

Let ω be a commutation operator for a Γ-graded associative algebra $A = \oplus_{\gamma\in\Gamma} A_\gamma$. On the Γ-graded linear space A we define a new product

$$\langle\cdot,\cdot\rangle_\omega : A\times A \to A$$

by setting

$$\langle C, D\rangle_\omega = CD - \omega(\gamma,\delta)\big(DC\big) \in A_{\gamma+\delta}$$

for homogenous $C \in A_\gamma$ and $D \in A_\delta$ and extending linearly. We get a new Γ- graded algebra structure on the linear space A. This algebra will be denoted by $A(\omega)$.

Proposition 65. *If A is a Γ-graded associative algebra, then $A(\omega)$ is an ω Lie algebra.*

Proof. We must check that the conditions (177) and (178) are satisfied. Equation (177) holds because

$$\begin{aligned} -\omega(\gamma,\delta)\Big(\langle D,C\rangle_\omega\Big) &= -\omega(\gamma,\delta)\Big(DC-\omega(\delta,\gamma)\big(CD\big)\Big) = \\ &= \Big(\omega(\gamma,\delta)\omega(\delta,\gamma)\Big)(CD)-\omega(\gamma,\delta)\big(DC\big) = CD-\omega(\gamma,\delta)\Big(DC\Big) = \langle C,D\rangle_\omega \end{aligned}$$

where we used (180) from Lemma 64 in the third equality. Next, we prove (178) by direct calculation, using (176) and (175).

$$\begin{aligned} &\langle\langle C,D\rangle_\omega,E\rangle_\omega+\omega(\gamma+\delta,\eta)\Big(\langle\langle E,C\rangle_\omega,D\rangle_\omega\Big)+ \\ &\qquad + \big(\omega(\gamma+\delta,\eta)\omega(\eta+\gamma,\delta)\big)\Big(\langle\langle D,E\rangle_\omega,C\rangle_\omega\Big) = \end{aligned}$$

$$\begin{aligned} &= \langle C,D\rangle_\omega E-\omega(\gamma+\delta,\eta)\big(E\langle C,D\rangle_\omega\big)+ \\ &\quad +\omega(\gamma+\delta,\eta)\Big(\langle E,C\rangle_\omega D\Big)-\omega(\gamma+\delta,\eta)\Big(\omega(\eta+\gamma,\delta)\big(D\langle E,C\rangle_\omega\big)\Big)+ \\ &\quad +\big(\omega(\gamma+\delta,\eta)\omega(\eta+\gamma,\delta)\big)\Big(\langle D,E\rangle_\omega C\Big)- \\ &\quad -\big(\omega(\gamma+\delta,\eta)\omega(\eta+\gamma,\delta)\omega(\delta+\eta,\gamma)\big)\Big(C\langle D,E\rangle_\omega\Big) = \end{aligned}$$

$$\begin{aligned} &= CDE-\omega(\gamma,\delta)(DC)E-\omega(\gamma+\delta,\eta)(ECD)+\omega(\gamma+\delta,\eta)\Big(E\omega(\gamma,\delta)\big(DC\big)\Big)+ \\ &\quad +\omega(\gamma+\delta,\eta)(ECD)-\omega(\gamma+\delta,\eta)\Big(\omega(\eta,\gamma)(CE)D\Big)- \\ &\quad -\big(\omega(\gamma+\delta,\eta)\omega(\eta+\gamma,\delta)\big)(DEC)+\big(\omega(\gamma+\delta,\eta)\omega(\eta+\gamma,\delta)\big)\Big(D\omega(\eta,\gamma)(CE)\Big)+ \\ &\quad +\big(\omega(\gamma+\delta,\eta)\omega(\eta+\gamma,\delta)\big)(DEC)-\big(\omega(\gamma+\delta,\eta)\omega(\eta+\gamma,\delta)\big)\Big(\omega(\delta,\eta)(ED)C\Big)- \\ &\quad -\big(\omega(\gamma+\delta,\eta)\omega(\eta+\gamma,\delta)\omega(\delta+\eta,\gamma)\big)(CDE)+ \\ &\quad +\big(\omega(\gamma+\delta,\eta)\omega(\eta+\gamma,\delta)\omega(\delta+\eta,\gamma)\big)\Big(C\omega(\delta,\eta)(ED)\Big) = \end{aligned}$$

$$\begin{aligned} &= -\omega(\gamma,\delta)(DC)E+\omega(\gamma+\delta,\eta)\Big(E\omega(\gamma,\delta)(DC)\Big)- \\ &\quad -\omega(\gamma+\delta,\eta)\Big(\omega(\eta,\gamma)(CE)D\Big)+\big(\omega(\gamma+\delta,\eta)\omega(\eta+\gamma,\delta)\big)\Big(D\omega(\eta,\gamma)(CE)\Big)- \\ &\quad -\big(\omega(\gamma+\delta,\eta)\omega(\eta+\gamma,\delta)\big)\Big(\omega(\delta,\eta)(ED)C\Big)+C\omega(\delta,\eta)(ED) = \end{aligned}$$

$$
\begin{aligned}
&= C\omega(\delta,\eta)(ED) - \omega(\gamma+\delta,\eta)\Big(\omega(\eta,\gamma)(CE)D\Big)+\\
&\quad + \omega(\gamma+\delta,\eta)\Big\{E\omega(\gamma,\delta)(DC) - \omega(\eta+\gamma,\delta)\Big(\omega(\delta,\eta)(ED)C\Big)\Big\}+\\
&\quad + \big(\omega(\gamma+\delta,\eta)\omega(\eta+\gamma,\delta)\big)\Big\{D\omega(\eta,\gamma)(CE) - \omega(\delta+\eta,\gamma)\Big(\omega(\gamma,\delta)(DC)E\Big)\Big\} =
\end{aligned}
$$

$$
\begin{aligned}
&= \Big(\mathtt{L}_C\omega(\delta,\eta)\mathtt{R}_D - \omega(\gamma+\delta,\eta)\mathtt{R}_D\omega(\eta,\gamma)\mathtt{L}_C\Big)(E)+\\
&\quad + \omega(\gamma+\delta,\eta)\Big\{\Big(\mathtt{L}_E\omega(\gamma,\delta)\mathtt{R}_C - \omega(\eta+\gamma,\delta)\mathtt{R}_C\omega(\delta,\eta)\mathtt{L}_E\Big)(D)\Big\}+\\
&\quad + \omega(\gamma+\delta,\eta)\omega(\eta+\gamma,\delta)\Big\{\Big(\mathtt{L}_D\omega(\eta,\gamma)\mathtt{R}_E - \omega(\delta+\eta,\gamma)\mathtt{R}_E\omega(\gamma,\delta)\mathtt{L}_D\Big)(C)\Big\} =\\
&= 0
\end{aligned}
$$

where we could use (176) in the last equality, because $\mathtt{R}_D$ and $\mathtt{L}_D$ are homogenous linear operators on A of degree δ, whenever D is homogenous of degree δ. Also, right and left multiplication operators commute, since A is associative. We have shown that (177) and (178) holds in $A(\omega)$, hence it is a Γ-graded ω Lie algebra. □

Proposition 66. *Let V be a Γ-graded linear space and suppose $\omega : \Gamma \times \Gamma \to \mathrm{Lgr}(V)_0$ is a function such that*

$$\omega(\alpha,\beta)\omega(\beta,\alpha) = \mathrm{id}_V \tag{186}$$

$$\omega(\alpha+\beta,\gamma) = \omega(\alpha,\gamma)\omega(\beta,\gamma) \tag{187}$$

$$\omega(\alpha,\beta)A\omega(\beta,\gamma)C = \omega(\beta,\gamma)C\omega(\alpha,\beta)A \tag{188}$$

for all $\alpha,\beta,\gamma \in \Gamma$ and $A \in \mathrm{Lgr}(V)_\alpha, C \in \mathrm{Lgr}(V)_\gamma$ such that $AC = CA$. Then ω is a commutation operator for V. Furthermore, if A is an ω Lie algebra with this ω, equation (178) is equivalent to

$$\omega(\gamma,\alpha)\Big(\langle\langle A,B\rangle,C\rangle\Big) + \omega(\beta,\gamma)\Big(\langle\langle C,A\rangle,B\rangle\Big) + \omega(\alpha,\beta)\Big(\langle\langle B,C\rangle,A\rangle\Big) = 0 \tag{189}$$

Proof. We check that conditions (175) and (176) are satisfied. We have for all $\gamma,\delta,\eta \in \Gamma$,

$$
\begin{aligned}
\omega(\gamma+\delta,\eta)\omega(\eta+\gamma,\delta)\omega(\delta+\eta,\gamma) &= \omega(\gamma,\eta)\omega(\delta,\eta)\omega(\eta,\delta)\omega(\gamma,\delta)\omega(\delta,\gamma)\omega(\eta,\gamma) =\\
&= \omega(\gamma,\eta)\,\mathrm{id}_V\,\mathrm{id}_V\,\omega(\eta,\gamma) = \mathrm{id}_V
\end{aligned}
$$

where we used (186) and (187). Thus (175) is true. Also, when $D \in \mathrm{Lgr}(V)_\delta$ and $E \in \mathrm{Lgr}(V)_\eta$ are two commuting homogenous linear mappings on V,

$$
\begin{aligned}
\omega(\delta+\eta,\gamma)E\omega(\gamma,\delta)D &= \omega(\delta,\gamma)\omega(\eta,\gamma)E\omega(\gamma,\delta)D =\\
&= \omega(\delta,\gamma)\omega(\gamma,\delta)D\omega(\eta,\gamma)E = D\omega(\eta,\gamma)E
\end{aligned}
$$

where we used equations (186)-(188). Therefore (176) holds, and we have shown that ω is a commutation operator for V.

Now let A be an ω Lie algebra with multiplication $\langle\cdot,\cdot\rangle$, where the mapping ω satisfies equations (186)-(188). Then, by definition of an ω Lie algebra, equation (178) is fulfilled. That is, we have for all homogenous $C \in \mathrm{Lgr}(A)_\gamma$, $D \in \mathrm{Lgr}(A)_\delta$ and $E \in \mathrm{Lgr}(A)_\eta$,

$$\langle\langle C,D\rangle,E\rangle + \omega(\gamma+\delta,\eta)\Big(\langle\langle E,C\rangle,D\rangle\Big) + (\omega(\gamma+\delta,\eta)\omega(\eta+\gamma,\delta))\Big(\langle\langle D,E\rangle,C\rangle\Big) = 0$$

If we apply the invertible operator $\omega(\eta,\gamma)$ to both sides of this equation and use (187) we get the equivalent equation

$$\omega(\eta,\gamma)\Big(\langle\langle C,D\rangle,E\rangle\Big) + (\omega(\eta,\gamma)\omega(\gamma,\eta)\omega(\delta,\eta))\Big(\langle\langle E,C\rangle,D\rangle\Big) + \\ + (\omega(\eta,\gamma)\omega(\gamma,\eta)\omega(\delta,\eta)\omega(\eta,\delta)\omega(\gamma,\delta))\Big(\langle\langle D,E\rangle,C\rangle\Big) = 0$$

Finally, using (186), this is equivalent to

$$\omega(\eta,\gamma)\Big(\langle\langle C,D\rangle,E\rangle\Big) + \omega(\delta,\eta)\Big(\langle\langle E,C\rangle,D\rangle\Big) + \omega(\gamma,\delta)\Big(\langle\langle D,E\rangle,C\rangle\Big) = 0$$

which was to be proved. □

The following example shows in which sence commutation operators generalize commutation factors.

Example 6. Let $V = \oplus_{\gamma\in\Gamma} V_\gamma$ be a Γ-graded linear space and let ϵ be a commutation factor on Γ. Then define a map

$$\omega_\epsilon : \Gamma \times \Gamma \to \mathrm{Lgr}(V)_0 \tag{190}$$

by setting

$$\omega_\epsilon(\gamma,\delta) = \epsilon(\gamma,\delta)\cdot \mathrm{id}_V \tag{191}$$

for all $\gamma,\delta \in \Gamma$ and $v \in V$, where id_V is the identity operator on V. Clearly $\omega_\epsilon(\gamma,\delta)$ is homogenous of degree zero for any $\gamma,\delta\in\Gamma$. We claim that ω_ϵ is a commutation operator for V. To prove this, we can use Proposition 66. It is easy to see, using the properties of the commutation factor ϵ, that (186) and (187) are satisfied with $\omega = \omega_\epsilon$. We check the third condition (188). Let $\alpha,\beta,\gamma\in\Gamma$ and let $A \in \mathrm{Lgr}(V)_\alpha$, $C \in \mathrm{Lgr}(V)_\gamma$ be commuting homogenous linear operators. Then

$$\omega_\epsilon(\alpha,\beta)A\omega_\epsilon(\beta,\gamma)C = \epsilon(\alpha,\beta)A\epsilon(\beta,\gamma)C = \epsilon(\beta,\gamma)C\epsilon(\alpha,\beta)A = \omega_\epsilon(\beta,\gamma)C\omega_\epsilon(\alpha,\beta)A$$

Thus, by Proposition 66, ω_ϵ is a commutation operator.

Furthermore, any Γ-graded ϵ Lie algebra is a Γ-graded ω_ϵ Lie algebra. This follows since in any Γ-graded ϵ Lie algebra, the two identities (21) and (22) are true. Using this, and the definition (190)-(191) of ω_ϵ, one can easily verify that (177) and (189) are true. But by Proposition 66, (189) is equivalent to (178).

Remark 15. It would be interesting to construct examples of commutation operators and Γ-graded ω Lie algebras which does not come from commutation factors and Γ-graded ϵ-Lie algebras.

10 Homogenous (σ,τ)-derivations on the Witt Lie algebra

In Section 10.1 we consider (σ,τ)-derivations from a groupoid with unit to a field. Then in Section 10.2 we show how these (σ,τ)-derivations can induce (σ,τ)-derivations on more complicated structures. In the final subsection we prove that any homogenous (σ,τ)-derivation of degree zero on the Witt algebra can be obtained via such a construction.

10.1 (σ,τ)-Derivations from a groupoid with unit to a field

A *groupoid* B is a set with binary operation. By a nonassociative ring we mean a ring whose multiplication is not necessarily associative.

Definition 11. Let B be a groupoid, R be a nonassociative ring and let $\sigma,\tau : B \to R$ be any two functions. Then a map $D : B \to R$ which satisfies

$$D(xy) = D(x)\tau(y) + \sigma(x)D(y) \tag{192}$$

is called a (σ,τ)-*derivation from* B *to* R. Denote by $\mathfrak{D}_{(\sigma,\tau)}(B,R)$ the set of all (σ,τ)-derivations from B to R.

Since the zero map from B to R, which takes every $x \in B$ to the zero element in R, is a (σ,τ)-derivation for any σ and τ, we are interested in the pairs (σ,τ) of maps $B \to R$ for which there exist nonzero (σ,τ)-derivations from B to R. The set of such pairs (σ,τ) will be denoted by $\mathcal{M}(B,R)$:

$$\mathcal{M}(B,R) = \{(\sigma,\tau) \in R^B \times R^B \mid \mathfrak{D}_{(\sigma,\tau)}(B,R) \neq \{0\}\}, \tag{193}$$

where R^B denotes the set of functions from B to R.

In this section we will investigate $\mathcal{M}(B,R)$ and the corresponding sets $\mathfrak{D}_{(\sigma,\tau)}(B,R)$ for $(\sigma,\tau) \in \mathcal{M}(B,R)$, in the case when the groupoid B has a neutral element $e \in B$ and R is a field F. The set of functions from B to F, which we will denote by F^B, has a natural structure of a ring by pointwise addition and multiplication:

$$(f \cdot g)(x) = f(x) \cdot g(x), \qquad (f+g)(x) = f(x) + g(x).$$

Embedding F into F^B by identifying each $a \in F$ with the constant function $f_a \in F^B$ defined by $f_a(x) = a$ for all $x \in B$, and restricting the ring multiplication in F^B to

$F \times F^B$, F^B can be viewed as a linear space over F. We have the following simple lemma.

Lemma 67. *Let B be a groupoid, and F a field. Let $\sigma, \tau : B \to F$ be any maps. Then $\mathfrak{D}_{(\sigma,\tau)}(B, F)$ is an F-linear subspace of F^B.*

Proof. Let $c, d \in F$ and $D, E \in \mathfrak{D}_{(\sigma,\tau)}(B, F)$. Then

$$\begin{aligned}(c \cdot D + d \cdot E)(xy) &= c \cdot D(xy) + d \cdot E(xy) = \\ &= cD(x)\tau(y) + c\sigma(x)D(y) + dE(x)\tau(y) + d\sigma(x)E(y) = \\ &= cD(x)\tau(y) + dE(x)\tau(y) + \sigma(x)cD(y) + \sigma(x)dE(y) = \\ &= (c \cdot D + d \cdot E)(x)\tau(y) + \sigma(x)(c \cdot D + d \cdot E)(y)\end{aligned}$$

for all $x, y \in B$, since F is commutative. Thus $c \cdot D + d \cdot E \in \mathfrak{D}_{(\sigma,\tau)}(B, F)$. □

Next we will prove a proposition which will allow us to split $\mathcal{M}(B, F)$ into two disjoint sets.

Proposition 68. *Let B be a groupoid with unit e, F be a field, and let $(\sigma, \tau) \in \mathcal{M}(B, F)$. Then the following three statements are equivalent*

1) $\sigma(e) = \tau(e) = 1$,
2) $\sigma(e) + \tau(e) \neq 1$,
3) $D(e) = 0$ *for every* $D \in \mathfrak{D}_{(\sigma,\tau)}(B, F)$.

Proof. Clearly 1) implies 2). Suppose that 2) holds. Then for every (σ, τ)-derivation D from B to F we have

$$D(e) = D(e \cdot e) = D(e)\tau(e) + \sigma(e)D(e)$$

so that,

$$D(e)(1 - \tau(e) - \sigma(e)) = 0,$$

and therefore $D(e) = 0$, since F has no zero-divizors. Thus 2) implies 3). Finally, suppose that 3) holds. Then since $(\sigma, \tau) \in \mathcal{M}(B, F)$, there is a nonzero (σ, τ)-derivation D from B to F. Pick an $x \in B$ such that $D(x) \neq 0$. Then, since we assumed $D(e) = 0$ we get

$$(1 - \sigma(e))D(x) = D(e \cdot x) - \sigma(e)D(x) = D(e)\tau(x) + \sigma(e)D(x) - \sigma(e)D(x) = 0$$

and

$$D(x)(1 - \tau(e)) = D(x \cdot e) - D(x)\tau(e) = D(x)\tau(e) + \sigma(x)D(e) - D(x)\tau(e) = 0.$$

Thus $\sigma(e) = 1$ and $\tau(e) = 1$, because F has no zero-divizors. □

Remark 16. The same proof holds when F is replaced by an arbitrary nonassociative ring with unit and no zero-divizors.

Corollary 69. *Let B be a groupoid with unit e, F be a field, and let $(\sigma,\tau) \in \mathcal{M}(B,F)$. Then either $\sigma(e)+\tau(e)=1$ or $\sigma(e)=\tau(e)=1$.*

Form the following two subsets of $\mathcal{M}(B,F)$:

$$\mathcal{N}_1(B,F) = \{(\sigma,\tau) \in \mathcal{M}(B,F) \bigm| \sigma(e)+\tau(e)=1\} \tag{194}$$

$$\mathcal{N}_2(B,F) = \{(\sigma,\tau) \in \mathcal{M}(B,F) \bigm| \sigma(e)=\tau(e)=1\} \tag{195}$$

Corollary 69 shows that $\mathcal{N}_1(B,F)$ and $\mathcal{N}_2(B,F)$ form a partition of $\mathcal{M}(B,F)$ into disjoint sets. We will now study the first of these cases. It would of course also be interesting to study the other case.

10.1.1 The case $\sigma(e)+\tau(e)=1$.

Theorem 70. *Let B be a groupoid with unit e, F be a field, and $(\sigma,\tau) \in \mathcal{N}_1(B,F)$. Then $\mathfrak{D}_{(\sigma,\tau)}(B,F)$ is one-dimensional as a linear space over F, and $\sigma+\tau$ is a basisvector. Furthermore, the coordinate of $D \in \mathfrak{D}_{(\sigma,\tau)}(B,F)$ in the basis $\{\sigma+\tau\}$ equals $D(e)$.*

Proof. If $(\sigma,\tau) \in \mathcal{N}_1(B,F) \subseteq \mathcal{M}(B,F)$ we have that $\mathfrak{D}_{(\sigma,\tau)}(B,F) \neq \{0\}$. Thus it is enough to show that $\mathfrak{D}_{(\sigma,\tau)}(B,F) \subseteq F\cdot(\sigma+\tau)$. Let $D \in \mathfrak{D}_{(\sigma,\tau)}(B,F)$ be arbitrary. First we observe, using $\sigma(e)+\tau(e)=1$, that

$$\begin{aligned}\sigma(x)D(e) - D(x)\sigma(e) &= \sigma(x)D(e) - D(x)(1-\tau(e)) = \\ &= D(x)\tau(e) + \sigma(x)D(e) - D(x) = D(x\cdot e) - D(x) = 0\end{aligned}$$

and therefore

$$D(e)\sigma(x) = D(x)\sigma(e). \tag{196}$$

Similarly, we have

$$\begin{aligned}D(e)\tau(x) - \tau(e)D(x) &= D(e)\tau(x) - (1-\sigma(e))D(x) = \\ &= D(e)\tau(x) + \sigma(e)D(x) - D(x) = D(e\cdot x) - D(x) = 0\end{aligned}$$

which imply

$$D(e)\tau(x) = D(x)\tau(e). \tag{197}$$

Adding (196) and (197) yields

$$D(e)(\sigma(x)+\tau(x)) = D(x)(\sigma(e)+\tau(e)) = D(x).$$

In other words, $D = D(e)\cdot(\sigma+\tau) \in F\cdot(\sigma+\tau)$. $\square$

Corollary 71. *Let B be a groupoid with unit e, F be a field, and $(\sigma,\tau) \in \mathcal{N}_1(B,F)$. Then σ and τ are both (σ,τ)-derivations from B to F, and*

$$\sigma(x)\tau(e) = \sigma(e)\tau(x) \tag{198}$$

holds for all $x \in B$. In particular, if $\sigma(e) = 0$, then $\sigma = 0$ and if $\tau(e) = 0$, then $\tau = 0$.

Proof. Let D be a (σ,τ)-derivation from B to F such that $D(e) \neq 0$. Such a D exists, since otherwise, by Proposition 68, we would have $\sigma(e) + \tau(e) \neq 1$ which contradicts that $(\sigma,\tau) \in \mathcal{N}_1(B,F)$. From (196) and (197) follows

$$\sigma(x) = \big(D(e)^{-1}\sigma(e)\big)D(x) \tag{199}$$

$$\tau(x) = \big(D(e)^{-1}\tau(e)\big)D(x) \tag{200}$$

for all $x \in B$. Thus σ and τ are also (σ,τ)-derivations from B to F. Therefore, by Theorem 70,

$$\begin{aligned}\sigma(x) = \sigma(e)(\sigma+\tau)(x) &= \sigma(e)\sigma(x) + \sigma(e)\tau(x) = \\ &= (1-\tau(e))\sigma(x) + \sigma(e)\tau(x) = \sigma(x) - \tau(e)\sigma(x) + \sigma(e)\tau(x),\end{aligned}$$

for all $x \in B$, so that

$$\sigma(x)\tau(e) = \sigma(e)\tau(x)$$

for all $x \in B$. Finally, if $\sigma(e) = 0$, then $\tau(e) = 1$, and if $\tau(e) = 0$, then $\sigma(e) = 1$, so the last part follows directly from equation (198). □

If R is a ring, let $\langle R,\cdot\rangle$ denote the multiplicative groupoid of R whose underlying set is R and whose operation is the multiplication $\cdot$ from the ring R.

Theorem 72. *Let B be a groupoid with unit e, and F be a field. Then*

$$\mathcal{N}_1(B,F) = \{(p\varphi, q\varphi) \mid p,q \in F, p+q = 1, \varphi \in \mathrm{Hom}(B,\langle F,\cdot\rangle), \varphi \neq 0\}.$$

Proof. Let $(\sigma,\tau) \in \mathcal{N}_1(B,F)$. Assume first that $\sigma(e) = 0$. Then by Corollary 71, $\sigma = 0$. Also, by Corollary 71, τ is a (σ,τ)-derivation from B to F. Thus for $x,y \in B$,

$$\tau(xy) = \tau(x)\tau(y) + \sigma(x)\tau(y) = \tau(x)\tau(y),$$

which shows that τ is a homomorphism from B to $\langle F,\cdot\rangle$. Also τ is nonzero, since $\tau(e) = 1 - \sigma(e) = 1$. So $(\sigma,\tau) = (p\varphi, q\varphi)$ with $p = 0$, $q = 1$, and $\varphi = \tau$. Now assume $\sigma(e) \neq 0$. Then by equation (198) in Corollary 71 we have

$$\tau(x) = \sigma(e)^{-1}\tau(e)\sigma(x) = \sigma(e)^{-1}(1-\sigma(e))\sigma(x) = (\sigma(e)^{-1} - 1)\sigma(x) \tag{201}$$

for all $x \in B$. By Corollary 71, σ is a (σ, τ)-derivation, so for any $x, y \in B$,

$$\begin{aligned}\sigma(xy) &= \sigma(x)\tau(y) + \sigma(x)\sigma(y) = \\ &= \sigma(x)(\sigma(e)^{-1} - 1)\sigma(y) + \sigma(x)\sigma(y) = \\ &= \sigma(e)^{-1}\sigma(x)\sigma(y),\end{aligned} \tag{202}$$

where we used (201) in the second equality. Now, define $\varphi : B \to F$ by $\varphi(x) = \sigma(e)^{-1}\sigma(x)$. Then, φ is nonzero, since $\varphi(e) = 1$ and from (202) follows

$$\varphi(xy) = \sigma(e)^{-1}\sigma(xy) = \sigma(e)^{-1}\sigma(e)^{-1}\sigma(x)s(y) = \sigma(e)^{-1}\sigma(x)\sigma(e)^{-1}\sigma(y) = \varphi(x)\varphi(y)$$

which shows that φ is a homomorphism from B to $\langle F, \cdot \rangle$. Let $p = \sigma(e)$ and $q = 1 - \sigma(e)$. Then $\sigma(x) = p\varphi(x)$ by definition of φ, and $\tau(x) = q\varphi(x)$ by (201).

Conversely, let $p, q \in F$ be such that $p+q = 1$, and let φ be a nonzero homomorphism from B to $\langle F, \cdot \rangle$. Let $x \in B$ be such that $\varphi(x) \neq 0$. Then

$$1 = \varphi(x)^{-1}\varphi(x) = \varphi(x)^{-1}\varphi(x \cdot e) = \varphi(x)^{-1}\varphi(x)\varphi(e) = \varphi(e)$$

so that

$$p\varphi(e) + q\varphi(e) = p + q = 1. \tag{203}$$

Also, for all $x, y \in B$ we have

$$\varphi(xy) = \varphi(x)\varphi(y) = \varphi(x)q\varphi(y) + p\varphi(x)\varphi(y)$$

which shows that φ is a $(p\varphi, q\varphi)$-derivation. Equation (203) together with the existence of the nonzero $(p\varphi, q\varphi)$-derivation φ, shows that indeed $(p\varphi, q\varphi) \in \mathcal{N}_1(B, F)$ which was to be proved. $\square$

10.2 (σ, τ)-Derivations induced from grading groupoid

Let B be a groupoid and R an (associative) commutative ring. Consider the R-module

$$A = \bigoplus_{b \in B} A_b \tag{204}$$

where each A_b is a cyclic R-module generated by some element which we denote by x_b:

$$A_b = \{rx_b + nx_b \mid r \in R, n \in \mathbb{Z}\} \tag{205}$$

When α and β are functions from B to R, we can consider (α, β)-derivations δ from B to R. In this subsection we show that we can associate to every such δ, an R-linear map from A to A which is a (σ, τ)-derivation, with σ and τ depending on α and β.

Let $\mathrm{ModEnd}_R(A)$ be the R-algebra of all R-module endomorphisms on A, and let $\mathrm{ModEnd}_R(A)_0$ denote the subalgebra of $\mathrm{ModEnd}_R(A)$ consisting of those R-module endomorphisms which map each A_b into itself. Let R^B denote the commutative R-algebra of functions $B \to R$, with pointwise operations. Define

$$\Phi : R^B \to \mathrm{ModEnd}_R(A)_0 \tag{206}$$

by letting $\Phi(f), f \in R^B$, be the R-module endomorphism determined by

$$\Phi(f)(x_b) = f(b)x_b \qquad \text{for } b \in B \tag{207}$$

It is clear that $\Phi(f)$ maps each A_b into itself.

Proposition 73. *Φ is a homomorphism of R-algebras.*

Proof. For any $b \in B$, $f, g \in R^B$ and $r \in R$ we have

$$\begin{aligned}\Phi(f+g)(x_b) &= (f+g)(b)x_b = (f(b)+g(b))x_b = f(b)x_b + g(b)x_b = \\ &= \Phi(f)(x_b) + \Phi(g)(x_b) = \big(\Phi(f)+\Phi(g)\big)(x_b),\end{aligned}$$

$$\Phi(r\cdot f)(x_b) = (r\cdot f)(b)x_b = r\cdot f(b)x_b = r\cdot\Phi(f)(x_b) = \big(r\cdot\Phi(f)\big)(x_b),$$

$$\begin{aligned}\Phi(f\cdot g)(x_b) &= (f\cdot g)(b)x_b = (f(b)\cdot g(b))x_b = f(b)g(b)x_b = g(b)f(b)x_b = \\ &= g(b)\Phi(f)(x_b) = \Phi(f)(g(b)x_b) = \Phi(f)\big(\Phi(g)(x_b)\big) = \big(\Phi(f)\circ\Phi(g)\big)(x_b).\end{aligned}$$

□

Remark 17. Note that Φ is surjective if R is a ring with unity 1, and Φ is injective if and only if all the cyclic R-modules A_b are free R-modules. In particular, when R is a field, Φ is bijective.

Suppose that the R-module A has been given the structure of a B-graded R-algebra, i.e. suppose we have introduced an R-algebra multiplication on A such that $A_bA_c \subseteq A_{bc}$ for all $b, c \in B$. Then we let $\mathrm{AlgEnd}_R(A)_0$ be the subset of $\mathrm{ModEnd}_R(A)_0$ consisting of the R-algebra endomorphisms on A which map each A_b into itself. Let $\langle R, \cdot\rangle$ denote the multiplicative groupoid of R.

Proposition 74. *If $f \in R^B$ is a groupoid homomorphism from B to $\langle R, \cdot\rangle$, that is, if*

$$f(bc) = f(b)f(c) \tag{208}$$

for all $b, c \in B$, then $\Phi(f)$ is an R-algebra endomorphism of A. In other words,

$$\Phi\big(\,\mathrm{Hom}(B, \langle R, \cdot\rangle)\big) \subseteq \mathrm{AlgEnd}_R(A)_0. \tag{209}$$

Proof. Let $b, c \in B$. Then, since A is B-graded, $x_b x_c \in A_{bc}$ and thus for $f \in \mathrm{Hom}(B, \langle R, \cdot \rangle)$ we have

$$\Phi(f)(x_b x_c) = f(bc)x_b x_c = f(b)f(c)x_b x_c = f(b)x_b f(c)x_c = \Phi(f)(x_b) \cdot \Phi(f)(x_c),$$

which proves the claim. □

Proposition 75. *If $\alpha, \beta \in R^B$ are maps and $\delta \in R^B$ is an (α, β)-derivation from B to R, that is,*

$$\delta(bc) = \delta(b)\beta(c) + \alpha(b)\delta(c) \tag{210}$$

for $b, c \in B$, then $\Phi(\delta)$ is a $(\Phi(\alpha), \Phi(\beta))$-derivation on A, mapping each A_b into itself. In other words

$$\Phi\big(\mathfrak{D}_{(\alpha,\beta)}(B, R)\big) \subseteq \mathfrak{D}_{(\Phi(\alpha),\Phi(\beta))}(A)_0 \tag{211}$$

Proof. Let $b, c \in B$. Then, since A is B-graded, $x_b x_c \in A_{bc}$ and thus for $\delta \in \mathfrak{D}_{(\alpha,\beta)}(B, R)$ we have

$$\begin{aligned}\Phi(\delta)(x_b x_c) &= \delta(bc)x_b x_c = \big(\delta(b)\beta(c) + \alpha(b)\delta(c)\big)x_b x_c = \\ &= \delta(b)x_b\beta(c)x_c + \alpha(b)x_b\delta(c)x_c = \Phi(\delta)(x_b)\Phi(\beta)(x_c) + \Phi(\alpha)(x_b)\Phi(\delta)(x_c).\end{aligned}$$

□

10.3 Homogenous (σ, τ)-derivations on the Witt algebra

The Witt algebra $\mathfrak{d}$ is an algebra with basis $\{d_n \mid n \in \mathbb{Z}\}$ and the relation

$$[d_m, d_n] = (m - n)d_{m+n}. \tag{212}$$

It is easy to check that $\mathfrak{d}$ is a Lie algebra. Note also that $\mathfrak{d}$ is $\mathbb{Z}$-graded with the grading

$$\mathfrak{d} = \oplus_{n\in\mathbb{Z}}\mathbb{C}d_n.$$

If we let B be the group $\langle \mathbb{Z}, + \rangle$ of integers under addition, R be the field $\mathbb{C}$ of complex numbers, A be the Witt algebra $\mathfrak{d}$, and $A_n = \mathbb{C}d_n$, then we can construct the map Φ, as defined in (206)-(207), which, since $\mathbb{C}$ is a field, is bijective (see Remark 17).

Proposition 76. *If σ is a nonzero homogenous endomorphism on $\mathfrak{d}$ of degree s, then $s = 0$.*

Proof. Write $\sigma(d_n) = \alpha_n d_{n+s}$, where $\alpha_n \in \mathbb{C}$ for $n \in \mathbb{Z}$. Then

$$\begin{aligned}0 &= \sigma([d_m, d_n] - (m - n)d_{m+n}) = [\sigma(d_m), \sigma(d_n)] - (m - n)\sigma(d_{m+n}) = \\ &= [\alpha_m d_{m+s}, \alpha_n d_{n+s}] - (m - n)\alpha_{m+n}d_{m+n+s} = \\ &= \alpha_m\alpha_n(m + s - n - s)d_{m+n+2s} - (m - n)\alpha_{m+n}d_{m+n+s}.\end{aligned} \tag{213}$$

Thus, if $s \neq 0$, we must have in particular $(m-n)\alpha_{m+n} = 0$ for all integers m, n. Taking $m = 1$, $n = -1$ we see that $2\alpha_0 = 0$, and for $n = 0$, and m arbitrary nonzero, we have $m\alpha_m = 0$. Thus $\alpha_n = 0$ for all integers n, which contradicts the assumption of σ being nonzero. □

Theorem 77. *Let σ be a homogenous endomorphism on $\mathfrak{d}$ of degrees zero. Then there exists a unique groupoid homomorphism $\varphi \in \mathrm{Hom}(\langle \mathbb{Z}, +\rangle, \langle \mathbb{C}, \cdot\rangle)$ such that $\sigma = \Phi(\varphi)$, where Φ is the map defined in (206)-(207).*

Remark 18. In other words, when B is the group $\langle \mathbb{Z}, +\rangle$ of integers under addition, R is the field $\mathbb{C}$ of complex numbers, and A is the Witt algebra $\mathfrak{d} = \oplus_{n\in\mathbb{Z}}\mathbb{C}d_n$ we have equality in (209).

Proof. Write $\sigma(d_n) = \alpha_n d_n$. Uniqueness of φ is clear, since Φ is injective. In fact, if there is such a φ we must have $\sigma(d_n) = \Phi(\varphi)(d_n) = \varphi(n)d_n$. Thus $\varphi(n) = \alpha_n$. We must now prove that if we define φ in this way, it is a homomorphism from $\langle \mathbb{Z}, +\rangle$ to $\langle \mathbb{C}, \cdot\rangle$. From (213) follows that

$$(m-n)(\alpha_m\alpha_n - \alpha_{m+n}) = 0$$

for all integers m, n. Thus $\alpha_m\alpha_n = \alpha_{m+n}$ for all $m \neq n$, which imply

$$\alpha_m(\alpha_0 - 1) = 0 \qquad \text{for } m \neq 0.$$

If $\alpha_m = 0$ for all $m \neq 0$, then $\alpha_0 = \alpha_1\alpha_{-1} = 0$ so that φ is the zero map, which is a homomorphism. Otherwise, we must have $\alpha_0 = 1$. Then $\alpha_1\alpha_{-1} = \alpha_0 = 1$ so $\alpha_1 \neq 0$ and $\alpha_{-1} \neq 0$. Therefore, we have for $n \neq 1$,

$$\alpha_n\alpha_n = \alpha_n\alpha_n\alpha_1\alpha_1^{-1} = \alpha_n\alpha_{n+1}\alpha_1^{-1} = \alpha_{2n+1}\alpha_1^{-1} = \alpha_{2n}\alpha_1\alpha_1^{-1} = \alpha_{2n},$$

and for $n \neq -1$,

$$\alpha_n\alpha_n = \alpha_n\alpha_n\alpha_{-1}\alpha_{-1}^{-1} = \alpha_n\alpha_{n-1}\alpha_{-1}^{-1} = \alpha_{2n-1}\alpha_{-1}^{-1} = \alpha_{2n}\alpha_{-1}\alpha_{-1}^{-1} = \alpha_{2n}.$$

Thus $\alpha_m\alpha_n = \alpha_{m+n}$ for all integers m, n. Therefore φ is a homomorphism from $\langle \mathbb{Z}, +\rangle$ to $\langle \mathbb{C}, \cdot\rangle$, satisfying $\Phi(\varphi) = \sigma$. □

Theorem 78. *Let f and g be homomorphisms from $\langle \mathbb{Z}, +\rangle$ to $\langle \mathbb{C}, \cdot\rangle$, and let D be a homogenous $(\Phi(f), \Phi(g))$-derivation on $\mathfrak{d}$ of degree zero. Then there is a unique (f, g)-derivation δ from $\langle \mathbb{Z}, +\rangle$ to $\mathbb{C}$ such that $D = \Phi(\delta)$ where Φ is the map defined in (206)-(207).*

Remark 19. In other words, when B is the group $\langle \mathbb{Z}, +\rangle$ of integers under addition, R is the field $\mathbb{C}$ of complex numbers, and A is the Witt algebra $\mathfrak{d} = \oplus_{n\in\mathbb{Z}}\mathbb{C}d_n$ we have equality in (211).

Proof. Write $D(d_n) = \gamma_n d_n$. Uniqueness of δ is clear, since Φ is injective. In fact, as before, $\delta(n) = \gamma_n$. We must now prove that if we define δ in this way, it is an (f, g)-derivation from $\langle \mathbb{Z}, + \rangle$ to $\mathbb{C}$.

Suppose first that $f = g = 0$. Then $\Phi(f) = \Phi(g) = 0$ also, so that

$$D([d_m, d_n]) = [D(d_m), \Phi(g)(d_n)] + [\Phi(f)(d_m), D(d_n)] = 0.$$

But we also have

$$D([d_m, d_n]) = D((m-n)d_{m+n}) = (m-n)\delta(m+n)d_{m+n}, \tag{214}$$

for all $m, n \in \mathbb{Z}$. Therefore we have

$$(m-n)\delta(m+n) = 0, \quad \text{for } n, m \in \mathbb{Z}. \tag{215}$$

If we choose $n = 0$ and $m \neq 0$ in (215), we obtain $\delta(m) = 0$ for all $m \neq 0$. Taking $n = -1$, $m = 1$ in (215) we also get $\delta(0) = 0$. Thus $\delta(n) = 0$ for all $n \in \mathbb{Z}$ so δ is trivially an (f, g)-derivation from $\langle \mathbb{Z}, + \rangle$ to $\mathbb{C}$.

Next, suppose that $f \neq 0$ (the case $g \neq 0$ is symmetric). Then, since $f(0)^2 = f(0+0) = f(0)$, we have either $f(0) = 0$ or $f(0) = 1$. But if $f(0) = 0$ we would have

$$f(n) = f(n+0) = f(n)f(0) = 0$$

for all n which contradicts $f \neq 0$. Thus $f(0) = 1$. This fact implies that

$$f(1)f(-1) = f(0) = 1, \tag{216}$$

which in particular means that $f(1)$ and $f(-1)$ both are nonzero. Now

$$\begin{aligned} D([d_m, d_n]) &= [D(d_m), \Phi(g)(d_n)] + [\Phi(f)(d_m), D(d_n)] = \\ &= [\delta(m)d_m, g(n)d_n] + [f(m)d_m, \delta(n)d_n] = \\ &= \big(\delta(m)g(n) + f(m)\delta(n)\big)(m-n)d_{m+n}. \end{aligned}$$

Using this and (214) we obtain

$$\delta(m+n) = \delta(m)g(n) + f(m)\delta(n), \quad \text{for all } m, n \in \mathbb{Z}, m \neq n. \tag{217}$$

The proof is finished if we can show that (217) implies that

$$\delta(2n) = \delta(n)g(n) + f(n)\delta(n) \quad \text{for any } n \in \mathbb{Z}.$$

It can be done as follows. Let $n \in \mathbb{Z}$ be arbitrary, and let $x \in \{1, -1\}$ be such that $x \neq n$. Then $2n \neq x$ so it follows from (217) that

$$\begin{aligned} \delta(x)g(2n) + f(x)\delta(2n) &= \delta(x+2n) = \\ &= \delta((n+x)+n) = \\ &= \delta(n+x)g(n) + f(n+x)\delta(n) = \\ &= \delta(n+x)g(n) + f(n)f(x)\delta(n). \end{aligned}$$

Using that (216) implied $f(x) \neq 0$, we can solve for $\delta(2n)$ and obtain

$$\begin{aligned}
\delta(2n) &= \frac{1}{f(x)}\big(\delta(n+x)g(n) + f(n)f(x)\delta(n) - \delta(x)g(2n)\big) = \\
&= \frac{1}{f(x)}\big(\delta(x+n)g(n) + f(n)f(x)\delta(n) - \delta(x)g(n)g(n)\big) = \\
&= \frac{1}{f(x)}\Big(\big(\delta(x)g(n) + f(x)\delta(n)\big)g(n) + f(n)f(x)\delta(n) - \delta(x)g(n)g(n)\Big) = \\
&= \frac{1}{f(x)}\big(\delta(x)g(n)g(n) + f(x)\delta(n)g(n) + f(n)f(x)\delta(n) - \delta(x)g(n)g(n)\big) = \\
&= \frac{1}{f(x)}\big(f(x)\delta(n)g(n) + f(n)f(x)\delta(n)\big) = \\
&= \delta(n)g(n) + f(n)\delta(n),
\end{aligned}$$

where we in the third equality used $x \neq n$ and (217). This completes the proof. $\square$

11 A Generalization of the Witt algebra

Let $\mathbb{C}[t, t^{-1}]$ denote the algebra of all complex Laurent polynomials:

$$\mathbb{C}[t, t^{-1}] = \{\sum_{k\in\mathbb{Z}} a_k t^k \mid a_k \in \mathbb{C}, \text{only finitely many nonzero}\}.$$

The Witt algebra defined in Section 10.3 is isomorphic to the Lie algebra $\mathfrak{D}(\mathbb{C}[t, t^{-1}])$ of all derivations of $\mathbb{C}[t, t^{-1}]$. In this section we will use this fact as a starting point for a generalization of the Witt algebra to an algebra consisting of σ-derivations.

Let A be a commutative associative algebra over $\mathbb{C}$ with unity 1. If $\sigma : A \to A$ is a homomorphism of algebras, we denote as usual by $\mathfrak{D}_\sigma(A)$ the A-module of all σ-derivations on A. For clarity we will denote the module multiplication by $\cdot$ and the algebra multiplication in A by juxtaposition. The *annihilator* $\mathrm{Ann}(D)$ of an element $D \in \mathfrak{D}_\sigma(A)$ is the set of all $a \in A$ such that $a \cdot D = 0$. It is easy to see that $\mathrm{Ann}(D)$ is an ideal in A for any $D \in \mathfrak{D}_\sigma(A)$.

We fix now a homomorphism $\sigma : A \to A$, an element $\Delta \in \mathfrak{D}_\sigma(A)$, and an element $\delta \in A$, and we assume that these objects satisfy the following two conditions.

$$\sigma(\mathrm{Ann}(\Delta)) \subseteq \mathrm{Ann}(\Delta), \tag{218}$$

$$\Delta(\sigma(a)) = \delta\sigma(\Delta(a)), \quad \text{for } a \in A. \tag{219}$$

Let

$$A \cdot \Delta = \{a \cdot \Delta \mid a \in A\}$$

denote the cyclic A-submodule of $\mathfrak{D}_\sigma(A)$ generated by Δ. In this setting, we have the following theorem, which introduces a $\mathbb{C}$-algebra structure on $A \cdot \Delta$.

Theorem 79. *The map*

$$[\cdot,\cdot]_\sigma : A\cdot\Delta \times A\cdot\Delta \to A\cdot\Delta \tag{220}$$

defined by setting

$$[a\cdot\Delta, b\cdot\Delta]_\sigma = (\sigma(a)\cdot\Delta)\circ(b\cdot\Delta) - (\sigma(b)\cdot\Delta)\circ(a\cdot\Delta), \quad \textit{for } a,b\in A, \tag{221}$$

where $\circ$ denotes composition of functions, is a well defined $\mathbb{C}$-algebra product on the linear space $A\cdot\Delta$, and it satisfies the following identities for $a,b,c\in A$:

$$[a\cdot\Delta, b\cdot\Delta]_\sigma = \big(\sigma(a)\Delta(b) - \sigma(b)\Delta(a)\big)\cdot\Delta, \tag{222}$$

$$[a\cdot\Delta, b\cdot\Delta]_\sigma = -[b\cdot\Delta, a\cdot\Delta]_\sigma, \tag{223}$$

and

$$\begin{aligned} &[\sigma(a)\cdot\Delta,[b\cdot\Delta,c\cdot\Delta]_\sigma]_\sigma + \delta\cdot[a\cdot\Delta,[b\cdot\Delta,c\cdot\Delta]_\sigma]_\sigma + \\ &+[\sigma(b)\cdot\Delta,[c\cdot\Delta,a\cdot\Delta]_\sigma]_\sigma + \delta\cdot[b\cdot\Delta,[c\cdot\Delta,a\cdot\Delta]_\sigma]_\sigma + \\ &+[\sigma(c)\cdot\Delta,[a\cdot\Delta,b\cdot\Delta]_\sigma]_\sigma + \delta\cdot[c\cdot\Delta,[a\cdot\Delta,b\cdot\Delta]_\sigma]_\sigma = 0. \end{aligned} \tag{224}$$

Proof. We must first show that $[\cdot,\cdot]_\sigma$ is a well defined function. That is, if $a_1\cdot\Delta = a_2\cdot\Delta$, then

$$[a_1\cdot\Delta, b\cdot\Delta]_\sigma = [a_2\cdot\Delta, b\cdot\Delta]_\sigma, \tag{225}$$

and

$$[b\cdot\Delta, a_1\cdot\Delta]_\sigma = [b\cdot\Delta, a_2\cdot\Delta]_\sigma, \tag{226}$$

for $b\in A$. Now $a_1\cdot\Delta = a_2\cdot\Delta$ is equivalent to $a_1 - a_2 \in \operatorname{Ann}(\Delta)$. Therefore, using the assumption (218), we also have $\sigma(a_1-a_2)\in\operatorname{Ann}(\Delta)$. Hence

$$\begin{aligned} [a_1\cdot\Delta, b\cdot\Delta]_\sigma - [a_2\cdot\Delta, b\cdot\Delta]_\sigma &= (\sigma(a_1)\cdot\Delta)\circ(b\cdot\Delta) - (\sigma(b)\cdot\Delta)\circ(a_1\cdot\Delta) \\ &\quad - (\sigma(a_2)\cdot\Delta)\circ(b\cdot\Delta) + (\sigma(b)\cdot\Delta)\circ(a_2\cdot\Delta) = \\ &= (\sigma(a_1-a_2)\cdot\Delta)\circ(b\cdot\Delta) \\ &\qquad - (\sigma(b)\cdot\Delta)\circ((a_1-a_2)\cdot\Delta) = \\ &= 0, \end{aligned}$$

which shows (225). The proof of (226) is symmetric.

Next we prove (222), which also shows that $A\cdot\Delta$ is closed under $[\cdot,\cdot]_\sigma$. Let $a,b,c\in A$ be arbitrary. Then, since Δ is a σ-derivation of A we have

$$\begin{aligned} [a\cdot\Delta, b\cdot\Delta]_\sigma(c) &= (\sigma(a)\cdot\Delta)\Big((b\cdot\Delta)(c)\Big) - (\sigma(b)\cdot\Delta)\Big((a\cdot\Delta)(c)\Big) = \\ &= \sigma(a)\Delta\big(b\Delta(c)\big) - \sigma(b)\Delta\big(a\Delta(c)\big) = \\ &= \sigma(a)\big(\Delta(b)\Delta(c) + \sigma(b)\Delta(\Delta(c))\big) \\ &\quad - \sigma(b)\big(\Delta(a)\Delta(c) + \sigma(a)\Delta(\Delta(c))\big) = \\ &= \big(\sigma(a)\Delta(b) - \sigma(b)\Delta(a)\big)\Delta(c) + (\sigma(a)\sigma(b) - \sigma(b)\sigma(a))\Delta(\Delta(c)). \end{aligned}$$

Since A is commutative, the last term is zero. Thus (222) is true. (223) is clear from the definition. Using the linearity of σ and Δ, and the definition of $[\cdot,\cdot]_\sigma$, or the formula (222), it is also easy to see that $[\cdot,\cdot]_\sigma$ is bilinear.

It remains to prove (224). For this we shall introduce some convenient notation. If $f : A \times A \times A \to A \cdot \Delta$ is a function, we will write

$$\circlearrowleft_{a,b,c} f(a,b,c)$$

for the cyclic sum

$$f(a,b,c) + f(b,c,a) + f(c,a,b).$$

We note the following properties of the cyclic sum:

$$\circlearrowleft_{a,b,c} \big(x \cdot f(a,b,c) + y \cdot g(a,b,c)\big) = x \cdot \circlearrowleft_{a,b,c} f(a,b,c) + y \cdot \circlearrowleft_{a,b,c} g(a,b,c),$$
$$\circlearrowleft_{a,b,c} f(a,b,c) = \circlearrowleft_{a,b,c} f(b,c,a) = \circlearrowleft_{a,b,c} f(c,a,b),$$

where $f, g : A \times A \times A \to A \cdot \Delta$ are two functions, and $x, y \in A$. Combining these two identities we obtain

$$\circlearrowleft_{a,b,c} \big(f(a,b,c) + g(a,b,c)\big) = \circlearrowleft_{a,b,c} \big(f(a,b,c) + g(b,c,a)\big) = \circlearrowleft_{a,b,c} \big(f(a,b,c) + g(c,a,b)\big),$$

which we shall refer to as the *shifting property* of cyclic summation. With this notation, (224) can be written

$$\circlearrowleft_{a,b,c} \Big\{[\sigma(a) \cdot \Delta, [b \cdot \Delta, c \cdot \Delta]_\sigma]_\sigma + \delta \cdot [a \cdot \Delta, [b \cdot \Delta, c \cdot \Delta]_\sigma]_\sigma\Big\} = 0. \tag{227}$$

Using (222) and that Δ is a σ-derivation of A we get

$$\begin{aligned}
&[\sigma(a) \cdot \Delta, [b \cdot \Delta, c \cdot \Delta]_\sigma]_\sigma = \\
&\quad = [\sigma(a) \cdot \Delta, \big(\sigma(b)\Delta(c) - \sigma(c)\Delta(b)\big) \cdot \Delta]_\sigma = \\
&\quad = \Big\{\sigma^2(a)\Delta\big(\sigma(b)\Delta(c) - \sigma(c)\Delta(b)\big) - \sigma\big(\sigma(b)\Delta(c) - \sigma(c)\Delta(b)\big)\Delta\big(\sigma(a)\big)\Big\} \cdot \Delta = \\
&\quad = \Big\{\sigma^2(a)\Big(\Delta\big(\sigma(b)\big)\Delta(c) + \sigma^2(b)\Delta^2(c) - \Delta\big(\sigma(c)\big)\Delta(b) - \sigma^2(c)\Delta^2(b)\Big) \\
&\qquad - \Big(\sigma^2(b)\sigma\big(\Delta(c)\big) - \sigma^2(c)\sigma\big(\Delta(b)\big)\Big)\Delta\big(\sigma(a)\big)\Big\} \cdot \Delta = \\
&\quad = \sigma^2(a)\Delta\big(\sigma(b)\big)\Delta(c) \cdot \Delta + \sigma^2(a)\sigma^2(b)\Delta^2(c) \cdot \Delta \\
&\qquad - \sigma^2(a)\Delta\big(\sigma(c)\big)\Delta(b) \cdot \Delta - \sigma^2(a)\sigma^2(c)\Delta^2(b) \cdot \Delta \\
&\qquad - \sigma^2(b)\sigma\big(\Delta(c)\big)\Delta\big(\sigma(a)\big) \cdot \Delta + \sigma^2(c)\sigma\big(\Delta(b)\big)\Delta\big(\sigma(a)\big) \cdot \Delta,
\end{aligned} \tag{228}$$

where $\sigma^2 = \sigma \circ \sigma$ and $\Delta^2 = \Delta \circ \Delta$. Applying cyclic summation to the second and fourth term in (228) we get

$$\begin{aligned}
&\circlearrowleft_{a,b,c} \Big\{\sigma^2(a)\sigma^2(b)\Delta^2(c) \cdot \Delta - \sigma^2(a)\sigma^2(c)\Delta^2(b) \cdot \Delta\Big\} = \\
&\qquad = \circlearrowleft_{a,b,c} \Big\{\sigma^2(a)\sigma^2(b)\Delta^2(c) \cdot \Delta - \sigma^2(b)\sigma^2(a)\Delta^2(c) \cdot \Delta\Big\} = 0,
\end{aligned}$$

using the shifting property and that A is commutative. Similarly, if we apply cyclic summation to the fifth and sixth term in (228) and use the relation (219) we obtain

$$\begin{aligned}
&\circlearrowleft_{a,b,c}\Big\{-\sigma^2(b)\sigma\big(\Delta(c)\big)\Delta\big(\sigma(a)\big)\cdot\Delta+\sigma^2(c)\sigma\big(\Delta(b)\big)\Delta\big(\sigma(a)\big)\cdot\Delta\Big\}=\\
&\quad=\circlearrowleft_{a,b,c}\Big\{-\sigma^2(b)\sigma\big(\Delta(c)\big)\delta\sigma\big(\Delta(a)\big)\cdot\Delta+\sigma^2(c)\sigma\big(\Delta(b)\big)\delta\sigma\big(\Delta(a)\big)\cdot\Delta\Big\}=\\
&\quad=\delta\cdot\circlearrowleft_{a,b,c}\Big\{-\sigma^2(b)\sigma\big(\Delta(c)\big)\sigma\big(\Delta(a)\big)\cdot\Delta+\sigma^2(b)\sigma\big(\Delta(a)\big)\sigma\big(\Delta(c)\big)\cdot\Delta\Big\}=0,
\end{aligned}$$

where we again used the shifting property of cyclic summation. Consequently, the only terms in the right hand side of (228) which do not vanish when we take cyclic summation are the first and the third. In other words,

$$\begin{aligned}
&\circlearrowleft_{a,b,c}[\sigma(a)\cdot\Delta,[b\cdot\Delta,c\cdot\Delta]_\sigma]_\sigma=\\
&\qquad=\circlearrowleft_{a,b,c}\Big\{\sigma^2(a)\Delta\big(\sigma(b)\big)\Delta(c)\cdot\Delta-\sigma^2(a)\Delta\big(\sigma(c)\big)\Delta(b)\cdot\Delta\Big\}.
\end{aligned}\tag{229}$$

We now consider the other term in (227). First note that from (222) we have

$$[b\cdot\Delta,c\cdot\Delta]_\sigma=\big(\Delta(c)\sigma(b)-\Delta(b)\sigma(c)\big)\cdot\Delta$$

since A is commutative. Using first this and then (222) we get

$$\begin{aligned}
&\delta\cdot[a\cdot\Delta,[b\cdot\Delta,c\cdot\Delta]_\sigma]_\sigma=\\
&\quad=\delta\cdot[a\cdot\Delta,\big(\Delta(c)\sigma(b)-\Delta(b)\sigma(c)\big)\cdot\Delta]_\sigma=\\
&\quad=\delta\Big(\sigma(a)\Delta\big(\Delta(c)\sigma(b)-\Delta(b)\sigma(c)\big)-\sigma\big(\Delta(c)\sigma(b)-\Delta(b)\sigma(c)\big)\Delta(a)\Big)\cdot\Delta=\\
&\quad=\delta\Big\{\sigma(a)\Big(\Delta^2(c)\sigma(b)+\sigma\big(\Delta(c)\big)\Delta\big(\sigma(b)\big)-\Delta^2(b)\sigma(c)-\sigma\big(\Delta(b)\big)\Delta\big(\sigma(c)\big)\Big)\\
&\qquad-\Big(\sigma\big(\Delta(c)\big)\sigma^2(b)-\sigma\big(\Delta(b)\big)\sigma^2(c)\Big)\Delta(a)\Big\}\cdot\Delta=\\
&\quad=\delta\sigma(a)\Delta^2(c)\sigma(b)\cdot\Delta+\delta\sigma(a)\sigma\big(\Delta(c)\big)\Delta\big(\sigma(b)\big)\cdot\Delta\\
&\qquad-\delta\sigma(a)\Delta^2(b)\sigma(c)\cdot\Delta-\delta\sigma(a)\sigma\big(\Delta(b)\big)\Delta\big(\sigma(c)\big)\cdot\Delta\\
&\qquad-\delta\sigma\big(\Delta(c)\big)\sigma^2(b)\Delta(a)\cdot\Delta+\delta\sigma\big(\Delta(b)\big)\sigma^2(c)\Delta(a)\cdot\Delta.
\end{aligned}$$

Using (219), this is equal to

$$\begin{aligned}
&\delta\sigma(a)\Delta^2(c)\sigma(b)\cdot\Delta+\sigma(a)\Delta\big(\sigma(c)\big)\Delta\big(\sigma(b)\big)\cdot\Delta\\
&\quad-\delta\sigma(a)\Delta^2(b)\sigma(c)\cdot\Delta-\sigma(a)\Delta\big(\sigma(b)\big)\Delta\big(\sigma(c)\big)\cdot\Delta\\
&\quad-\Delta\big(\sigma(c)\big)\sigma^2(b)\Delta(a)\cdot\Delta+\Delta\big(\sigma(b)\big)\sigma^2(c)\Delta(a)\cdot\Delta=\\
&=\delta\sigma(a)\Delta^2(c)\sigma(b)\cdot\Delta-\delta\sigma(a)\Delta^2(b)\sigma(c)\cdot\Delta\\
&\quad-\Delta\big(\sigma(c)\big)\sigma^2(b)\Delta(a)\cdot\Delta+\Delta\big(\sigma(b)\big)\sigma^2(c)\Delta(a)\cdot\Delta.
\end{aligned}$$

The first two terms of this last expression vanish after a cyclic summation and using the shifting property, so

$$\begin{aligned}\circlearrowleft_{a,b,c} \delta \cdot [a \cdot \Delta, [b \cdot \Delta, c \cdot \Delta]_\sigma]_\sigma = \\ = \circlearrowleft_{a,b,c} \Big\{ - \Delta\big(\sigma(c)\big)\sigma^2(b)\Delta(a) \cdot \Delta + \Delta\big(\sigma(b)\big)\sigma^2(c)\Delta(a) \cdot \Delta \Big\}.\end{aligned} \tag{230}$$

Finally, combining this with (229) we deduce

$$\begin{aligned}
&\circlearrowleft_{a,b,c} \Big\{ [\sigma(a) \cdot \Delta, [b \cdot \Delta, c \cdot \Delta]_\sigma]_\sigma + \delta[a \cdot \Delta, [b \cdot \Delta, c \cdot \Delta]_\sigma]_\sigma \Big\} = \\
&= \circlearrowleft_{a,b,c} [\sigma(a) \cdot \Delta, [b \cdot \Delta, c \cdot \Delta]_\sigma]_\sigma + \circlearrowleft_{a,b,c} \delta[a \cdot \Delta, [b \cdot \Delta, c \cdot \Delta]_\sigma]_\sigma = \\
&= \circlearrowleft_{a,b,c} \Big\{ \sigma^2(a)\Delta\big(\sigma(b)\big)\Delta(c) \cdot \Delta - \sigma^2(a)\Delta\big(\sigma(c)\big)\Delta(b) \cdot \Delta \Big\} \\
&\quad + \circlearrowleft_{a,b,c} \Big\{ - \Delta\big(\sigma(c)\big)\sigma^2(b)\Delta(a) \cdot \Delta + \Delta\big(\sigma(b)\big)\sigma^2(c)\Delta(a) \cdot \Delta \Big\} = \\
&= \circlearrowleft_{a,b,c} \Big\{ \sigma^2(a)\Delta\big(\sigma(b)\big)\Delta(c) \cdot \Delta - \sigma^2(a)\Delta\big(\sigma(c)\big)\Delta(b) \cdot \Delta \Big\} \\
&\quad + \circlearrowleft_{a,b,c} \Big\{ - \Delta\big(\sigma(b)\big)\sigma^2(a)\Delta(c) \cdot \Delta + \Delta\big(\sigma(c)\big)\sigma^2(a)\Delta(b) \cdot \Delta \Big\} = \\
&= 0,
\end{aligned}$$

as was to be shown. The proof is complete. $\square$

Remark 20. If A is not assumed to be commutative, the construction still works if one impose on Δ the additional condition that

$$[a, b]\Delta(c) = 0 \quad \text{for all } a, b, c \in A.$$

Then the mapping $x \cdot \Delta : b \mapsto x\Delta(b)$ is again a σ-derivation for all $x \in A$. Furthermore, since $[A, A]$ is an ideal in A,

$$[a, b]x\Delta(c) = 0 \quad \text{for all } a, b, c, x \in A.$$

Hence $A \cdot \Delta$ is a left A-module. Then Theorem 79 remain valid with the same proof. We only need to note that, although A is not commutative we have $[a, b] \cdot \Delta = 0$ which is to say that

$$ab \cdot \Delta = ba \cdot \Delta.$$

Example 7. Take $A = \mathbb{C}[t, t^{-1}]$, $\sigma = \mathrm{id}_A$, the identity operator on A, $\Delta = \frac{d}{dt}$, and $\delta = 1$. In this case one can show that $A \cdot \Delta$ is equal to the whole $\mathfrak{D}_\sigma(A)$. The conditions (218) and (219) are trivial to check. The definition (221) coincides with the usual Lie bracket of derivations, and equation (224) reduces to twice the usual Jacobi identity. Hence we recover the ordinary Witt algebra.

Example 8. Let A be a unique factorization domain, and let $\sigma : A \to A$ be a homomorphism, different from the identity. Then by Theorem 27,

$$\mathfrak{D}_\sigma(A) = A \cdot \Delta,$$

where $\Delta = \frac{\mathrm{id} - \sigma}{g}$ and $g = GCD\big((\mathrm{id} - \sigma)(A)\big)$. Furthermore, let $y \in A$ and set

$$x = \frac{\mathrm{id} - \sigma}{g}(y) = \frac{y - \sigma(y)}{g}.$$

Then we have

$$\sigma(g)\sigma(x) = \sigma(gx) = \sigma(y) - \sigma^2(y) = (\mathrm{id} - \sigma)(\sigma(y)). \tag{231}$$

From the definition of g we know that it divides $(\mathrm{id} - \sigma)(g) = g - \sigma(g)$. Thus g also divides $\sigma(g)$. When we divide (231) by g and substitute the expression for x we obtain

$$\frac{\sigma(g)}{g}\sigma\big(\frac{\mathrm{id} - \sigma}{g}(y)\big) = \frac{\mathrm{id} - \sigma}{g}(\sigma(y)),$$

or, with our notation $\Delta = \frac{\mathrm{id} - \sigma}{g}$,

$$\frac{\sigma(g)}{g}\sigma\big(\Delta(y)\big) = \Delta(\sigma(y)).$$

This shows that (219) holds with $\delta = \sigma(g)/g$. Since A has no zero-divizors and $\sigma \neq \mathrm{id}$, it follows that $\mathrm{Ann}(\Delta) = 0$ so the equation (218) is clearly true. Hence we can use Theorem 79 to define an algebra structure on $\mathfrak{D}_\sigma(A) = A \cdot \Delta$ which satisfies (223) and (224) with $\delta = \sigma(g)/g$.

Let us now make the following definition.

Definition 12. A *homomorphism Lie algebra* is a nonassociative algebra L together with an algebra homomorphism $h : L \to L$, such that

$$[x, y]_h = -[y, x]_h, \tag{232}$$

$$[(\mathrm{id} + h)(x), [y, z]_h]_h + [(\mathrm{id} + h)(y), [z, x]_h]_h + [(\mathrm{id} + h)(z), [x, y]_h]_h = 0, \tag{233}$$

where $[\cdot, \cdot]_h$ denotes the product in L.

If L is a Lie algebra, it is a homomorphism Lie algebra with its homomorphism $h = \mathrm{id}_L$ equal to the identity operator on L.

Example 9. Suppose A is a commutative associative algebra, $\sigma : A \to A$ a homomorphism, $\Delta \in \mathfrak{D}_\sigma(A)$ and $\delta \in A$ satisfy the equations (218)-(219). Then since $\sigma(\mathrm{Ann}(\Delta)) \subseteq \mathrm{Ann}(\Delta)$, the mapping σ induces a map

$$\bar{\sigma} : A \cdot \Delta \to A \cdot \Delta,$$

$$\bar{\sigma} : a \cdot \Delta \mapsto \sigma(a) \cdot \Delta.$$

This map has the following property

$$\begin{aligned}
[\bar{\sigma}(a \cdot \Delta), \bar{\sigma}(b \cdot \Delta)]_\sigma &= [\sigma(a) \cdot \Delta, \sigma(b) \cdot \Delta]_\sigma = \\
&= \big(\sigma^2(a)\Delta(\sigma(b)) - \sigma^2(b)\Delta(\sigma(a))\big) \cdot \Delta = \\
&= \big(\sigma^2(a)\delta\sigma(\Delta(b)) - \sigma^2(b)\delta\sigma(\Delta(a))\big) \cdot \Delta = \\
&= \delta\sigma\big(\sigma(a)\Delta(b) - \sigma(b)\Delta(a)\big) \cdot \Delta = \\
&= \delta \cdot \bar{\sigma}([a \cdot \Delta, b \cdot \Delta]_\sigma).
\end{aligned}$$

We suppose now that $\delta \in \mathbb{C}\backslash\{0\}$. Dividing both sides of the above calculation by δ^2 and using bilinearity of the product, we obtain that

$$\frac{1}{\delta}\bar{\sigma}$$

is an algebra homomorphism $A \cdot \Delta \to A \cdot \Delta$, and Theorem 79 makes $A \cdot \Delta$ into a homomorphism Lie algebra with $(1/\delta)\bar{\sigma}$ as its homomorphism.

References

[An] D. Anick, *"Differential Algebras in Topology"*, A K Peters, 1993.
(Differential algebras.)

[BFKL] K. I. Beidar, Y. Fong, W. F. Ke, C. H. Lee, *Posner's Theorem for Generalized (σ, τ)-derivations,* Second International Tainan-Moscow Algebra Workshop.
(Commutativity theorems in rings and (σ, τ)-derivations.)

[Bell] A. D. Bell, *Skew Differential Operators on Commutative Rings,* Contamporary Mathematics, vol. 130, 1992, pp. 49–67.
(Skew Differential operators on commutative rings.)

[Ber] F. A. Berezin, *"Introduction to algebra and analysis with anticommuting variables",* Izdatelstvo Moskovskogo Universiteta, 1983. (Russian)
(Superderivations on Lie superalgebras.)

[Bl] P. E. Block, , Ann. Math., **90** 3, 1980, 433–459.
(Differential rings.)

[BLH] L. A. Bokut, I. V. Lvov, V. K. Harchenko, *"Noncommutative Rings"*, Moscow, VINITI, 1987, 5–116, (Russian).
(σ-Derivations in rings and algebras, σ-derivations in rings, and skew-polynomial ring or ring of differential polynomials.)

[BR] O. Bratteli, D. W. Robinson, *"Operator Algebras and Statistical Mechanics"*, Springer-Verlag, vol. I, 1979, vol. II, 1981.
(Derivations on C^*-algebras and von Neumann algebras.)

[Bur] N. Burbaki, *"Algebra. Polynomials and fields. Ordered groups"*, Moscow, Nauka, 1965, (Russian translation).
(Derivations on rings of polynomials and power series.)

[ChS] F. Chyzak, B. Salvy *Non-commutative Elimination in Ore Proves Multivariate Identities*, J. Symbolic Computation, **26**, 1998, 187–227.
(Gröbner bases and algoritms for summation and construction of proofs of identities in skew-polynomial rings, Ore algebras and Ore rings with applications in q-analysis. See [W].)

[C1] P. M. Cohn, *"Difference algebra"*, New York, Intersci. Publ., 1965.
(Difference algebra.)

[C2] P. M. Cohn, *"Skew fields constructions"*, London Math. Soc. Lect. Notes Ser., 27, 1977.
(Skew fields, Ore rings, σ-derivations on rings.)

[Co] A. Connes, *"Non-commutative Geometry"*, Academic Press, 1994.
(Non-commutative geometry.)

[Da] J. Dauns, *"A Concrete Approach to Division Rings"*, Heldermann Verlag Berlin, 1982.
(Skew-polynomial rings, skew fields, division rings, σ-derivations.)

[Dr] P. K. Draxl, *"Skew fields"*, London Math. Soc. Lect. Notes Ser., 81, 1983.
(Skew fields.)

[Es] A. van den Essen, *"Polynomial Automorphisms and the Jacobian Conjecture"*, Birkhäuser Verlag, 2000.
(Connection of σ-derivations to derivations and exponentiation mapping.)

[Ex] H. Exton, *"q-Hypergeometric Functions and Applications"*, Cambridge University Press, Halsted Press, Chichister, 1983.
(q-Derivatives and q-analysis.)

[F] C. Faith, *"Algebra. Rings, modules and categories"*, Mir, vol. I, 1977, vol. II, 1979. (Russian translation).
(σ-Derivations in rings, and skew-polynomial ring or ring of differential polynomials (p. 7.27-7.30). Derivations in rings (p. 7.37-7.40).)

[FD] B. Farb, R. K. Dennis *"Non-commutative algebra"*, Springer-Verlag, 1993.
(σ-Derivations and derivations on algebras, twisted polynomial ring, Ore condition.)

[GR] G. Gasper, M. Rahman *"Basic Hypergeometric Series"*, Cambridge University Press, 1990.
(q-Derivatives and q-analysis.)

[GW] K. R. Goodearl, R. B. Warfield, Jr. *"An introduction to Non-commutative Noetherian Rings"*, London Mathematical Society Student Texts, 16, Cambridge University Press, Cambridge, 1989.
(σ-Derivations in skew polynomial ring in the theory of Noetherian rings.)

[GVF] J. M. Gracia-Bondia, J. C. Varilly, H. Figueroa, *"Elements of Non-commutative Geometry"*, Birkhäuser, 2001.
(Non-commutative geometry.)

[Har1] V. K. Harchenko, *Action of groups and algebras Lie on non-commutative rings*, Uspechi Mat. Nauk, vol.35, no. 2, 1980, pp. 67–90, (Russian).
(Derivations of associative rings and algebras.)

[Har2] V. K. Harchenko, *"Automorphisms and derivations of associative rings"*, Novosibirsk, 1989.
(Derivations of associative rings and algebras.)

[HaS] J. T. Hartwig, S. D. Silvestrov, *Generalized derivations on algebras*, to appear.

[HS] L. Hellström, S. D. Silvestrov, *Commuting elements q-deformed Heisenberg algebras*, World Scientific, 2000, 256 pp (ISBN: 981-02-4403-7).
(q-Deformed Heisenberg algebras and q-difference operators.)

[Her1] I. N. Herstein, *"Non-commutative rings"*, Carus Monograph, 15, 1968. (Russian translation: Moscow, Mir, 1972.)
(Derivations of associative rings.)

[Her2] I. N. Herstein, *"Rings with involution"*, The Univ. of Chicago Press, 1976.
(Derivations of associative rings with involution.)

[J1] N. Jacobson, *"Structure of rings"*, Colloquium Publications, **37**, Amer. Math. Soc., 1956. (Russian translation: Moscow, Inostr. Liter., 1961.)
(Derivations of associative rings and algebras.)

[J2] N. Jacobson, *"Lie Algebras"*, Interscience Publishers, 1962 (Dover, 1979).

[Jor1] D. A. Jordan, *Krull and Global Dimension of Certain Iterated Skew Polynomial Rings,* Contemporary Mathematics, vol. 130, 1992, pp. 49–67.
(Krull dimension and global dimension for Ore type rings.)

[Jor2] D. A. Jordan, *Iterated skew polynomial rings and quantum groups,* J. Algebra, **156**, (1993), pp. 194-218
(Ore type rings and quantum groups.)

[Jos] A. Joseph, *"Some Ring Theoretic Techniques and Open Problems in Enveloping Algebras"*, in S. Montgomery, L. Small, eds., "Noncommutative Rings", Springer-Verlag, 1992.
(σ-Derivations, enveloping algebras, homological algebra, ring theory.)

[Kac] V. Kac, P. Cheung *"Quantum Calculus"*, Springer, 2002.
(q-Derivatives and q-analysis.)

[Ka] S. Kaijser, *Derivation-type Operators on Algebras*, Research report, Department of mathematics, Uppsala University, 1969.

[K] I. Kaplansky, *"Introduction to differential algebra"*, Moscow, Inostr. Liter., 1959, (Russian translation).
(Differential rings and fields.)

[Ka] C. Kassel, *Quantum groups*, Springer-Verlag, 1995. Russian translation: Fazis, Moscow, 1999.
(σ-Derivations and quantum groups.)

[Kol1] E. R. Kolchin, *"Differential algebra and algebraic groups"*, Acad. Press, 1973.
(Differential rings and fields.)

[Kol2] E. R. Kolchin, *"Differential algebraic groups"*, Acad. Press, 1985.
(Differential rings and fields.)

[LLe1] T. Y. Lam, A. Leroy, *Algebraic conjugacy classes and skew polynomial rings,* in F. van Oystaeyen and L. LeBruyn, eds., "Perspectives in NATO ASI Series, Kluwer Publishers, 1988, 153-203.
(σ-Derivations and skew polynomial rings.)

[LLe2] T. Y. Lam, A. Leroy, *Vandermonde and Wronskian Matrices over Division Rings,* J. Algebra, **119**, 1988, 308–336.
(Theory of Vandermonde and Wronskian matrices based on σ-derivations and skew polynomial rings.)

[Lee] T.-K. Lee, *σ-commuting mappings in semiprime rings,* Communications in Algebra, **29**, 1 (2001), 2945-2951.
(Commutativity theorems in rings extended to (σ, τ)-derivations cont. of Beidar, et.al. [BFKL].)

[Le1] A. Leroy, *Dérivées logarithmiques pour une S-dérivation algébrique,* Comm. Algebra, **13**, 1985, 85–99.
(σ-Derivations and skew polynomial rings.)

[Le2] A. Leroy, *(S)-dérivations algébriques sur les corps gauches et sur les anneaux premiers,* Comm. Algebra, **14**, 1986, 1473–1479.
(σ-Derivations and skew polynomial rings.)

[Man] Yu. I. Manin, *"Topics in Non-commutative Geometry",* Princeton University Press, 1991.
(Super-derivaitons on graded supercommutative rings and their pseudodifferential calculus in non-commutative geometry.)

[Mat] M. Matsuda, *"First order algebraic differential equation",* LNM 804, Springer-Verlag, 1980.
(Differential rings and fields.)

[MR] O.B. Melnikov, V. N. Remeslennikov, V. A. Romankov, L. A. Skornjakov, P. Shestakov, *"General algebra", vol. I,* Moscow, Nauka, 1990, (Russian).
(Derivations and σ-derivations, derivations in rings and algebras (ch. III.1.3, pp. 304 – 306), differential rings (ch. III.5.5, p. 556 – 557), σ-derivations in group rings (ch. II.1.2, p. 105 – 106), σ-derivations in rings, and skew-polynomial ring or ring of differential polynomials (ch. II.2.3, pp. 335).)

[MP] A. V. Michalev, E. V. Pankratjev, *"Differential and difference algebra",* Moscow, VINITI, vol. 25, 1987, pp. 67–139, (Russian).
(Derivations of associative rings and algebras.)

[Mon] S. Montgomery, S. P. Smith, *Skew derivations and $U_q(sl(2))$,* Israel J. Math., **130**, 1990, pp. 158-166.
(Skew Differential operators play important role in quantum groups.)

[NSS] V. E. Nazaikinskii, V. E. Shatalov, B. Yu. Sternin, *Methods of Noncommutative Analysis. Theory and Applications*, De Gruyter studies in mathematics 22, De

Gruyter, Berlin, New York, 1976.
(Difference and q-difference operators and non-commutative analysis.)

[Ore1] O. Ore, *Theory of non-commutative polynomials,* Ann. Math., **34**, 1933, 480-508.
(The article where Ore rings where introduced.)

[P] R. S. Pierce, *"Associative algebras",* Springer-Verlag, New York, 1982. (Russian translation: Moscow, Mir, 1986.)
(Derivations of associative rings.)

[Po] J.-F. Pommaret, *"Differential Galois theory",* Gordon and Breach Sci. Publ., 1983.
(Differential rings and fields.)

[Pr] P. van Pragg, *S-dérivations internes dans les corps,* J. Algebra, **17**, 1971, 470–473.
(σ-Derivations. See [LLe1, LLe2, Le1, Le2].)

[Ri] J. F. Ritt, *"Differential algebra",* AMS Coll. Pub., 33, 1950.
(Differential rings and fields.)

[Sak] S. Sakai, *Operator algebras in Dynamical Systems*, Cambridge University Press, 1991.
(Derivations on operator algebras.)

[Sam] A. A. Samarskii, *Introduction to Numerical Methods*, Nauka, Moscow 1987 (Russian).

[S] A. N. Sharkovsky, *Coexistence of cycles of a continuous transformation of a line into itself*, Ukrain. Mat. Zhurn. **16**, 1 (1964), 61-71 (Russian).

[Smi] S. P. Smith, *"Quantum Groups: An Introduction and Survey for Ring Theorists",* in S. Montgomery, L. Small, eds., "Noncommutative Rings", Springer-Verlag, 1992.
(Quantum qroups, quantum algebras, σ-Derivations.)

[Sm] T. H. M. Smits, *Nilpotent S-derivations,* Indag. Math., **30**, 1968, 72–86.
(σ-Derivations.)

[W] V. Weispfenning, , Finite Gröbner bases in non-Noetherian skew polynomial rings, in P. S. Wang, ed., Symbolic and Algebraic Computation, 329-334, New York. ACM Press. Proceedings ISSAC'92, July 1992, Berkley, California.
(Gröbner bases in skew-polynomial rings, Ore algebras and Ore rings. See [ChS].)

Highest weight representations of the Virasoro algebra

Jonas T. Hartwig

May 22, 2002

Abstract. We consider representations of the Virasoro algebra, a one-dimensional central extension of the Lie algebra of vectorfields on the unit circle. Positive-energy, highest weight and Verma representations are defined and investigated. The Shapovalov form is introduced, and we study Kac formula for its determinant and some consequences for unitarity and degeneracy of irreducible highest weight representations. In the last section we realize the centerless Ramond algebra as a super Lie algebra of superderivations.

Contents

1 Introduction

In this second part of the master thesis we review some of the representation theory for the Virasoro algebra. It is the unique nontrivial one-dimensional central extension of the Witt algebra, which is the Lie algebra of all vectorfields on the unit circle. More specifically we will study highest weight representations, which is an important class of representations. Shapovalov ([5]) defined a Hermitian form on any highest weight representation. This in particular induces a nondegenerate form on the irreducible quotient. Thus properties of irreducible highest weight representations can be studied in terms of this form. In [2], [3] Kac gave a formula for the determinant of the Shapovalov form. The formula was proved by Feigin and Fuchs in [1].

In Section 2 we introduce some notation that will be used throughout the article. The Witt algebra is defined algebraically as the Lie algebra of all derivations of Laurent polynomials. We show that it has a unique nontrivial one-dimensional central extension, namely the Virasoro algebra. We define highest weight, positive energy, and Verma representations in Section 3. Conditions for an irreducible highest weight representation to be degenerate or unitary are considered in Section 4. We also provide some lemmas to support the main theorem (Theorem 28), the Kac determinant formula, although we do not give a complete proof. Finally, in Section 5 we consider a supersymmetric extension of the Witt algebra, and we show that it has a representation as superderivations on $\mathbb{C}[t, t^{-1}, \epsilon \mid \epsilon^2 = 0]$. Superderivations are special cases of σ-derivations, as described in the first part of the master thesis.

2 Definitions and notations

For a Lie algebra $\mathfrak{g}$, let $\mathcal{U}(\mathfrak{g})$ denote its universal enveloping algebra.

Definition 1 (Extension)**.** Let $\mathfrak{g}$ and I be Lie algebras. An *extension* $\widetilde{\mathfrak{g}}$ of $\mathfrak{g}$ by I is a short exact sequence

$$0 \longrightarrow I \longrightarrow \widetilde{\mathfrak{g}} \longrightarrow \mathfrak{g} \longrightarrow 0$$

of Lie algebras. The extension is *central* if the image of I is contained in the center of $\widetilde{\mathfrak{g}}$, and *one-dimensional* if I is.

Note that $\widetilde{\mathfrak{g}}$ is isomorphic to $\mathfrak{g} \oplus I$ as linear spaces. Given two Lie algebras $\mathfrak{g}$ and I, one may always give $\mathfrak{g} \oplus I$ a Lie algebra structure by defining $[x+a, y+b]_{\mathfrak{g}\oplus I} = [x, y]_{\mathfrak{g}} + [a, b]_I$ for $x, y \in \mathfrak{g}, a, b \in I$. This extension is considered to be trivial.

Definition 2 (Antilinear anti-involution)**.** An antilinear anti-involution ω on a complex algebra A is a map $A \to A$ such that

$$\omega(\lambda x + \mu y) = \overline{\lambda}\omega(x) + \overline{\mu}\omega(y) \quad \text{for } \lambda, \mu \in \mathbb{C}, x, y \in A \tag{1}$$

and

$$\omega(xy) = \omega(y)\omega(x) \quad \text{for } x, y \in A, \tag{2}$$

$$\omega(\omega(x)) = x \quad \text{for } x \in A. \tag{3}$$

Definition 3 (Unitary representation)**.** Let $\mathfrak{g}$ be a Lie algebra with an antilinear anti-involution $\omega : \mathfrak{g} \to \mathfrak{g}$. Let $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation of $\mathfrak{g}$ in a linear space V equipped with an Hermitian form $\langle \cdot, \cdot \rangle$. The form $\langle \cdot, \cdot \rangle$ is called *contravariant* with respect to ω if

$$\langle \pi(x)u, v \rangle = \langle u, \pi(\omega(x))v \rangle \quad \text{for all } x \in \mathfrak{g}, u, v \in V.$$

The representation π is said to be *unitary* if in addition $\langle v, v \rangle > 0$ for all nonzero $v \in V$.

Remark 1. If only one representation is considered, we will often use module notation and write xu for $\pi(x)u$ whenever it is convenient to do so.

The following Lemma will be used a number of times.

Lemma 1. *Let V be a representation of a Lie algebra $\mathfrak{g}$ which decomposes as a direct sum of eigenspaces of a finite dimensional commutative subalgebra $\mathfrak{h}$:*

$$V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda \tag{4}$$

where $V_\lambda = \{v \in V \mid hv = \lambda(h)v \text{ for all } h \in \mathfrak{h}\}$, and $\mathfrak{h}^*$ is the dual vector space of $\mathfrak{h}$. Then every subrepresentation U of V respects this decomposition in the sence that

$$U = \bigoplus_{\lambda \in \mathfrak{h}^*} (U \cap V_\lambda).$$

Proof. Any $v \in V$ can be written in the form $v = \sum_{j=1}^m w_j$, where $w_j \in V_{\lambda_j}$ according to (4). Since $\lambda_i \neq \lambda_j$ for $i \neq j$ there is an $h \in \mathfrak{h}$ such that $\lambda_i(h) \neq \lambda_j(h)$ for $i \neq j$. Now if $v \in U$, then

$$\begin{array}{rcccccccc}
v & = & w_1 & + & w_2 & + & \dots & + & w_m \\
h(v) & = & \lambda_1(h)w_1 & + & \lambda_2(h)w_2 & + & \dots & + & \lambda_m(h)w_m \\
\vdots & & & & & & & & \\
h^{m-1}(v) & = & \lambda_1(h)^{m-1}w_1 & + & \lambda_2(h)^{m-1}w_2 & + & \dots & + & \lambda_m(h)^{m-1}w_m
\end{array}$$

The coefficient matrix in the right hand side is a Vandermonde matrix, and thus invertible. Therefore each w_j is a linear combination of vectors of the form $h^i(v)$, all of which lies in U, since $v \in U$ and U is a representation of $\mathfrak{g}$. Thus each $w_j \in U \cap V_{\lambda_j}$ and the proof is finished. $\square$

2.1 The Witt algebra

The Witt algebra $\mathfrak{d}$ can be defined as the complex Lie algebra of derivations of the algebra $\mathbb{C}[t, t^{-1}]$ of complex Laurent polynomials. Explicitly,

$$\mathbb{C}[t, t^{-1}] = \{\sum_{k\in\mathbb{Z}} a_k t^k \mid a_k \in \mathbb{C}, \text{only finitely many nonzero}\}$$

and

$$\mathfrak{d} = \{D : \mathbb{C}[t, t^{-1}] \to \mathbb{C}[t, t^{-1}] \mid D \text{ is linear and } D(pq) = D(p)q + pD(q)\} \tag{5}$$

with the usual Lie bracket: $[D, E] = DE - ED$. One can check that $\mathfrak{d}$ is closed under this product. The following proposition reveals the structure of $\mathfrak{d}$.

Proposition 2. *Consider the elements d_n of $\mathfrak{d}$ defined by*

$$d_n = -t^{n+1}\frac{d}{dt} \quad \textit{for } n \in \mathbb{Z}.$$

Then

$$\mathfrak{d} = \bigoplus_{n\in\mathbb{Z}} \mathbb{C}d_n \tag{6}$$

and

$$[d_m, d_n] = (m - n)d_{m+n} \quad \textit{for } m, n \in \mathbb{Z}. \tag{7}$$

Proof. Clearly $\mathfrak{d} \supseteq \bigoplus_{n\in\mathbb{Z}} \mathbb{C}d_n$. To show the reverse inclusion, let $D \in \mathfrak{d}$ be arbitrary. Then, using (5), i.e. that D is a derivation of $\mathbb{C}[t, t^{-1}]$, we obtain

$$D(1) = D(1 \cdot 1) = D(1) \cdot 1 + 1 \cdot D(1) = 2D(1).$$

Hence $D(1) = 0$, which implies that

$$0 = D(t \cdot t^{-1}) = D(t) \cdot t^{-1} + t \cdot D(t^{-1}),$$

which shows that

$$D(t^{-1}) = D(t) \cdot (-t^{-2}). \tag{8}$$

Now define the element $E \in \bigoplus_{n\in\mathbb{Z}} \mathbb{C}d_n$ by

$$E = D(t)\frac{d}{dt},$$

and note that $E(t) = D(t)$. Note further that $E(t^{-1}) = D(t) \cdot (-t^{-2})$ and thus, by (8), that the derivations E and D coincide on the other generator t^{-1} of $\mathbb{C}[t, t^{-1}]$ also. Using the easily proved fact that a derivation of an algebra is uniquely determined by the value on the generators of the algebra, we conclude that $D = E$. Therefore $\mathfrak{d} \subseteq \bigoplus_{n\in\mathbb{Z}} \mathbb{C}d_n$ and the proof of (6) is finished.

We now show the relation (7). For any $p(t) \in \mathbb{C}[t, t^{-1}]$, we have

$$\begin{aligned}(d_m d_n)(p(t)) &= d_m(-t^{n+1} \cdot p'(t)) = \\ &= d_m(-t^{n+1}) \cdot p'(t) + (-t^{n+1}) \cdot d_m(p'(t)) = \\ &= -t^{m+1} \cdot (-(n+1))t^n \cdot p'(t) + (-t^{n+1})(-t^{m+1})p''(t) = \\ &= (n+1)t^{m+n+1} \cdot p'(t) + t^{m+n+2}p''(t).\end{aligned}$$

The second of these terms is symmetric in m and n, and therefore vanishes when we take the commutator, yielding

$$[d_m, d_n](p(t)) = \big((n+1) - (m+1)\big)t^{m+n+1}p'(t) = (m-n) \cdot d_{m+n}(p(t)),$$

as was to be shown. □

Remark 2. Note that the commutation relation (7) shows that $\mathfrak{d}$ is $\mathbb{Z}$-graded as a Lie algebra with the grading (6).

2.2 Existence and uniqueness of Vir

Theorem 3. *The Witt algebra $\mathfrak{d}$ has a unique nontrivial one-dimensional central extension $\widetilde{\mathfrak{d}} = \mathfrak{d} \oplus \mathbb{C}\overline{c}$, up to isomorphism of Lie algebras. This extension has a basis $\{c\} \cup \{d_n \mid n \in \mathbb{Z}\}$, where $c \in \mathbb{C}\overline{c}$, such that the following commutation relations are satisfied:*

$$[c, d_n] = 0 \qquad \text{for } n \in \mathbb{Z}, \tag{9}$$

$$[d_m, d_n] = (m-n)d_{m+n} + \delta_{m,-n}\frac{m^3 - m}{12}c \qquad \text{for } m, n \in \mathbb{Z}. \tag{10}$$

The extension $\widetilde{\mathfrak{d}}$ is called the *Virasoro algebra*, and is denoted by Vir.

Proof. We first prove uniqueness. Suppose $\widetilde{\mathfrak{d}} = \mathfrak{d} \oplus \mathbb{C}\overline{c}$ is a nontrivial one-dimensional central extension of $\mathfrak{d}$. Let $\overline{d}_n, n \in \mathbb{Z}$ denote the standard basis elements of $\mathfrak{d}$, then we have

$$\begin{aligned}[\overline{d}_m, \overline{d}_n] &= (m-n)\overline{d}_{m+n} + a(m,n)\overline{c} \\ [\overline{c}, \overline{d}_n] &= 0\end{aligned} \tag{11}$$

for $m, n \in \mathbb{Z}$, where $a : \mathbb{Z} \times \mathbb{Z} \to \mathbb{C}$ is some function. Note that we must have $a(m,n) = -a(n,m)$ because $\widetilde{\mathfrak{d}}$ is a Lie algebra and thus has an anti-symmetric product:

$$0 = [\overline{d}_m, \overline{d}_n] + [\overline{d}_n, \overline{d}_m] = (m - n + n - m)\overline{d}_0 + (a(m,n) + a(n,m))\overline{c}.$$

Define new elements

$$d'_n = \begin{cases} \overline{d}_0 & \text{if } n = 0 \\ \overline{d}_n - \frac{1}{n}a(0,n)\overline{c} & \text{if } n \neq 0 \end{cases}$$
$$c' = \overline{c}$$

Then $\{c'\} \cup \{d'_n \mid n \in \mathbb{Z}\}$ is a new basis for $\widetilde{\mathfrak{d}}$. The new commutation relations are

$$\begin{aligned} [c', d'_n] &= 0 \\ [d'_m, d'_n] &= (m-n)d_{m+n} + a(m,n)c = \\ &= (m-n)d'_{m+n} + a'(m,n)c' \end{aligned} \tag{12}$$

for $m, n \in \mathbb{Z}$, where $a' : \mathbb{Z} \times \mathbb{Z} \to \mathbb{C}$ is defined by

$$a'(m,n) = \begin{cases} a(m,n) & \text{if } m+n = 0 \\ a(m,n) + \frac{m-n}{m+n}a(0,m+n) & \text{if } m+n \neq 0 \end{cases} \tag{13}$$

Note that since a is antisymmetric, so is a', and therefore in particular $a'(0,0) = 0$. From (13) follows that $a'(0,n) = 0$ for any nonzero n. These facts together with (12) shows that

$$[d'_0, d'_n] = -nd'_n \tag{14}$$

Using now the Jacobi identity which holds in $\widetilde{\mathfrak{d}}$ we obtain

$$[[d'_0, d'_n], d'_m] + [[d'_n, d'_m], d'_0] + [[d'_m, d'_0], d'_n] = 0$$

$$[-nd'_n, d'_m] + [(n-m)d'_{n+m} + a'(n,m)c', d'_0] - [d'_n, md'_m] = 0$$

$$-(n+m)(n-m)d'_{n+m} - (n+m)a'(n,m)c' + (n-m)(n+m)d'_{n+m} = 0$$

which shows that $a'(n,m) = 0$ unless $n + m = 0$. Thus, setting $b(m) = a'(m, -m)$, equation (12) can be written

$$\begin{aligned} [c', d'_n] &= 0 \\ [d'_m, d'_n] &= (m-n)d'_{m+n} + \delta_{m+n,0}b(m)c' \end{aligned}$$

Again we use Jacobi identity

$$[[d'_n, d'_1], d'_{-n-1}] + [[d'_1, d'_{-n-1}], d'_n] + [[d'_{-n-1}, d'_n], d'_1] = 0$$

$$[(n-1)d'_{n+1}, d'_{-n-1}] + [(n+2)d'_{-n}, d'_n] + [(-2n-1)d'_{-1}, d'_1] = 0$$

$$(n-1)(2(n+1)d'_0 + b(n+1)c') + (n+2)(-2nd'_0 + b(-n)c') + (-2n-1)(-2d'_0 + b(-1)c') = 0$$

$$(2n^2 - 2 - 2n^2 - 4n + 4n + 2)d'_0 + \big((n-1)b(n+1) - (n+2)b(n) + (2n+1)b(1)\big)c' = 0,$$

which is equivalent to

$$(n-1)b(n+1) = (n+2)b(n) - (2n+1)b(1).$$

This is a second order linear recurrence equation in b. One verifies that $b(m) = m$ and $b(m) = m^3$ are two solutions, obviously linear independent. Thus there are $\alpha, \beta \in \mathbb{C}$ such that

$$b(m) = \alpha m^3 + \beta m.$$

Finally, we set

$$d_n = d'_n + \delta_{n,0}\frac{\alpha+\beta}{2}c',$$

and

$$c = 12\alpha c'.$$

If $\alpha \neq 0$, this is again a change of basis. Then, for $m+n \neq 0$,

$$\begin{aligned}[d_m, d_n] &= (m-n)d'_{m+n} + \delta_{m+n,0}(\alpha m^3 + \beta m)c' = \\ &= (m-n)d_{m+n} + \delta_{m+n,0}\frac{m^3-m}{12}c,\end{aligned}$$

and for $m+n=0$,

$$\begin{aligned}[d_m, d_n] &= (m-n)d'_{m+n} + (\alpha m^3 + \beta m)c' = \\ &= 2md'_{m+n} + 2m\frac{\alpha+\beta}{2}c' + (\alpha m^3 - \alpha m)c' = \\ &= 2md_{m+n} + \frac{m^3-m}{12}c = \\ &= (m-n)d_{m+n} + \delta_{m+n,0}\frac{m^3-m}{12}c.\end{aligned}$$

From these calculations we also see that $\alpha = 0$ corresponds to the trivial extension. The proof of uniqueness is finished. To prove existence, it is enough to check that the relations (9)-(10) define a Lie algebra, which is easy. □

The antilinear map $\omega : \mathrm{Vir} \to \mathrm{Vir}$ defined by requiring

$$\omega(d_n) = d_{-n} \tag{15}$$

$$\omega(c) = c \tag{16}$$

is an antilinear anti-involution on Vir. Indeed

$$\begin{aligned}[\omega(d_n), \omega(d_m)] = [d_{-n}, d_{-m}] &= (-n+m)d_{-n-m} + \delta_{-n,m}\frac{-n^3+n}{12}c = \\ &= (m-n)d_{-(m+n)} + \delta_{m,-n}\frac{m^3-m}{12}c = \omega([d_m, d_n]).\end{aligned}$$

Contravariance of Hermitian forms on representations of Vir, and unitarity of the representations will always be considered with respect to this ω.

Note that Vir has the following triangular decomposition into Lie subalgebras:

$$n^- = \bigoplus_{i=1}^{\infty} \mathbb{C}d_{-i} \qquad \mathfrak{h} = \mathbb{C}c \oplus \mathbb{C}d_0 \qquad n^+ = \bigoplus_{i=1}^{\infty} \mathbb{C}d_i \tag{17}$$

3 Representations of Vir

3.1 Positive-energy and highest weight representations

Definition 4 (Positive-energy representation of Vir)**.** Let $\pi : \mathrm{Vir} \to \mathfrak{gl}(V)$ be a representation of Vir in a linear space V such that

a) V admits a basis consisting of eigenvectors of $\pi(d_0)$,

b) all eigenvalues of the basis vectors are non-negative, and

c) the eigenspaces of $\pi(d_0)$ are finite-dimensional.

Then π is said to be a *positive-energy* representation of Vir.

Definition 5 (Highest weight representation of Vir)**.** A representation of Vir in a linear space V is a *highest weight* representation if there is an element $v \in V$ and two numbers $C, h \in \mathbb{C}$, such that

$$cv = Cv, \tag{18}$$

$$d_0 v = hv, \tag{19}$$

$$V = \mathcal{U}(\mathrm{Vir})v = \mathcal{U}(n^-)v, \tag{20}$$

$$n^+ v = 0. \tag{21}$$

The vector v is called a *highest weight vector* and (C, h) is the *highest weight*.

Remark 3. The second equality in condition (20) follows from (18), (19) and (21). To see this, use the Poincaré-Birkhoff-Witt theorem:

$$\mathcal{U}(\mathrm{Vir}) = \mathcal{U}(n^-)\mathcal{U}(\mathfrak{h})\mathcal{U}(n^+),$$

and write $\mathcal{U}(n^+) = \mathbb{C} \cdot 1 + \mathcal{U}(n^+)n^+$. Then

$$\mathcal{U}(\mathrm{Vir})v = \mathcal{U}(n^-)\mathcal{U}(\mathfrak{h})(\mathbb{C} \cdot 1 + \mathcal{U}(n^+)n^+)v = \mathcal{U}(n^-)\mathcal{U}(\mathfrak{h})v = \mathcal{U}(n^-)v,$$

where we used (21) in the second equality, and (18)-(19) in the last.

Proposition 4. *Any highest weight representation V with highest weight (C, h) has the decomposition*

$$V = \bigoplus_{k \in \mathbb{Z}_{\geq 0}} V_{h+k} \tag{22}$$

where V_{h+k} is the $(h+k)$-eigenspace of d_0 spanned by vectors of the form

$$d_{-i_s} \dots d_{-i_1}(v) \quad \textit{with} \quad 0 < i_1 \leq \dots \leq i_s, \quad i_1 + \dots + i_s = k.$$

Proof. Using that $[d_0, \cdot]$ is a derivation of $\mathcal{U}(\mathrm{Vir})$ we get

$$\begin{aligned} d_0 d_{-i_s} \dots d_{-i_1} - d_{-i_s} \dots d_{-i_1} d_0 &= \sum_{m=1}^{s} d_{-i_s} \dots d_{-i_{m+1}} [d_0, d_{-i_m}] d_{-i_{m-1}} \dots d_{-i_1} = \\ &= \sum_{m=1}^{s} i_m d_{-i_s} \dots d_{-i_{m+1}} d_{-i_m} d_{-i_{m-1}} \dots d_{-i_1} = \\ &= (i_1 + \dots + i_s) d_{-i_s} \dots d_{-i_1}. \end{aligned} \tag{23}$$

Therefore we have

$$\begin{aligned} d_0(d_{-i_s} \dots d_{-i_1}(v)) &= (i_1 + \dots + i_s) d_{-i_s} \dots d_{-i_1}(v) + d_{-i_s} \dots d_{-i_1} d_0(v) = \\ &= (i_1 + \dots + i_s + h) d_{-i_s} \dots d_{-i_1}(v). \end{aligned}$$

□

Proposition 5. *An irreducible positive energy representation of* Vir *is a highest weight representation.*

Proof. Let $\mathrm{Vir} \to \mathfrak{gl}(V)$ be an irreducible positive energy representation of Vir in a linear space V, and let $w \in V$ be a nontrivial eigenvector for d_0. Then $d_0 w = \lambda w$ for some $\lambda \in \mathbb{R}_{\geq 0}$. Now for any $t \in \mathbb{Z}_{\geq 0}$ and $(j_t, \dots, j_1) \in \mathbb{Z}^t$ we have, using the same calculation as in Proposition 4,

$$d_0 d_{j_t} \dots d_{j_1} w = (\lambda - (j_t + \dots + j_1)) d_{j_t} \dots d_{j_1} w.$$

Since V is a positive energy representation, this shows that the set

$$M = \{ j \in \mathbb{Z} \mid d_{j_t} \dots d_{j_1} w \neq 0 \text{ for some } t \geq 0, (j_t, \dots, j_1) \in \mathbb{Z}^t \text{ with } j_t + \dots + j_1 = j \}$$

is bounded from above by λ. It is also nonempty, because $0 \in M$. Let $t \geq 0$ and $(j_t, \dots, j_1) \in \mathbb{Z}^t$ with $j_t + \dots + j_1 = \max M$ be such that $v = d_{j_t} \dots d_{j_1} w \neq 0$. Then

$$d_j v = d_j d_{j_t} \dots d_{j_1} w = 0 \quad \text{for } j > 0 \tag{24}$$

since otherwise $j + \max M = j + j_t + \ldots + j_1 \in M$, which is impossible. We also have

$$d_0 v = d_0 d_{j_t} \ldots d_{j_1} w = (\lambda - (j_t + \ldots + j_1)) d_{j_t} \ldots d_{j_1} w = hv \tag{25}$$

where we set $h = \lambda - (j_t + \ldots + j_1)$. Using some argument involving restrictions to eigenspaces, it can be shown using Schur's Lemma that c acts by some multiple $C \in \mathbb{C}$ of the identity operator on V. In particular we have

$$cv = Cv. \tag{26}$$

Consider the submodule V' of V defined by

$$V' = \mathcal{U}(\mathrm{Vir})v. \tag{27}$$

It is nontrivial, since $0 \neq v \in V'$. Therefore, since V is irreducible, we must have $V = V'$. Recalling Remark 3 and using (24)-(27), it now follows that V is a highest weight representation, and the proof is finished. □

Proposition 6. *A unitary highest weight representation V of* Vir *is irreducible.*

Proof. If U is a subrepresentation of V, then $V = U \oplus U^\perp$. Using the decomposition (22) of V and Lemma 1 we obtain

$$U = \bigoplus_{k \geq 0} U \cap V_{h+k} \qquad\qquad U^\perp = \bigoplus_{k \geq 0} U^\perp \cap V_{h+k}$$

In particular, since V_h is one-dimensional and spanned by some nonzero highest weight vector v, we have either $v \in U$ or $v \in U^\perp$. Thus either $U = V$ or $U = 0$. □

3.2 Verma representations

Definition 6 (Verma representation of Vir)**.** A highest weight representation $M(C, h)$ of Vir with highest weight vector v and highest weight (C, h) is called a *Verma representation* if it satisfies the following universal property:

For any highest weight representation V of Vir with heighest weight vector u and highest weight (C, h), there exists a unique epimorphism $\varphi : M(C, h) \to V$ of Vir-modules which maps v to u.

Proposition 7. *For each $C, h \in \mathbb{C}$ there exists a unique Verma representation $M(C, h)$ of* Vir *with highest weight (C, h). Furthermore, the map $\mathcal{U}(n^-) \to M(C, h)$ sending x to xv is not only surjective, but also injective.*

Proof. To prove existence, let $I(C,h)$ denote the left ideal in $\mathfrak{U}(\mathrm{Vir})$ generated by the elements $\{d_n \mid n>0\} \cup \{d_0 - h\cdot 1_{\mathfrak{U}(\mathrm{Vir})},\ c - C\cdot 1_{\mathfrak{U}(\mathrm{Vir})}\}$, where $1_{\mathfrak{U}(\mathrm{Vir})}$ is the identity element in $\mathfrak{U}(\mathrm{Vir})$. Form the linear space $M(C,h) = \mathfrak{U}(\mathrm{Vir})/I(C,h)$, and define a map $\pi : \mathrm{Vir} \to \mathfrak{gl}(M(C,h))$ by

$$\pi(x)(u + I(C,h)) = xu + I(C,h).$$

Then π is a representation of Vir. Furthermore, it is a highest weight representation of Vir with highest weight vector $v = 1_{\mathfrak{U}(\mathrm{Vir})} + I(C,h)$ and highest weight (C,h).

We now show that π is a Verma representation. Let $\rho : \mathrm{Vir} \to \mathfrak{gl}(V)$ be any highest weight representation with highest weight (C,h) and highest weight vector u. By restricting the multiplication we can view $\mathfrak{U}(\mathrm{Vir})$ as a left Vir-module. The action of $\mathfrak{U}(\mathrm{Vir})$ on V

$$\begin{aligned} \alpha : \mathfrak{U}(\mathrm{Vir}) &\to V \\ x &\to xu \end{aligned}$$

then becomes a Vir-module homomorphism. We claim that $\alpha(I(C,h)) = 0$. Indeed, it is enough to check that the image under α of the generators $d_n, n>0$, $d_0 - h\cdot 1_{\mathfrak{U}(\mathrm{Vir})}$, and $c - C\cdot 1_{\mathfrak{U}(\mathrm{Vir})}$ of the left ideal are zero, and this follows since V is a highest weight representation of Vir with highest weight vector u and highest weight (C,h). Thus α induces a Vir-module epimorphism $\varphi : \mathfrak{U}(\mathrm{Vir})/I(C,h) = M(C,h) \to V$ which clearly maps v to u. This shows existence of the map φ.

Next we prove that there can exist at most one Vir-module epimorphism $\varphi : M(C,h) \to V$ which maps v to u. Since $M(C,h)$ is a highest weight module, any element is a linear combination of elements of the form

$$d_{-i_s} \dots d_{-i_1} + I(C,h),$$

where $i_j > 0$ and $s \geq 0$. We show by induction on s that φ is uniquely defined on each such element. If $s = 0$, we must have $\varphi(1_{\mathfrak{U}(\mathrm{Vir})} + I(C,h)) = \varphi(v) = u$. If $s > 0$ we have

$$\begin{aligned} \varphi(d_{-i_s} \dots d_{-i_1} + I(C,h)) &= \varphi(\pi(d_{-i_s})(d_{-i_{s-1}} \dots d_{-i_1} + I(C,h) = \\ &= \rho(d_{-i_s})\varphi(d_{-i_{s-1}} \dots d_{-i_1} + I(C,h)) \end{aligned}$$

since φ is a Vir-module homomorphism. By induction on s, φ is uniquely defined on $M(C,h)$. Consequently, π is a Verma representation.

Uniqueness of the Verma representaion $M(C,h)$ is a standard exercise in abstract nonsense. Injectivity of the map $\mathfrak{U}(n^-) \ni x \mapsto \pi(x)(1_{\mathfrak{U}(\mathrm{Vir})} + I(C,h)) = x + I(C,h)$ follows from the Poincaré-Birkhoff-Witt theorem. □

In the rest of the article, v shall always denote a fixed choice of a nonzero highest weight vector in $M(C,h)$.

Proposition 8. *a) The Verma representation $M(C,h)$ has the decomposition*

$$M(C,h) = \bigoplus_{k\in\mathbb{Z}_{\geq 0}} M(C,h)_{h+k} \tag{28}$$

where $M(C,h)_{h+k}$ is the $(h+k)$-eigenspace of d_0 of dimension $p(k)$ spanned by vectors of the form

$$d_{-i_s}\dots d_{-i_1}(v) \quad \text{with} \quad 0 < i_1 \leq \dots \leq i_s, i_1+\dots+i_s = k$$

b) M(C,h) is indecomposable, i.e. we cannot find nontrivial subrepresentations W_1, W_2 of $M(C,h)$ such that

$$M(C,h) = W_1 \oplus W_2.$$

c) M(C,h) has a unique maximal proper subrepresentation $J(C,h)$, and

$$V(C,h) = M(C,h)/J(C,h)$$

is the unique irreducible highest weight representation with highest weight (C,h).

Proof. Part (a) is a restatement of Proposition 4 for Verma modules. It remains to determine the dimension of an eigenspace V_{h+k} of d_0. Note that in a Verma representation, the set of all the vectors

$$d_{-i_s}\dots d_{-i_1}(v), \quad i_s \geq \dots \geq i_1 \geq 1, \quad i_1+\dots+i_s = k$$

form a basis for V_{h+k} because a vanishing linear combination would contradict the injectivity of the linear map $\mathcal{U}(n^-) \ni x \mapsto xv \in M(C,h)$. The number of such vectors are precisely the number of partitions of k into positive integers.

For part b), assume that $M(C,h) = W_1 \oplus W_2$ is a decomposition into subrepresentations. Using Lemma 1 with $\mathfrak{g} = \text{Vir}$ and $\mathfrak{h} = \mathbb{C}d_0$ and $V = M(C,h)$ and $U = W_1$ and $U = W_2$, we would have

$$W_1 = \bigoplus_{k\geq 0} W_1 \cap M(C,h)_{h+k} \qquad\qquad W_2 = \bigoplus_{k\geq 0} W_2 \cap M(C,h)_{h+k}$$

respectively. Since $\dim M(C,h)_h = 1$, we have either $M(C,h)_h \subseteq W_1$ or $M(C,h)_h \subseteq W_2$. In the former case, $v \in W_1$ which imply, since W_1 is a representation of Vir, that $M(C,h) = \mathcal{U}(Vir)v \subseteq W_1$. In other words, $W_1 = M(C,h)$ and $W_2 = 0$. The other case is symmetric. Thus no nontrivial decompositions can exist.

To prove c), we observe from the proof of part b) that a subrepresentation of $M(C,h)$ is proper if and only if it does not contain the highest weight vector v. Thus if we form the sum $J(C,h)$ of all proper subrepresentations of $M(C,h)$, it is itself a proper subrepresentation of $M(C,h)$. Clearly $J(C,h)$ is maximal among all proper subrepresentations.

It is also unique, because it contains and is contained in any other maximal proper subrepresentation of $M(C,h)$.

For the uniqueness of $V(C,h)$, let $V'(C,h)$ be any irreducible highest weight module with the same highest weight (C,h). Then by definition of the Verma module there is a submodule $J'(C,h)$ of $M(C,h)$ such that

$$V'(C,h) = M(C,h)/J'(C,h).$$

Since $V'(C,h)$ is irreducible, $J'(C,h)$ must be maximal and proper, and hence equal to $J(C,h)$. Thus $V'(C,h) = V(C,h)$, and the proof is finished. $\square$

3.3 Shapovalov's form

Proposition 9. *Let $C, h \in \mathbb{R}$. Then*

a) there is a unique contravariant Hermitian form $\langle\cdot|\cdot\rangle$ on M(C,h) such that $\langle v|v\rangle = 1$,

b) the eigenspaces of d_0 are pairwise orthogonal with respect to this form,

c) $J(C,h) = \ker\langle\cdot|\cdot\rangle \equiv \{u \in M(C,h) \mid \langle u|w\rangle = 0$ for all $w \in M(C,h)\}$.

The form is called *Shapovalov's form*.

Proof. a) We first prove uniqueness of the form. The antilinear anti-involution $\omega : \mathrm{Vir} \to \mathrm{Vir}$ defined in equations (15)-(16) extends uniquely to an antilinear anti-involution $\widetilde{\omega} : \mathcal{U}(\mathrm{Vir}) \to \mathcal{U}(\mathrm{Vir})$ on the universal enveloping algebra as follows:

$$\widetilde{\omega}(x_1 \dots x_m) = \omega(x_m) \dots \omega(x_1)$$

for elements $x_i \in \mathrm{Vir}$. If $x, y \in \mathcal{U}(\mathrm{Vir})$, then

$$\langle xv|yv\rangle = \langle v|\widetilde{\omega}(x)yv\rangle \tag{29}$$

since the form is contravariant.

The universal enveloping algebra $\mathcal{U}(\mathrm{Vir})$ of Vir has the following decomposition:

$$\mathcal{U}(\mathrm{Vir}) = (n^-\mathcal{U}(\mathrm{Vir}) + \mathcal{U}(\mathrm{Vir})n^+) \oplus \mathcal{U}(\mathfrak{h}).$$

Since $\mathfrak{h}$ is commutative, we can identify $\mathcal{U}(\mathfrak{h})$ with $S(\mathfrak{h})$, the symmetric algebra on the vectorspace $\mathfrak{h} = \mathbb{C}c \oplus \mathbb{C}d_0$. Let $P : \mathcal{U}(\mathrm{Vir}) \to S(\mathfrak{h}) = \mathcal{U}(\mathfrak{h})$ be the projection, and let $e_{(C,h)} : S(\mathfrak{h}) \to \mathbb{C}$ be the algebra homomorphism determined by

$$e_{(C,h)}(c) = C \qquad\qquad e_{(C,h)}(d_0) = h$$

Then we have for $x \in \mathcal{U}(\mathrm{Vir})$,

$$P(x)v = e_{(C,h)}(P(x))v$$

Since $M(C,h)$ is a highest weight representation, we have

$$\langle v|n^-\mathcal{U}(\mathrm{Vir})v + \mathcal{U}(\mathrm{Vir})n^+v\rangle = \langle n^+v|\mathcal{U}(\mathrm{Vir})v\rangle + \langle v|\mathcal{U}(\mathrm{Vir})n^+v\rangle = 0$$

Therefore

$$\langle xv|yv\rangle = \langle v|\widetilde{\omega}(x)yv\rangle = \langle v|P(\widetilde{\omega}(x)y)v\rangle = e_{(C,h)}\big(P(\widetilde{\omega}(x)y)\big). \tag{30}$$

This shows that the form is unique, if it exists.

To show existence, we recall the construction of $M(C,h)$ as a quotient of $\mathcal{U}(\mathrm{Vir})$ by a left ideal $I(C,h)$. Clearly $P(n^+) = P(n^-) = 0$, but we also have

$$e_{(C,h)}(P(c - C\cdot 1)) = e_{(C,h)}(c - C\cdot 1) = C - C = 0$$

$$e_{(C,h)}(P(d_0 - h\cdot 1)) = e_{(C,h)}(d_0 - h\cdot 1) = h - h = 0$$

where $1 = 1_{\mathcal{U}(\mathrm{Vir})}$. Note further that

$$P(xy) = P(x)y \qquad P(yx) = yP(x)$$

for $x \in \mathcal{U}(\mathrm{Vir})$, $y \in \mathcal{U}(\mathfrak{h})$. Combining these observations we deduce

$$e_{(C,h)}(P(x)) = 0 \qquad \text{for} \quad x \in I(C,h) \text{ or } x \in \widetilde{\omega}(I(C,h)). \tag{31}$$

It is now clear that we may take (30) as the definition of the form, because if $xv = x'v$ and $yv = y'v$ for some $x, x', y, y' \in \mathcal{U}(\mathrm{Vir})$ then $x - x', y - y' \in I(C,h)$ so that

$$\begin{aligned}\langle xv|yv\rangle - \langle x'v|y'v\rangle &= \langle (x-x')v|yv\rangle + \langle x'v|(y-y')v\rangle = \\ &= \langle \widetilde{\omega}(y)(x-x')v|v\rangle + \langle v|\widetilde{\omega}(x')(y-y')v\rangle = \\ &= 0.\end{aligned}$$

It is easy to see that the form is Hermitian. Contravariance is also clear:

$$\langle xyv|zv\rangle = e_{(C,h)}\big(P(\widetilde{\omega}(xy)z)\big) = e_{(C,h)}\big(P(\widetilde{\omega}(y)\widetilde{\omega}(x)z)\big) = \langle yv|\widetilde{\omega}(x)zv\rangle.$$

Finally, we have

$$\langle v|v\rangle = e_{(C,h)}(P(1\cdot 1)) = 1,$$

which concludes the proof of part a).

b) If $x \in M(C,h)_{h+k}$ and $y \in M(C,h)_{h+l}$ with $k \neq l$ we have

$$(k-l)\langle x|y\rangle = \langle (h+k)x|y\rangle - \langle x|(h+l)y\rangle = \langle d_0x|y\rangle - \langle x|d_0y\rangle = \langle x|\omega(d_0)y - d_0y\rangle = 0$$

since $\omega(d_0) = d_0$, and therefore we must have $\langle x|y\rangle = 0$.

c) It is easy to see, using contravariance of the form, that $\ker\langle\cdot|\cdot\rangle$ is a Vir subrepresentation of $M(C,h)$. Since $\langle v|v\rangle = 1$, it is a proper subrepresentation. Hence $\ker\langle\cdot|\cdot\rangle \subseteq J(C,h)$.

Conversely, suppose $x \in \mathcal{U}(\mathrm{Vir})$ is such that $xv \in J(C,h)$, but $xv \notin \ker\langle\cdot|\cdot\rangle$. Then there is a $y \in \mathcal{U}(\mathrm{Vir})$ such that

$$0 \neq \langle yv|xv\rangle = e_{(C,h)}(P(\widetilde{\omega}(y)x)).$$

Since $J(C,h)$ is a representation of Vir, we have found $z = \widetilde{\omega}(y)xv \in J(C,h)$ with a nonzero component in $M(C,h)_h = \mathbb{C}v$. Therefore, using Lemma 1, we must have $v \in J(C,h)$. This contradicts $J(C,h) \neq M(C,h)$ and the proof is finished. □

Corollary 10. *If $C, h \in \mathbb{R}$, then V(C,h)=M(C,h)/J(C,h) carries a unique contravariant Hermitian form $\langle\cdot|\cdot\rangle$ such that $\langle v + J(C,h)|v + J(C,h)\rangle = 1$.*

From now on we will always assume that $C, h \in \mathbb{R}$ so that the Shapovalov form is always defined.

4 Unitarity and degeneracy of representations

The unique irreducible highest weight representation $V(C,h)$ with highest weight (C,h) is called a *degenerate representation* if $V(C,h) \neq M(C,h)$. In this section we will investigate for which highest weights (C,h) the representation $V(C,h)$ is degenerate.

We will also study unitary highest weight representations. From the preceeding section we can draw some simple but important conclusions.

Proposition 11. *There exists at most one unitary highest weight representation of* Vir *for a given highest weight (C,h), namely $V(C,h)$.*

Proof. Use Proposition 6, and Proposition 8 part c). □

Thus to study unitary highest weight representations, it is enough to consider those of the irreducible representations $V(C,h)$ which are unitary. This leads to the question: For which highest weights (C,h) is $V(C,h)$ unitary? We have the following preliminary result.

Proposition 12. *If $V(C,h)$ is unitary, then $C \geq 0$ and $h \geq 0$.*

Proof. A necessary condition for unitarity of $V(C,h)$ is that

$$c_n = \langle d_{-n}v|d_{-n}v\rangle \geq 0 \qquad \text{for } n > 0.$$

Since the form is contravariant we have

$$c_n = \langle v | d_n d_{-n} v \rangle = \langle v | \big(d_{-n} d_n + 2nd_0 + \frac{n^3 - n}{12} c \big) v \rangle = 2nh + \frac{n^3 - n}{12} C$$

Since $c_1 = 2h$, we must have $h \geq 0$. Also, if n is sufficently large, c_n has the same sign as C, so $C \geq 0$ is also necessary. $\square$

To give a more detailed answer, we consider the matrix $S(C,h)$ of the Shapovalov form on $M(C,h)$.

$$S(C,h) = \big(\langle d_{-i_s} \dots d_{-i_1} v | d_{-j_t} \dots d_{-j_1} v \rangle \big)_{1 \leq i_1 \leq \dots \leq i_s,\ 1 \leq j_1 \leq \dots \leq j_t}$$

Since $M(C,h)$ is a direct sum of finite-dimensional pairwise orthogonal subspaces $M(C,h)_{h+n}$, $n \geq 0$, the matrix $S(C,h)$ is also a direct sum of matrices $S_n(C,h)$, $n \geq 0$, where $S_n(C,h)$ is the matrix of the Shapovalov form restricted to $M(C,h)_{h+n}$.

$$S_n(C,h) = \big(\langle d_{-i_s} \dots d_{-i_1} v | d_{-j_t} \dots d_{-j_1} v \rangle \big)_{(i_1,\dots,i_s),(j_1,\dots,j_t) \in P(n)}, \tag{32}$$

where $P(n)$ denotes the set of all partitions of n. We now define

$$\det_n(C,h) = \det S_n(C,h) \tag{33}$$

A necessary and sufficient condition for the degeneracy of $V(C,h)$ is that $J(C,h) \neq 0$, and this happens if and only if $\det_n(C,h) = 0$ for some $n \geq 0$. If $V(C,h)$ is unitary, $S_n(C,h)$ must be positive semi-definite for each $n \geq 0$, and thus $\det_n(C,h)$ must be non-negative for $n \geq 0$.

The following proposition shows that the representation theory for Vir is more interesting than that of the Witt algebra.

Proposition 13 (Gomes)**.** *If $C = 0$, the only unitary highest weight representation π with heighest weight (C,h) is the trivial one which satisfies $\pi(d_n) = 0$ for all $n \in \mathbb{Z}$.*

Proof. Suppose $V(0,h)$ is unitary, and let $N \in \mathbb{Z}_{\geq 0}$. Then it is necessary that $S_{2N}(0,h)$ is positive semi-definite. In particular the matrix

$$\begin{bmatrix} \langle d_{-2N} v | d_{-2N} v \rangle & \langle d_{-N}^2 v | d_{-2N} v \rangle \\ \langle d_{-2N} v | d_{-N}^2 v \rangle & \langle d_{-N}^2 v | d_{-N}^2 v \rangle \end{bmatrix} \tag{34}$$

must be positive semi-definite. Since $C = 0$ we have

$$\langle d_{-2N} v | d_{-2N} v \rangle = \langle v | (4Nd_0 + \frac{(2N)^3 - 2N}{12} c) v \rangle = 4Nh,$$

$$\begin{aligned}\langle d_{-N}^2 v | d_{-2N} v\rangle = \langle d_{-2N} v | d_{-N}^2 v\rangle &= \langle v | d_{2N} d_{-N}^2 v\rangle = \\ &= \langle v | (3N d_N d_{-N} + d_{-N} 3N d_N) v\rangle = \\ &= 3N \cdot 2Nh = \\ &= 6N^2 h,\end{aligned}$$

$$\begin{aligned}\langle d_{-N}^2 v | d_{-N}^2 v\rangle &= \langle d_{-N} v | (2N d_0 d_{-N} + d_{-N} 2N d_0) v\rangle = \\ &= 2N(h + N + h)\langle d_{-N} v | d_{-N} v\rangle = \\ &= (4Nh + 2N^2) \cdot 2Nh = \\ &= 8N^2 h^2 + 4N^3 h.\end{aligned}$$

Consequently the matrix (34) has the determinant

$$(4Nh)(8N^2h^2 + 4N^3) - (6N^2h)^2 = 32N^3h^3 + 16N^4h^2 - 36N^4h^2 = 4N^3h^2(8h - 5N),$$

which is negative for sufficently large N, unless $h = 0$. By uniqueness, $V(0,0)$ must be the trivial one-dimensional representation. $\square$

Our next goal is to find a general formula for $\det_n(C,h)$. For this we will need a series of lemmas.

4.1 Some lemmas

The universal enveloping algebra $\mathcal{U}(n^-)$ of n^- has a natural filtration

$$\mathcal{U}(n^-) = \bigcup_{k=0}^{\infty} \mathcal{U}(n^-)_{(k)} \tag{35}$$

$$\mathcal{U}(n^-)_{(0)} \subseteq \mathcal{U}(n^-)_{(1)} \subseteq \dots \tag{36}$$

$$\mathcal{U}(n^-)_{(k)}\mathcal{U}(n^-)_{(l)} \subseteq \mathcal{U}(n^-)_{(k+l)} \quad \text{for } k,l \in \mathbb{Z}_{\geq 0} \tag{37}$$

where

$$\mathcal{U}(n^-)_{(k)} = \sum_{0 \leq r \leq k} (n^-)^r = \sum_{\substack{0 \leq r \leq k \\ j_r \geq \dots j_1 \geq 1}} \mathbb{C} d_{-j_r} \dots d_{-j_1}. \tag{38}$$

For simplicity we will also use the notation

$$K_{(s)} = \mathcal{U}(\mathrm{Vir})n^+ + \mathcal{U}(n^-)_{(s-1)}d_0 + \mathcal{U}(n^-)_{(s-1)}c + \mathcal{U}(n^-)_{(s)} \qquad \text{for } s \geq 1,$$

and we note that

$$\mathcal{U}(n^-)_{(t)}K_{(s)} \subseteq K_{(t+s)} \qquad \text{for } t \geq 0, s \geq 1, \tag{39}$$

$$K_{(s)} \subseteq K_{(s+1)} \qquad \text{for } s \geq 1. \tag{40}$$

Lemma 14. *Let $i \geq 1$ and $j_s, \ldots, j_1 \geq 1$ be integers, where $s \geq 1$. Then*

$$d_i d_{-j_s} \ldots d_{-j_1} \in K_{(s)}. \tag{41}$$

Furthermore, if $i \notin \{j_1, \ldots, j_s\}$, then (41) can be replaced by the stronger conclusion

$$d_i d_{-j_s} \ldots d_{-j_1} \in \mathcal{U}(\mathrm{Vir})n^+ + \mathcal{U}(n^-)_{(s-2)} d_0 + \mathcal{U}(n^-)_{(s-2)} c + \mathcal{U}(n^-)_{(s)}, \tag{42}$$

where $\mathcal{U}(n^-)_{(-1)}$ is to be interpreted as zero.

Proof. We mainly consider (41), the case (42) being analogous. We use induction on s. If $s = 1$, we have

$$d_i d_{-j_1} = d_{-j_1} d_i + (i + j_1) d_{i-j_1} + \delta_{i,-j_1} \frac{i^3 - i}{12} c.$$

Now $d_{-j_1} d_i \in \mathcal{U}(\mathrm{Vir})n^+$ and $\delta_{i,-j_1} = 0$ since $i, j_1 \geq 1$. For the middle term $(i + j_1) d_{i-j_1}$ there are three cases. First, if $i < j_1$, then $(i + j_1) d_{i-j_1} \in \mathcal{U}(n^-)_{(1)} = \mathcal{U}(n^-)_{(s)}$. Secondly, if $i > j_1$, then $(i + j_1) d_{i-j_1} \in \mathcal{U}(\mathrm{Vir})n^+$. Finally, if $i = j_1$ (this case does not occur when proving (42)), then $(i + j_1) d_{i-j_1} = (i + j_1) d_0 \in \mathcal{U}(n^-)_{(0)} d_0 = \mathcal{U}(n^-)_{(s-1)} d_0$.

For the induction step, first note that

$$d_i d_{-j_s} \ldots d_{-j_1} = d_{-j_s} d_i d_{-j_{s-1}} \ldots d_{-j_1} + [d_i, d_{-j_s}] d_{-j_{s-1}} \ldots d_{-j_1}.$$

Using the induction hypothesis and (39) we have

$$d_{-j_s} d_i d_{-j_{s-1}} \ldots d_{-j_1} \in \mathcal{U}(n^-)_{(1)} K_{(s-1)} \subseteq K_{(s)}.$$

Therefore it is enough to show that

$$[d_i, d_{-j_s}] d_{-j_{s-1}} \ldots d_{-j_1} \in K_{(s)}. \tag{43}$$

This is clear if $i - j_s < 0$, since $\mathcal{U}(n^-)_s \subseteq K_{(s)}$. But (43) is also true if $i - j_s > 0$, using the induction hypothesis and (40). It remains to consider the case $i = j_s$ (this case does not occur when proving (42)). Since $[d_i, d_{-i}] = 2id_0 + \frac{i^3-i}{12} c$, we get

$$\begin{aligned}
[d_i, d_{-j_s}] d_{-j_{s-1}} \ldots d_{-j_1} &= (2id_0 + \frac{i^3 - i}{12} c) d_{-j_{s-1}} \ldots d_{-j_1} = \\
&= \frac{i^3 - i}{12} d_{-j_{s-1}} \ldots d_{-j_1} c + 2i d_{-j_{s-1}} \ldots d_{-j_1} d_0 \\
&\quad + 2i(j_{s-1} + \ldots + j_1) d_{-j_{s-1}} \ldots d_{-j_1}.
\end{aligned}$$

Each of these terms belongs to the desired linear space $K_{(s)}$. □

In the next lemmas, $\langle\cdot|\cdot\rangle$ will denote the Shapovalov form on $M(C,h)$. We will fix $C \in \mathbb{R}$, and consider an expression of the form

$$\langle d_{-i_s} \dots d_{-i_1} v | d_{-j_t} \dots d_{-j_1} v\rangle$$

as a polynomial in h. We will use the notation $\deg_h p$ for the degree of p as a polynomial in h.

Lemma 15. *Suppose we have some integers $s, t \geq 1$ and $i_{t-1}, \dots, i_1 \geq 1$. If $x \in K_{(s)}$, then*

$$\deg_h \langle d_{-i_{t-1}} \dots d_{-i_1} v | xv\rangle \leq \min\{t, s\}. \tag{44}$$

Proof. To show (44), we use induction on $t+s$. If $t+s=2$, then $t=s=1$ and we have

$$xv = \alpha d_0 v + \beta c v + (\gamma d_{-k} + \delta)v = (\alpha h + \beta C + \delta)v + \gamma d_{-k} v$$

for some $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ and $k \geq 1$. Thus

$$\langle v | xv\rangle = \alpha h + \beta C + \delta.$$

The degree of this as a polynomial in h is less than or equal to $1 = \min\{t, s\}$.

The induction step can be carried out by noting that xv is a linear combination of elements of the form

$$\begin{aligned} w_1 &= d_{-k_{r-1}} \dots d_{-k_1} d_0 v = h d_{-k_{r-1}} \dots d_{-k_1} v, \\ w_2 &= d_{-k_{r-1}} \dots d_{-k_1} c v = C d_{-k_{r-1}} \dots d_{-k_1} v, \\ w_3 &= d_{-k_r} \dots d_{-k_1} v, \end{aligned}$$

where $r \leq s$. By Lemma 14 we have

$$d_{i_{t-1}} d_{-k_{r-1}} \dots d_{-k_1} \in K_{(r-1)} \subseteq K_{(s-1)}$$

$$d_{i_{t-1}} d_{-k_r} \dots d_{-k_1} \in K_{(r)} \subseteq K_{(s)}$$

and therefore,

$$\begin{aligned} \deg_h \langle d_{-i_{t-1}} \dots d_{-i_1} v | w_1\rangle = \deg_h \Big(h \cdot \langle d_{-i_{t-2}} \dots d_{-i_1} v | d_{i_{t-1}} d_{-k_{r-1}} \dots d_{-k_1} v\rangle \Big) \leq \\ \leq 1 + \min\{t-1, s-1\} \leq \min\{t, s\} \end{aligned}$$

by the induction hypothesis. Similarly,

$$\begin{aligned} \deg_h \langle d_{-i_{t-1}} \dots d_{-i_1} | w_2\rangle = \deg_h \Big(C \cdot \langle d_{-i_{t-2}} \dots d_{-i_1} v | d_{i_{t-1}} d_{-k_{r-1}} \dots d_{-k_1} v\rangle \Big) \leq \\ \leq \min\{t-1, s-1\} \leq \min\{t, s\} \end{aligned}$$

Finally,

$$\deg_h \langle d_{-i_{t-1}} \dots d_{-i_1} | w_3 \rangle = \deg_h \langle d_{-i_{t-2}} \dots d_{-i_1} v | d_{i_{t-1}} d_{-k_r} \dots d_{-k_1} v \rangle \le \\ \le \min\{t-1, s\} \le \min\{t, s\}$$

This proves the induction step. □

Corollary 16. *If* $i_t, \dots, i_1 \ge 1$ *and* $j_s, \dots, j_1 \ge 1$, *where* $s, t \ge 1$, *then*

$$\deg_h \langle d_{-i_t} \dots d_{-i_1} v | d_{-j_s} \dots d_{-j_1} v \rangle \le \min\{t, s\}. \tag{45}$$

Proof. Take $x = d_{i_t} d_{-j_s} \dots d_{-j_1}$ which is in $K_{(s)}$ by Lemma 14. □

We now consider the case $s = t$.

Lemma 17. *Let* $t \ge 1$ *be an integer.*

i) If $i_t \ge \dots \ge i_1 \ge 1$ *then*

$$\deg_h \langle d_{-i_t} \dots d_{-i_1} v | d_{-i_t} \dots d_{-i_1} v \rangle = t. \tag{46}$$

and the coefficient of h^t *is positive.*

ii) If $i_t \ge \dots \ge i_1 \ge 1$ *and* $j_t \ge \dots \ge j_1 \ge 1$ *but* $(i_t, \dots, i_1) \ne (j_t, \dots, j_1)$, *then*

$$\deg_h \langle d_{-i_t} \dots d_{-i_1} v | d_{-j_t} \dots d_{-j_1} v \rangle < t \tag{47}$$

Proof. We show part *i)* by induction on t. For $t = 1$ we have

$$d_{i_1} d_{-i_1} v = 2 i_1 d_0 v + \frac{i_1^3 - i_1}{12} c v = 2 i_1 h v + \frac{i_1^3 - i_1}{12} C v$$

and therefore

$$\langle d_{-i_1} v | d_{-i_1} v \rangle = \langle v | d_{i_1} d_{-i_1} v \rangle = \langle v | 2 i_1 h v + \frac{i_1^3 - i_1}{12} C v \rangle = 2 i_1 h + \frac{i_1^3 - i_1}{12} C$$

For the induction step, use the formula

$$d_{i_t} d_{-i_t} \dots d_{-i_1} v = \sum_{r=1}^{t} d_{-i_t} \dots d_{-i_{r+1}} [d_{i_t}, d_{-i_r}] d_{-i_{r-1}} \dots d_{-i_1} v$$

and note that $i_t - i_r \ge 0$. We consider each term separately. If r is such that $i_r = i_t$, then

$$d_{-i_t} \dots d_{-i_{r+1}} [d_{i_t}, d_{-i_r}] d_{-i_{r-1}} \dots d_{-i_1} v = \\ = d_{-i_t} \dots d_{-i_{r+1}} \big(2 i_t d_0 + \frac{i_t^3 - i_t}{12} c \big) d_{-i_{r-1}} \dots d_{-i_1} v = \\ = \Big(2 i_t (h + i_{r-1} + \dots + i_1) + \frac{i_t^3 - i_t}{12} C \Big) d_{-i_t} \dots d_{-i_{r+1}} d_{-i_{r-1}} \dots d_{-i_1} v.$$

Thus, using the induction hypothesis,

$$\deg_h \langle d_{-i_{t-1}} \dots d_{-i_1} v | d_{-i_t} \dots d_{-i_{r+1}} [d_{i_t}, d_{-i_r}] d_{-i_{r-1}} \dots d_{-i_1} v \rangle = 1 + t - 1 = t$$

and the coefficient of h^t is positive.

If r is such that $i_r < i_t$, then by Lemma 14 we have

$$d_{-i_t} \dots d_{-i_{r+1}} [d_{i_t}, d_{-i_r}] d_{-i_{r-1}} \dots d_{-i_1} \in \mathfrak{U}(n^-)_{(t-r)} K_{(r-1)} \subseteq K_{(t-1)},$$

so it follows from Lemma 15 that

$$\deg_h \langle d_{-i_{t-1}} \dots d_{-i_1} v | d_{-i_t} \dots d_{-i_{r+1}} [d_{i_t}, d_{-i_r}] d_{-i_{r-1}} \dots d_{-i_1} v \rangle \leq \min\{t, t-1\} = t - 1.$$

Thus such terms do not contribute to the highest power of h.

To show (47), we use induction on t. For $t = 1$ we have $i_1 \neq j_1$ so $\langle d_{-i_1} v | d_{-j_1} v \rangle = 0$, since the eigenspaces of d_0 are pairwise orthogonal. For the induction step consider the calculation

$$\begin{aligned} &\deg_h \langle d_{-i_t} \dots d_{-i_1} v | d_{-j_t} \dots d_{-j_1} v \rangle = \\ &\quad = \deg_h \langle d_{-i_{t-1}} \dots d_{-i_1} v | \sum_{p=1}^{t} d_{-j_t} \dots d_{-j_{p+1}} [d_{i_t}, d_{-j_p}] d_{-j_{p-1}} \dots d_{-j_1} v \rangle \leq \\ &\qquad \leq \max_{1 \leq p \leq t} \{ \deg_h \langle d_{-i_{t-1}} \dots d_{-i_1} v | d_{-j_t} \dots d_{-j_{p+1}} [d_{i_t}, d_{-j_p}] d_{-j_{p-1}} \dots d_{-j_1} v \rangle \} \end{aligned}$$

For each $p \in \{1, \dots, t\}$ we consider three cases. First, if $i_t - j_p < 0$ then

$$d_{-j_t} \dots d_{-j_{p+1}} [d_{i_t}, d_{-j_p}] d_{-j_{p-1}} \dots d_{-j_1} \in \mathfrak{U}(n^-)_{(t-p)} \mathfrak{U}(n^-)_{(1)} \mathfrak{U}(n^-)_{(p-1)} \subseteq \mathfrak{U}(n^-)_{(t)}$$

so that

$$\deg_h \langle d_{-i_{t-1}} \dots d_{-i_1} v | d_{-j_t} \dots d_{-j_{p+1}} [d_{i_t}, d_{-j_p}] d_{-j_{p-1}} \dots d_{-j_1} v \rangle \leq t - 1 < t \tag{48}$$

by Corollary 16. Secondly, if $i_t - j_p > 0$ then

$$d_{-j_t} \dots d_{-j_{p+1}} [d_{i_t}, d_{-j_p}] d_{-j_{p-1}} \dots d_{-j_1} \in \mathfrak{U}(n^-)_{(t-p)} K_{(p-1)} \subseteq K_{(t-1)}$$

by Lemma 14, and therefore (48) holds again, using Lemma 15. For the third case, when $i_t - j_p = 0$, we have

$$d_{-j_t} \dots d_{-j_{p+1}} [d_{i_t}, d_{-j_p}] d_{-j_{p-1}} \dots d_{-j_1} v = \lambda d_{-j_t} \dots d_{-j_{p+1}} d_{-j_{p-1}} \dots d_{-j_1} v$$

where $\lambda = 2i_t(h + j_{p-1} + \ldots + j_1) + \frac{i_t^3 - i_t}{12} C$. We claim now that

$$(i_{t-1}, \dots, i_1) \neq (j_t, \dots, j_{p+1}, j_{p-1}, \dots, j_1). \tag{49}$$

Assume the contrary. Then in particular $i_{t-1} = j_t$, and since $j_t \geq \ldots \geq j_1$ and $i_t \geq \ldots \geq i_1$, $i_t = j_p$ we get

$$j_t \geq \ldots \geq j_{p+1} \geq j_p = i_t \geq i_{t-1} = j_t.$$

Thus all inequalities must be equalities. Hence

$$j_{p+1} = i_{t-1} \geq \ldots \geq i_p = j_{p+1}.$$

Again all inequalities must be equalities, and consequently

$$j_k = i_l \quad \text{whenever } k, l \geq p.$$

In addition we assumed that $i_k = j_k$ for $k < p$. This contradicts $(i_t, \ldots, i_1) \neq (j_t, \ldots, j_1)$, so (49) is true. Thus we can use the induction hypothesis to conclude that

$$\begin{aligned}\deg_h \langle d_{-i_{t-1}} \ldots d_{-i_1} v | d_{-j_t} \ldots d_{-j_{p+1}} [d_{i_t}, d_{-j_p}] d_{-j_{p-1}} \ldots d_{-j_1} v \rangle &= \\ = 1 + \deg_h \langle d_{-i_{t-1}} \ldots d_{-i_1} v | d_{-j_t} \ldots d_{-j_{p+1}} d_{-j_{p-1}} \ldots d_{-j_1} v \rangle &< 1 + (t-1) = t.\end{aligned}$$

The proof is finished. □

4.2 Kac determinant formula

If p and q are two complex polynomials in h, we will write

$$p \sim q$$

if their highest degree terms coincide. In other words, $p \sim q$ if and only if $\deg_h(p-q) < \min\{\deg_h p, \deg_h q\}$. It is easy to see that $\sim$ is an equivalence relation on the set of complex polynomials in h.

Proposition 18.

$$\det{}_n(C,h) \sim \prod_{\substack{1 \leq i_1 \leq \ldots \leq i_t \\ i_1 + \ldots + i_t = n}} \langle d_{-i_t} \ldots d_{-i_1} v | d_{-i_t} \ldots d_{-i_1} v \rangle \tag{50}$$

Proof. Let $P(n)$ denote the set of all partitions of n, and for $i \in P(n)$, let $\ell(i)$ denote the length of i. For $i = (i_1, \ldots, i_s), j = (j_1, \ldots, j_t) \in P(n)$, define

$$A_{ij} = \langle d_{-i_s} \ldots d_{-i_1} v | d_{-j_t} \ldots d_{-j_1} v \rangle$$

Then a standard formula for the determinant gives

$$\det{}_n(C,h) = \sum_{\sigma \in S_{P(n)}} (-1)^{\operatorname{sgn} \sigma} \prod_{i \in P(n)} A_{i\sigma(i)}. \tag{51}$$

We will show that the term with $\sigma = \mathrm{id}$ has strictly higher h-degree than the other terms in the sum. From Lemma 17 part i) follows that $\deg_h A_{ii} = \ell(i)$ for all $i \in P(n)$. Therefore, we have

$$\deg_h \prod_{i\in P(n)} A_{i\sigma(i)} = \sum_{i\in P(n)} \ell(i) \qquad \text{when } \sigma = \mathrm{id}\,. \tag{52}$$

It follows from Corollary 16 that

$$\deg_h A_{i\sigma(i)} \leq \min\{\ell(i), \ell(\sigma(i))\},$$

for any $\sigma \in S_{P(n)}$ and all $i \in P(n)$. Also, by trivial arithmetic,

$$\min\{\ell(i), \ell(\sigma(i))\} \leq \frac{\ell(i) + \ell(\sigma(i))}{2}, \tag{53}$$

so for any $\sigma \in S_{P(n)}$, $i \in P(n)$ it is true that

$$\deg_h A_{i\sigma(i)} \leq \frac{\ell(i) + \ell(\sigma(i))}{2}. \tag{54}$$

But when $\sigma \neq \mathrm{id}$, there is some $j \in P(n)$ such that $\sigma(j) \neq j$. If $\ell(\sigma(j)) \neq \ell(j)$, the inequality (53) is strict for $i = j$. On the other hand, if $\ell(\sigma(j)) = \ell(j)$, then we can use Lemma 17 part ii) to obtain

$$\deg_h A_{j\sigma(j)} < \ell(j) = \frac{\ell(j) + \ell(\sigma(j))}{2}$$

In either case we have

$$\deg_h A_{j\sigma(j)} < \frac{\ell(j) + \ell(\sigma(j))}{2}. \tag{55}$$

Therefore, if we sum the inequalities (54) for all partitions $i \neq j$, and add (55) to the result we get

$$\deg_h \prod_{i\in P(n)} A_{i\sigma(i)} = \sum_{i\in P(n)} \deg_h A_{i\sigma(i)} < \sum_{i\in P(n)} \frac{\ell(i) + \ell(\sigma(i))}{2} = \sum_{i\in P(n)} \ell(i), \tag{56}$$

when $\sigma \neq \mathrm{id}$. In the last equality we used that $\sigma : P(n) \to P(n)$ is a bijection. Hence, combining (52) and (56) with (51), we obtain (50), which was to be proved. □

Lemma 19. *Let $k \geq 1$ be an integer. Then*

$$[d_n, d_{-n}^k] = nkd_{-n}^{k-1}\big(n(k-1) + 2d_0 + \frac{n^2-1}{12}c\big) \tag{57}$$

for all $n \in \mathbb{Z}$.

Proof. We use induction on k. For $k = 1$, we have

$$nd_{-n}^0\big(n \cdot 0 + 2d_0 + \frac{n^2-1}{12}c\big) = 2nd_0 + \frac{n^3-n}{12}c = [d_n, d_{-n}]. \tag{58}$$

For the induction step, we assume that (57) holds for $k = l$. Then consider the following calculations:

$$\begin{aligned}
[d_n, d_{-n}^{l+1}] &= d_n d_{-n}^{l+1} - d_{-n}^{l+1} d_n = \\
&= \big(d_n d_{-n}^l - d_{-n}^l d_n\big) d_{-n} + d_{-n}^l \big(d_n d_{-n} - d_{-n} d_n\big) = \\
&= [d_n, d_{-n}^l] d_{-n} + d_{-n}^l [d_n, d_{-n}] = \\
&= nl d_{-n}^{l-1}\big(n(l-1) + 2d_0 + \frac{n^2-1}{12}c\big) d_{-n} + d_{-n}^l (2nd_0 + \frac{n^3-n}{12}) = \\
&= nd_{-n}^l\big(ln(l+1) + (l+1)(2d_0 + \frac{n^2-1}{12}c)\big) = \\
&= n(l+1) d_{-n}^l (nl + 2d_0 + \frac{n^2-1}{12}c)
\end{aligned}$$

This shows the induction step. ∎

Lemma 20. *Let $k \geq 1$ be an integer. Then*

$$\langle d_{-n}^k v | d_{-n}^k v\rangle = k! n^k (2h + \frac{n^2-1}{12}C)(2h + \frac{n^2-1}{12}C + n) \cdot \ldots \cdot (2h + \frac{n^2-1}{12}C + n(k-1)) \tag{59}$$

for all $n \in \mathbb{Z}$.

Proof. We use induction on k. For $k = 1$, the right hand side of (59) equals

$$1! n^1 \big(2h + \frac{n^2-1}{12}C + n(1-1)\big) = 2hn + \frac{n^3-n}{12}C$$

while the left hand side is

$$\begin{aligned}
\langle d_{-n} | d_{-n} v\rangle &= \langle v | d_n d_{-n} v\rangle = \\
&= \langle v | (d_{-n} d_n + 2nd_0 + \frac{n^3-n}{12}c) v\rangle = \\
&= \langle v | (2nh + \frac{n^3-n}{12}C) v\rangle = \\
&= 2hn + \frac{n^3-n}{12}C
\end{aligned}$$

So (59) holds for $k = 1$.

For the induction step, we suppose that (59) holds for $k = l$. Then we have

$$\begin{aligned}\langle d_{-n}^{l+1}v|d_{-n}^{l+1}v\rangle &= \langle d_{-n}^{l}v|d_n d_{-n}^{l+1}v\rangle = \\ &= \langle d_{-n}^{l}v|\Big(d_{-n}^{l+1}d_n + n(l+1)d_{-n}^{l}\big(nl + 2d_0 + \frac{n^2-1}{12}c\big)\Big)v\rangle = \\ &= n(l+1)(nl + 2h + \frac{n^2-1}{12}C)\langle d_{-n}^{l}v|d_{-n}^{l}v\rangle = \\ &= n(l+1)(nl + 2h + \frac{n^2-1}{12}C)\cdot l!n^l(2h + \frac{n^2-1}{12}C)(2h + \frac{n^2-1}{12}C + n)\cdot\ldots \\ &\quad \ldots\cdot(2h + \frac{n^2-1}{12}C + n(l-1)) = \\ &= (l+1)!n^{l+1}(2h + \frac{n^2-1}{12}C)(2h + \frac{n^2-1}{12}C + n)\cdot\ldots\cdot(2h + \frac{n^2-1}{12}C + nl)\end{aligned}$$

where we used Lemma 19 in the second equality. This shows the induction step and the proof is finished. □

Corollary 21.

$$\langle d_{-n}^{k}v|d_{-n}^{k}v\rangle \sim k!(2nh)^k$$

Lemma 22. *Let $i_1, \ldots, i_s, j_1, \ldots j_s \in \mathbb{Z}_{>0}$, where $i_p \neq i_q$ for $p \neq q$. Then*

$$\langle d_{-i_s}^{j_s}\ldots d_{-i_1}^{j_1}v|d_{-i_s}^{j_s}\ldots d_{-i_1}^{j_1}v\rangle \sim \langle d_{-i_s}^{j_s}v|d_{-i_s}^{j_s}v\rangle\ldots\langle d_{-i_1}^{j_1}v|d_{-i_1}^{j_1}v\rangle. \tag{60}$$

Proof. We use induction on $\sum_k j_k$. If $\sum_k j_k = 1$, then we must have $s = 1$ so (60) is trivial.

To carry out the induction step, we will use that

$$\langle d_{-i_s}^{j_s}\ldots d_{-i_1}^{j_1}v|d_{-i_s}^{j_s}\ldots d_{-i_1}^{j_1}v\rangle = \langle d_{-i_s}^{j_s-1}\ldots d_{-i_1}^{j_1}v|d_{i_s}d_{-i_s}^{j_s}\ldots d_{-i_1}^{j_1}v\rangle.$$

First we use the Leibniz rule to obtain

$$\begin{aligned}d_{i_s}d_{-i_s}^{j_s}\ldots d_{-i_1}^{j_1}v &= \Big(\sum_{p=1}^{j_s} d_{-i_s}^{j_s-p}[d_{i_s}, d_{-i_s}]d_{-i_s}^{p-1}\Big)d_{-i_{s-1}}^{j_{s-1}}\ldots d_{-i_1}^{j_1}v \\ &\quad + d_{-i_s}^{j_s}d_{i_s}d_{-i_{s-1}}^{j_{s-1}}\ldots d_{-i_1}^{j_1}v = \\ &= \Big(\sum_{p=1}^{j_s} 2i_s\big(h + (p-1)i_s + j_{s-1}i_{s-1} + \ldots + j_1 i_1\big) + \frac{i_s^3 - i_s}{12}C\Big) \\ &\quad \cdot d_{-i_s}^{j_s-1}d_{-i_{s-1}}^{j_{s-1}}\ldots d_{-i_1}^{j_1}v + d_{-i_s}^{j_s}d_{i_s}d_{-i_{s-1}}^{j_{s-1}}\ldots d_{-i_1}^{j_1}v = \\ &= (2i_s j_s h + A)\cdot d_{-i_s}^{j_s-1}d_{-i_{s-1}}^{j_{s-1}}\ldots d_{-i_1}^{j_1}v + d_{-i_s}^{j_s}d_{i_s}d_{-i_{s-1}}^{j_{s-1}}\ldots d_{-i_1}^{j_1}v,\end{aligned}$$

where A is a constant independent of h. Consequently

$$\begin{aligned}\langle d_{-i_s}^{j_s}\dots d_{-i_1}^{j_1}v|d_{-i_s}^{j_s}\dots d_{-i_1}^{j_1}v\rangle &\sim 2i_sj_sh\langle d_{-i_s}^{j_s-1}\dots d_{-i_1}^{j_1}v|d_{-i_s}^{j_s-1}\dots d_{-i_1}^{j_1}v\rangle \\ &+\langle d_{-i_s}^{j_s-1}\dots d_{-i_1}^{j_1}v|d_{-i_s}^{j_s}d_{i_s}d_{-i_{s-1}}^{j_{s-1}}\dots d_{-i_1}^{j_1}v\rangle. \end{aligned} \tag{61}$$

By the induction hypothesis,

$$\begin{aligned} &2i_sj_sh\langle d_{-i_s}^{j_s-1}\dots d_{-i_1}{}^{j_1}v|d_{-i_s}^{j_s-1}\dots d_{-i_1}^{j_1}v\rangle\sim \\ &\sim 2i_sj_sh\langle d_{-i_s}^{j_s-1}v|d_{-i_s}^{j_s-1}v\rangle\cdot\langle d_{-i_{s-1}}^{j_{s-1}}v|d_{-i_{s-1}}^{j_{s-1}}v\rangle\dots\langle d_{-i_1}^{j_1}v|d_{-i_1}^{j_1}v\rangle\sim \\ &\sim 2i_sj_sh(j_s-1)!(2i_sh)^{j_s-1}\cdot\langle d_{-i_{s-1}}^{j_{s-1}}v|d_{-i_{s-1}}^{j_{s-1}}v\rangle\dots\langle d_{-i_1}^{j_1}v|d_{-i_1}^{j_1}v\rangle\sim \\ &\sim\langle d_{-i_s}^{j_s}v|d_{-i_s}^{j_s}v\rangle\dots\langle d_{-i_1}^{j_1}v|d_{-i_1}^{j_1}v\rangle. \end{aligned} \tag{62}$$

where we used Corollary 21 two times. The result will now follow from (61)-(62) if we can show that

$$\deg_h\langle d_{-i_s}^{j_s-1}\dots d_{-i_1}^{j_1}v|d_{-i_s}^{j_s}d_{i_s}d_{-i_{s-1}}^{j_{s-1}}\dots d_{-i_1}^{j_1}v\rangle<j_1+\ldots+j_s. \tag{63}$$

Since $i_s\neq i_p$ for $p<s$ we have by Lemma 14 that

$$d_{-i_s}^{j_s}d_{i_s}d_{-i_{s-1}}^{j_{s-1}}\dots d_{-i_1}^{j_1}\in\mathfrak{U}(\mathrm{Vir})n^+ +\mathfrak{U}(n^-)_{(k-2)}d_0+\mathfrak{U}(n^-)_{(k-2)}c+\mathfrak{U}(n^-)_{(k)},$$

where $k=j_1+\ldots+j_s$. If $x\in\mathfrak{U}(\mathrm{Vir})n^+$, then $xv=0$. If $x\in\mathfrak{U}(n^-)_{(k-2)}$, then

$$\deg_h\langle d_{-i_s}^{j_s-1}\dots d_{-i_1}^{j_1}v|xd_0v\rangle=1+\deg_h\langle d_{-i_s}^{j_s-1}\dots d_{-i_1}^{j_1}v|xv\rangle\leq 1+j_1+\ldots+j_s-2,$$

$$\deg_h\langle d_{-i_s}^{j_s-1}\dots d_{-i_1}^{j_1}v|xcv\rangle=\deg_h\langle d_{-i_s}^{j_s-1}\dots d_{-i_1}^{j_1}v|xv\rangle\leq j_1+\ldots+j_s-2,$$

by Corollary 16. Finally, if $y\in\mathfrak{U}(n^-)_{(k)}$, then

$$\deg_h\langle d_{-i_s}^{j_s-1}\dots d_{-i_1}^{j_1}v|yv\rangle\leq j_1+\ldots+j_s-1,$$

again by Corollary 16. These inequalities finishes the proof of (63) and we are done. □

Lemma 23.

$$\det_n(C,h)\sim\prod_{\substack{r,s\in\mathbb{Z}_{>0}\\ 1\leq rs\leq n}}\langle d_{-r}^s v|d_{-r}^s v\rangle^{m(r,s)},$$

where $m(r,s)$ is the number of partitions of n in which r appears exactly s times.

Proof. Use Proposition 18 and Lemma 22. □

Proposition 24. *For fixed C, $\det_n(C,h)$ is a polynomial in h of degree*

$$\sum_{\substack{r,s\in\mathbb{Z}_{>0}\\ 1\le rs\le n}} p(n-rs).$$

The coefficient K of the highest power of h is given by

$$K=\prod_{\substack{r,s\in\mathbb{Z}_{>0}\\ 1\le rs\le n}} \left((2r)^s s!\right)^{m(r,s)}, \tag{64}$$

and $m(r,s)$ can be calculated in terms of the partition function as follows:

$$m(r,s)=p(n-rs)-p(n-r(s+1)). \tag{65}$$

Proof. We first show (65). It is easy to see that the number of partitions of n in which r appears at least s times is $p(n-rs)$. But the number of partitions in which r appears exactly s times is equal to the number those which appears at least s times minus the number of those that appears at least $s+1$ times. Thus (65) is true.

From Lemma 23 and Corollary 21 follows that the coefficient of the highest power of h is equal to (64) and that

$$\begin{aligned}
\deg_h \det_n(C,h) &= \sum_{\substack{r,s\in\mathbb{Z}_{>0}\\ 1\le rs\le n}} s m(r,s) = \\
&= \sum_{1\le r\le n}\sum_{s=1}^{[n/r]} s\big(p(n-rs)-p(n-r(s+1))\big) = \\
&= \sum_{1\le r\le n}\sum_{s=1}^{[n/r]} \Big(p(n-rs)+(s-1)\cdot p(n-rs)-s\cdot p(n-r(s+1))\Big) = \\
&= \sum_{1\le r\le n}\sum_{s=1}^{[n/r]} \Big(p(n-rs)-[n/r]\cdot p(n-r([n/r]+1))\Big) = \\
&= \sum_{\substack{r,s\in\mathbb{Z}_{>0}\\ 1\le rs\le n}} p(n-rs)
\end{aligned}$$

□

Lemma 25. *Let V be a linear space of dimension n, and let $A\in \mathrm{End}(V)[t]$. Then* $\det A(t)$ *is divisible by t^k, where k is the dimension of* $\ker A(0)$.

Proof. Choose a basis $\{e_1, \dots e_k\}$ for the subspace $\ker A(0)$ of V and extend it to a basis $B = \{e_1, \dots e_k, e_{k+1}, \dots e_n\}$ for V. Write

$$A(t) = A_0 + A_1 t + \dots A_m t^m,$$

where $A_i \in \mathrm{End}(V)$. Let M_0 and $M(t)$ be the matrices of A_0 and $A(t)$ respectively in the basis B. Since $M_0 e_i = A(0)e_i = 0$ for $1 \leq i \leq k$, the first k columns of M_0 in the basis $\{e_1, \dots, e_n\}$ are zero, and therefore the first k columns of $M(t)$ are divisible by t. The result follows. □

Lemma 26. *If* $\det_n(C,h)$ *has a zero at* $h = h_0$*, then* $\det_n(C,h)$ *is divisible by*

$$(h - h_0)^{p(n-k)}$$

where k *is the smallest positive integer for which* $\det_k(C,h)$ *vanishes at* $h = h_0$*.*

Proof. Set $J_n(C,h) = J(C,h) \cap M(C,h)_{h+n} = \ker S_n(C,h)$. For $m \geq 1$, we have

$$\det{}_m(C,h_0) = 0 \quad \Longleftrightarrow \quad J_m(C,h_0) \neq 0.$$

Since $\det_k(C,h_0) = 0$ we can thus pick $u \in J_k(C,h_0)$, $u \neq 0$. This u must satisfy

$$d_n u = 0 \quad \text{for } n > 0,$$

since otherwise we would have for any $w \in M(C,h_0)$,

$$\langle w | d_n u \rangle = \langle d_{-n} w | u \rangle = 0,$$

because $u \in J(C,h_0)$. But $0 \neq d_n u \in M(C,h_0)_{h_0+k-n}$:

$$d_0 d_n u = [d_0, d_n]u + d_n d_0 u = (h_0 + k - n) d_n u$$

and this contradicts the minimality of k. Then $\mathcal{U}(\mathrm{Vir})u$ is a subrepresentation of $J(C,h_0)$. The subspace $\mathcal{U}(\mathrm{Vir})u \cap M(C,h)_{h+n}$ is spanned by the elements

$$d_{-i_s} \dots d_{-i_1} u, \qquad i_s \geq \dots i_1 \geq 1, \quad i_s + \dots + i_1 = n - k.$$

These are also linearly independent, since $\mathcal{U}(\mathrm{Vir})$ has no divizors of zero. Therefore $J_n(C,h_0)$ has a subspace of dimension $p(n-k)$, so $S_n(C,h_0)$ has a kernel of at least dimension $p(n-k)$. The result now follows from Lemma 25. □

We will need the following fact, which we will not prove.

Fact 27. $\det_n(C,h)$ *has a zero at* $h = h_{r,s}(C)$*, where*

$$h_{r,s}(C) = \frac{1}{48}\Big((13 - C)(r^2 + s^2) + \sqrt{(C-1)(C-25)}(r^2 - s^2) - 24rs - 2 + 2C\Big), \tag{66}$$

for each pair (r,s) *of positive integers such that* $1 \leq rs \leq n$*.*

The following is the main theorem of this article.

Theorem 28 (Kac determinant formula)**.**

$$\det_n(C,h) = K \prod_{\substack{r,s\in\mathbb{Z}_{>0}\\ 1\le rs\le n}} (h - h_{r,s}(C))^{p(n-rs)}, \tag{67}$$

where

$$K = \prod_{\substack{r,s\in\mathbb{Z}_{>0}\\ 1\le rs\le n}} ((2r)^s s!)^{m(r,s)} \tag{68}$$

and

$$m(r,s) = p(n-rs) - p(n-r(s+1))$$

and $h_{r,s}$ is given by (66).

Proof. From Fact 27 follows that $\det_n(C,h)$ has a zero at $h = h_{r,s}(C)$ for each $r,s\in\mathbb{Z}_{>0}$ satisfying $1\le rs\le n$. Using Lemma 26 we deduce that $\det_n(C,h)$ is divisible by $(h-h_{r,s}(C))^{p(n-rs)}$ for each $r,s\in\mathbb{Z}_{>0}$ with $1\le rs\le n$. Hence $\det_n(C,h)$ is divisible by the right hand side of (67), as polynomials in h. But we know from Proposition 24 that the degree in h of the determinant $\det_n(C,h)$ equals the degree in h of the right hand side of (67), and that the coefficient of the highest power of h is given by (68). Therefore equality holds in (67), and the proof is finished. □

If we set

$$\varphi_{r,r}(C) = h - h_{r,r}(C), \tag{69}$$

and

$$\varphi_{r,s}(C) = (h - h_{r,s}(C))(h - h_{s,r}(C)), \tag{70}$$

for $r\neq s$, then (67) can be written

$$\det_n(C,h) = K \prod_{\substack{r,s\in\mathbb{Z}_{>0}\\ s\le r\\ 1\le rs\le n}} \varphi_{r,s}(C)^{p(n-rs)}. \tag{71}$$

We will also use the following notation

$$\begin{aligned}\alpha_{r,s} &= \frac{1}{48}\big((13-C)(r^2+s^2) - 24rs - 2 + 2C\big) = \\ &= \frac{1}{4}(r-s)^2 - \frac{1}{48}(C-1)(r^2+s^2-2),\end{aligned} \tag{72}$$

$$\beta_{r,s} = \frac{1}{48}\sqrt{(C-1)(C-25)}(r^2-s^2). \tag{73}$$

Then

$$h_{r,s} = \alpha_{r,s} + \beta_{r,s}.$$

Note that α is symmetric in its indices, and β is antisymmetric. Therefore

$$\varphi_{r,s} = (h - h_{r,s})(h - h_{s,r}) = h^2 - 2\alpha_{r,s}h + \alpha_{r,s}^2 - \beta_{r,s}^2, \tag{74}$$

for $r \neq s$.

4.3 Consequences of the formula

Let us now return to the questions we asked at the beginning of Section 4.

Proposition 29. *a) $V(1,h) = M(1,h)$ if and only if $h \neq m^2/4$ for all $m \in \mathbb{Z}$.*
b) $V(0,h) = M(0,h)$ if and only if $h \neq (m^2-1)/24$ for all $m \in \mathbb{Z}$.

Proof. a) Putting $C = 1$ in (66) we get

$$h_{r,s}(1) = \frac{1}{48}\big(12(r^2+s^2) - 24rs\big) = \frac{(r-s)^2}{4}.$$

Thus, using (67) we obtain

$$\det_n(1,h) = K \prod_{\substack{r,s\in\mathbb{Z}_{>0}\\ 1\le rs\le n}} (h - \frac{(r-s)^2}{4})^{p(n-rs)}.$$

Therefore, $\det_n(1,h) \neq 0$ for all $n \in \mathbb{Z}$ if and only if $h \neq m^2/4$ for all $m \in \mathbb{Z}$.

b) When $C = 0$ we obtain

$$\begin{aligned} h_{r,s}(0) &= \frac{1}{48}\big(13(r^2+s^2) + 5(r^2-s^2) - 24rs - 2\big) = \\ &= \frac{9r^2 + 4s^2 - 12rs - 1}{24} = \\ &= \frac{(3r-2s)^2 - 1}{24}. \end{aligned}$$

Hence by formula (67) we have

$$\det_n(0,h) = K \prod_{\substack{r,s\in\mathbb{Z}_{>0}\\ 1\le rs\le n}} (h - \frac{(3r-2s)^2-1}{24})^{p(n-rs)}.$$

This shows that $\det_n(0,h) \neq 0$ for all $n \in \mathbb{Z}$ if and only if $h \neq (m^2-1)/24$ for all $m \in \mathbb{Z}$. $\square$

We need the following fact which we will not prove.

Fact 30. $V(1,3)$ *is unitary.*

Then we have the following proposition.

Proposition 31. *a)* $V(C,h) = M(C,h)$ *for* $C > 1,\ h > 0$.
b) $V(C,h)$ *is unitary for* $C \geq 1$ *and* $h \geq 0$.

Proof. a) It will be enough to show that $\det_n(C,h) > 0$ for all $C > 1, h > 0$ and $n \geq 1$. We prove in fact that each factor $\varphi_{r,s}$ of the product (71) is positive. For $s = r$, $1 \leq r \leq n$ we have

$$\varphi_{r,r}(C) = h - \frac{1}{48}\big((13-c)2r^2 - 24r^2 - 2 + 2C\big) = h + \frac{1}{24}(C-1)(r^2-1) > 0,$$

if $C > 1$ and $h > 0$. For $r \neq s$ we have

$$\begin{aligned}
\varphi_{r,s} &= h^2 - 2\alpha_{r,s}h + \alpha_{r,s}^2 - \beta_{r,s}^2 = \\
&= h^2 - \frac{1}{2}(r-s)^2 h + \frac{1}{24}(C-1)(r^2+s^2-2)h \\
&\quad + \frac{1}{16}(r-s)^4 - 2\frac{1}{4\cdot 48}(r-s)^2(C-1)(r^2+s^2-2) + \frac{1}{48^2}(C-1)^2(r^2+s^2-2)^2 \\
&\quad - \frac{1}{48^2}(C-1)(C-25)(r^2-s^2)^2 = \\
&= \Big(h - \frac{(r-s)^2}{4}\Big)^2 + \frac{1}{24}(C-1)(r^2+s^2-2)h \\
&\quad + \frac{1}{48^2}(C-1)^2\big((r^2+s^2-2)^2 - (r^2-s^2)^2\big) \\
&\quad + (C-1)\Big(\frac{24}{48^2}(r^2-s^2)^2 - \frac{1}{2\cdot 48}(r-s)^2(r^2+s^2-2)\Big) = \\
&= \Big(h - \frac{(r-s)^2}{4}\Big)^2 + \frac{1}{24}(C-1)(r^2+s^2-2)h \\
&\quad + \frac{1}{48^2}(C-1)^2\big(2r^2s^2 - 4(r^2+s^2) + 4 + 2r^2s^2\big) \\
&\quad + \frac{1}{96}(C-1)(r-s)^2\big(r^2 + 2rs + s^2 - r^2 - s^2 + 2\big) = \\
&= \Big(h - \frac{(r-s)^2}{4}\Big)^2 + \frac{1}{24}(C-1)(r^2+s^2-2)h \\
&\quad + \frac{1}{12\cdot 48}(C-1)^2(r^2-1)(s^2-1) \\
&\quad + \frac{1}{48}(C-1)(r-s)^2(rs+1).
\end{aligned}$$

This expression is strictly positive when $C > 1$ and $h > 0$. Therefore, when $C > 1, h > 0$, we have $\det_n(C, h) > 0$ for all $n > 0$, which proves part a).

b) Let $C \geq 1$ and $h \geq 0$ be arbitrary. Since $\mathbb{R}_{\geq 1} \times \mathbb{R}_{\geq 0}$ is pathwise connected, we can choose a path π from $(1, 3)$ to (C, h), i.e. a continuous function

$$\pi : [0, 1] \to \mathbb{R}_{\geq 1} \times \mathbb{R}_{\geq 0},$$

such that

$$p(0) = (1, 3) \quad \text{and} \quad p(1) = (C, h).$$

Moreover, this path can be chosen so that the image of the open interval $(0, 1)$ is contained in the open quadrant $\mathbb{R}_{>1} \times \mathbb{R}_{>0}$.

Let $n \in \mathbb{Z}_{\geq 0}$, and let

$$q(x, t) = a_n(x)t^{p(n)} + \ldots + a_0(x) = \det(S_n(\pi(x)) - tI)$$

be the characteristic polynomial of $S_n(\pi(x))$, the matrix of the Shapovalov form at level n on the Verma module with highest weight $\pi(x)$. Since $S_n(\pi(x))$ is Hermitian, each root of its characteristic equation is real. For $x \in [0, 1]$, we denote the roots by $\lambda_j(x)$, $j = 1, \ldots, p(n)$ such that

$$\lambda_1(x) \leq \ldots \leq \lambda_{p(n)}(x) \quad \text{for all } x \in [0, 1].$$

By a theorem on roots of polynomial equations, the roots are continuous functions of the coefficients. Thus, since the coefficients a_i in this case depend continuously on x, the roots $\lambda_j(x)$ of the characteristic equation of $S_n(\pi(x))$ are continuous functions of $x \in [0, 1]$. By the proof of part a) and the choice of π, we have $\det\big(S_n(\pi(x))\big) \neq 0$ for $x \in (0, 1)$. By Proposition 29 part a) we also have $\det(S_n(\pi(0))) = \det(S_n(1, 3)) \neq 0$, since $3 \neq m^2/4$ for all integers m. Thus none of the roots $\lambda_j(x)$ can be zero when $x < 1$. From Fact 30 follows that $\lambda_j(0) > 0$ for $j = 1, \ldots, p(n)$, so using the intermediate value theorem we obtain $\lambda_j(x) > 0$ for $j = 1, \ldots, p(n)$ and $x \in [0, 1)$. Hence $\lambda_j(1) \geq 0$ for $j = 1, \ldots, p(n)$. By spectral theory there is a unitary matrix U such that

$$\bar{U}^t S_n(\pi(1))U = U^{-1} S_n(\pi(1))U = \mathrm{diag}(\lambda_j(1)),$$

which shows that $S_n(\pi(1)) = S_n(C, h)$ is positive semi-definite for any $n \in \mathbb{Z}_{\geq 0}$. Thus $V(C, h)$ is unitary.

$\square$

4.4 Calculations for $n = 3$

In this section we calculate $\det_3(C, h)$ first by hand, and then by using Kac determinant formula.

4.4.1 By hand

We have

$$\det_3(C,h) = \begin{vmatrix} \langle d_{-3}v|d_{-3}v\rangle & \langle d_{-3}v|d_{-2}d_{-1}v\rangle & \langle d_{-3}v|d_{-1}^3v\rangle \\ \langle d_{-2}d_{-1}v|d_{-3}v\rangle & \langle d_{-2}d_{-1}v|d_{-2}d_{-1}v\rangle & \langle d_{-2}d_{-1}v|d_{-1}^3v\rangle \\ \langle d_{-1}^3v|d_{-3}v\rangle & \langle d_{-1}^3v|d_{-2}d_{-1}v\rangle & \langle d_{-1}^3v|d_{-1}^3v\rangle \end{vmatrix}.$$

We calculate the entries:

$$\begin{aligned} \langle d_{-3}v|d_{-3}v\rangle &= \langle v|(6d_0 + \frac{3^3-3}{12}c)v\rangle = \\ &= 6h + 2C \end{aligned}$$

$$\begin{aligned} \langle d_{-2}d_{-1}v|d_{-3}v\rangle &= \langle d_{-1}v|5d_{-1}v\rangle = \\ &= 10h \end{aligned}$$

$$\begin{aligned} \langle d_{-1}^3v|d_{-3}v\rangle &= \langle d_{-1}^2v|4d_{-2}v\rangle = \\ &= 4\langle d_{-1}v|3d_{-1}v\rangle = \\ &= 24h \end{aligned}$$

$$\begin{aligned} \langle d_{-2}d_{-1}v|d_{-2}d_{-1}v\rangle &= \langle d_{-1}v|(4d_0 + \frac{2^3-2}{12}c)d_{-1}v + d_{-2}3d_1v\rangle = \\ &= (4(h+1) + C/2)2h = \\ &= 8h^2 + (C+8)h \end{aligned}$$

$$\begin{aligned} \langle d_{-1}^3v|d_{-2}d_{-1}v\rangle &= \langle d_{-1}^2v|3d_{-1}d_{-1}v + d_{-2}2d_0v\rangle = \\ &= 3\langle d_{-1}v|2d_0d_{-1}v + d_{-1}2d_0v\rangle + 2h\langle d_{-1}v|3d_{-1}v\rangle = \\ &= 6(h+1)2h + 6h\cdot 2h + 6h\cdot 2h = \\ &= 36h^2 + 12h \end{aligned}$$

$$\begin{aligned} \langle d_{-1}^3v|d_{-1}^3v\rangle &= \langle d_{-1}^2v|2d_0d_{-1}^2v + d_{-1}2d_0d_{-1}v + d_{-1}^2 2d_0v\rangle = \\ &= 2(h+2+h+1+h)\langle d_{-1}v|2d_0d_{-1}v + d_{-1}2d_0v\rangle = \\ &= 6(h+1)\cdot 2(h+1+h)\cdot 2h = \\ &= 24h(2h^2+3h+1) = \\ &= 48h^3 + 72h^2 + 24h \end{aligned}$$

Thus the determinant is equal to

$$
\begin{aligned}
\det\nolimits_3(C,h) = \begin{vmatrix} 6h+2C & 10h & 24h \\ 10h & 8h^2+(C+8)h & 36h^2+12h \\ 24h & 36h^2+12h & 48h^3+72h^2+24h \end{vmatrix} = \\
&= 48h^2 \begin{vmatrix} 3h+C & 10h & 12h \\ 5 & 8h+C+8 & 18h+6 \\ 1 & 3h+1 & 2h^2+3h+1 \end{vmatrix} = \\
&= 48h^2\Big(12h\big(15h+5-(8h+C+8)\big) \\
&\quad -(18h+6)\big((3h+C)(3h+1)-10h\big) \\
&\quad +(2h^2+3h+1)\big((3h+C)(8h+C+8)-50h\big)\Big) = \\
&= 48h^2\Big(84h^2-(12C+36)h \\
&\quad -(18h+6)(9h^2+(3C-7)h+C) \\
&\quad +(2h^2+3h+1)(24h^2+(11C-26)h+C^2+8C)\Big) = \\
&= 48h^2\Big(84h^2-(12C+36)h \\
&\quad -\big(162h^3+(54C-72)h^2+(36C-42)h+6C\big) \\
&\quad +48h^4+(22C+20)h^3+(2C^2+49C-54)h^2 \\
&\qquad +(3C^2+35C-26)h+C^2+8C = \\
&= 48h^2\Big(48h^4+(22C-142)h^3+(2C^2-5C+102)h^2 \\
&\qquad +(3C^2-13C-20)h+C^2+2C\Big).
\end{aligned} \tag{75}
$$

4.4.2 Using the formula

To use the determinant formula, we first calculate the coefficient K for $n=3$. The partitions of 3 are (3), $(2,1)$ and $(1,1,1)$. Thus

$$
\begin{aligned}
K &= ((2\cdot 1)^1 1!)^1 \cdot ((2\cdot 1)^2 2!)^0 \cdot ((2\cdot 2)^1 1!)^1 \cdot ((2\cdot 1)^3 3!)^1 \cdot ((2\cdot 3)^1 1!)^1 = \\
&= 2\cdot 4\cdot 8\cdot 6\cdot 6 = 48^2.
\end{aligned}
$$

By (71) we now have

$$\det\nolimits_3(C,h) = 48^2 \varphi_{1,1}^2 \varphi_{2,1} \varphi_{3,1}. \tag{76}$$

First we have

$$\varphi_{1,1}(C) = h - h_{1,1}(C) = h. \tag{77}$$

We will use the notation introduced in (73)-(72). Then

$$\alpha_{2,1} = \frac{1}{4}(2-1)^2 - \frac{3}{48}(C-1) = \frac{5}{16} - \frac{1}{16}C,$$

$$\alpha_{2,1}^2 = \frac{1}{16^2}C^2 - \frac{10}{16^2}C + \frac{25}{16^2},$$

$$\beta_{2,1}^2 = \frac{9}{48^2}(C-1)(C-25) = \frac{1}{16^2}C^2 - \frac{26}{16^2}C + \frac{25}{16^2}.$$

Hence, using (74),

$$\varphi_{2,1}(C) = h^2 + (\frac{1}{8}C - \frac{5}{8})h + \frac{1}{16}C. \tag{78}$$

Also,

$$\alpha_{3,1} = \frac{1}{4}(3-1)^2 - \frac{8}{48}(C-1) = \frac{7}{6} - \frac{1}{6}C,$$

$$\alpha_{3,1}^2 = \frac{1}{36}C^2 - \frac{14}{36}C + \frac{49}{36},$$

$$\beta_{3,1}^2 = \frac{64}{48^2}(C-1)(C-25) = \frac{1}{36}C^2 - \frac{26}{36}C + \frac{25}{36}.$$

Therefore,

$$\varphi_{3,1}(C) = h^2 + (\frac{1}{3}C - \frac{7}{3})h + \frac{1}{3}C + \frac{2}{3}. \tag{79}$$

Consequently, using (76) we have

$$\begin{aligned}
\det{}_3(C,h) &= 48^2h^2\big(h^2 + (\frac{1}{8}C - \frac{5}{8})h + \frac{1}{16}C\big)\big(h^2 + (\frac{1}{3}C - \frac{7}{3})h + \frac{1}{3}C + \frac{2}{3}\big) = \\
&= 48h^2\big(16h^2 + (2C-10)h + C\big)\big(3h^2 + (C-7)h + C + 2\big) = \\
&= 48h^2\big(48h^4 + (16C - 112 + 6C - 30)h^3 \\
&\quad + (16C + 32 + 2C^2 - 14C - 10C + 70 + 3C)h^2 \\
&\quad + (2C^2 + 4C - 10C - 20 + C^2 - 7C)h + C^2 + 2C\big) = \\
&= 48h^2\big(48h^4 + (22C - 142)h^3 + (2C^2 - 5C + 102)h^2 \\
&\quad + (3C^2 - 13C - 20)h + C^2 + 2C\big).
\end{aligned}$$

This coincides with (75).

5 The centerless Ramond algebra

Let $\mathbb{C}[x, y, z]$ be the commutative associative algebra of polynomials in three indeterminates x, y, z. Form the ideal I generated by the two elements $xy - 1$ and z^2. Let

$$A = \mathbb{C}[x, y, z]/I$$

denote the quotient algebra. We will denote the images of x, y, and z under the canonical projection $\mathbb{C}[x,y,z] \to A$ by t, t^{-1} and ϵ respectively. Then we have

$$t^{-1}t = tt^{-1} = 1 \qquad \epsilon^2 = 0.$$

The algebra A can also be thought of as the tensor product algebra of $\mathbb{C}[t,t^{-1}]$ with the exterior algebra $\Lambda(\mathbb{C}\epsilon)$ on a one-dimensional linear space.

We have a $\mathbb{Z}_2$-grading

$$A = A_0 \oplus A_1, \tag{80}$$

$$A_iA_j \subset A_{i+j}, \tag{81}$$

defined by

$$A_0 = \mathbb{C}[t,t^{-1}], \qquad A_1 = \mathbb{C}[t,t^{-1}]\epsilon.$$

Since $A_1^2 = 0$, A can also be thought of as a supercommutative algebra:

$$ab = (-1)^{|a||b|}ba \quad \text{for } a,b \in A_0 \cup A_1,$$

where $|a| \in \mathbb{Z}_2$ denotes the degree of a homogenous element $a \in A_0 \cup A_1$.

For $n \in \mathbb{Z}$ we define the linear operators L_n, F_n on A by

$$L_n = -t^{n+1}\frac{d}{dt} - \frac{n}{2}t^n\epsilon\frac{d}{d\epsilon},$$

$$F_n = it^{n+1}\epsilon\frac{d}{dt} + it^n\frac{d}{d\epsilon}.$$

More explicitly we can define these mappings by requiring

$$L_n : t^k \mapsto -kt^{n+k},$$
$$L_n : t^k\epsilon \mapsto (-k - \frac{n}{2})t^{n+k}\epsilon,$$

and

$$F_n : t^k \mapsto ikt^{n+k}\epsilon,$$
$$F_n : t^k\epsilon \mapsto it^{n+k},$$

where $i = \sqrt{-1}$.

Proposition 32. *For $n \in \mathbb{Z}$, L_n is an even superderivation on A and F_n is an odd superderivation on A, in the sence that*

$$L_n(ab) = L_n(a)b + aL_n(b)$$

$$F_n(ab) = F_n(a)b + (-1)^{|a|}aF_n(b)$$

for homogenous $a,b \in A$.

Proof. A straightforward calculation yields

$$\begin{aligned}
L_n(t^k t^l) &= L_n(t^{k+l}) = (-k-l)t^{n+k+l} = -kt^{n+k}t^l - t^k \cdot lt^{n+l} = L_n(t^k)t^l + t^k L_n(t^l),\\
L_n(t^k \epsilon t^l) &= L_n(t^{k+l}\epsilon) = (-k-l-n/2)t^{n+k+l}\epsilon = (-k-n/2)t^{n+k}\epsilon \cdot t^l - t^k\epsilon \cdot lt^{n+l} =\\
&= L_n(t^k\epsilon)t^l + t^k\epsilon L_n(t^l),\\
L_n(t^k t^l \epsilon) &= L_n(t^{k+l}\epsilon) = (-k-l-n/2)t^{n+k+l}\epsilon = -kt^{n+k} \cdot t^l\epsilon + t^k \cdot (-l-n/2)t^{n+l}\epsilon =\\
&= L_n(t^k)t^l\epsilon + t^k L_n(t^l\epsilon),\\
L_n(t^k\epsilon t^l\epsilon) &= L_n(0) = 0 = (-k-n/2)t^{n+k}\epsilon \cdot t^l\epsilon + t^k\epsilon \cdot (-l-n/2)t^{n+l}\epsilon =\\
&= L_n(t^k\epsilon)t^l\epsilon + t^k\epsilon L_n(t^l\epsilon),
\end{aligned}$$

and

$$\begin{aligned}
F_n(t^k t^l) &= F_n(t^{k+l}) = i(k+l)t^{n+k+l}\epsilon = ikt^{n+k}\epsilon \cdot t^l + t^k \cdot ilt^{n+l}\epsilon = F_n(t^k)t^l + t^k F_n(t^l),\\
F_n(t^k\epsilon t^l) &= F_n(t^{k+l}\epsilon) = it^{n+k+l} = it^{n+k}t^l - t^k\epsilon \cdot ilt^{n+l}\epsilon = F_n(t^k\epsilon)t^l - t^k\epsilon F_n(t^l),\\
F_n(t^k t^l\epsilon) &= F_n(t^{k+l}\epsilon) = it^{n+k+l} = ikt^{n+k}\epsilon \cdot t^l\epsilon + t^k \cdot it^{n+l} = F_n(t^k)t^l\epsilon + t^k F_n(t^l\epsilon),\\
F_n(t^k\epsilon t^l\epsilon) &= F_n(0) = 0 = it^{n+k} \cdot t^l\epsilon - t^k\epsilon \cdot it^{n+l} = F_n(t^k\epsilon)t^l\epsilon - t^k\epsilon F_n(t^l\epsilon).
\end{aligned}$$

□

The *anticommutator* $[P, Q]_+$ of two linear operators P and Q on A is defined by

$$[P, Q]_+ = PQ + QP.$$

Proposition 33. *The operators L_n, F_n satisfy the following commutation relations:*

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n},\\
[L_m, F_n] &= (\frac{1}{2}m - n)F_{m+n},\\
[F_m, F_n]_+ &= 2L_{m+n}.
\end{aligned}$$

Remark 4. This shows that L_n and F_n generate a super Lie algebra. It is called the *centerless Ramond algebra*.

Proof. We have

$$\begin{aligned}
[L_m, L_n](t^k) &= (L_m L_n - L_n L_m)(t^k) =\\
&= L_m(-kt^{n+k}) - L_n(-kt^{m+k}) =\\
&= -k(-n-k)t^{m+n+k} + k(-m-k)t^{n+m+k} =\\
&= (m-n)(-k)t^{m+n+k} =\\
&= (m-n)L_{n+m}(t^k),
\end{aligned}$$

and

$$\begin{aligned}
[L_m, L_n](t^k\epsilon) &= (L_mL_n - L_nL_m)(t^k\epsilon) = \\
&= L_m((-k-n/2)t^{n+k}\epsilon) - L_n((-k-m/2)t^{m+k}\epsilon) = \\
&= (-k-n/2)(-n-k-m/2)t^{m+n+k}\epsilon \\
&\quad - (-k-m/2)(-m-k-n/2)t^{n+m+k}\epsilon = \\
&= (nk+n^2/2-mk-m^2/2)t^{m+n+k}\epsilon = \\
&= (m-n)(-k-(m+n)/2)t^{m+n+k}\epsilon = \\
&= (m-n)L_{m+n}(t^k\epsilon).
\end{aligned}$$

Also,

$$\begin{aligned}
[L_m, F_n](t^k) &= (L_mF_n - F_nL_m)(t^k) = \\
&= L_m(ikt^{n+k}\epsilon) - F_n(-kt^{m+k}) = \\
&= ik(-n-k-m/2)t^{m+n+k}\epsilon + ki(m+k)t^{n+m+k}\epsilon = \\
&= (m/2-n)ikt^{m+n+k}\epsilon = \\
&= (m/2-n)F_{m+n}(t^k),
\end{aligned}$$

and

$$\begin{aligned}
[L_m, F_n](t^k\epsilon) &= (L_mF_n - F_nL_m)(t^k\epsilon) = \\
&= L_m(it^{n+k}) - F_n((-k-m/2)t^{m+k}\epsilon) = \\
&= -i(n+k)t^{m+n+k} - (-k-m/2)it^{n+m+k} = \\
&= (m/2-n)it^{m+n+k} = \\
&= (m/2-n)F_{m+n}(t^k).
\end{aligned}$$

Finally we have,

$$\begin{aligned}
[F_m, F_n]_+(t^k) &= (F_mF_n + F_nF_m)(t^k) = \\
&= F_m(ikt^{n+k}\epsilon) + F_n(ikt^{m+k}\epsilon) = \\
&= ki^2t^{m+n+k} + ki^2t^{n+m+k} = \\
&= 2L_{m+n}(t^k),
\end{aligned}$$

and

$$\begin{aligned}
[F_m, F_n]_+(t^k\epsilon) &= (F_mF_n + F_nF_m)(t^k\epsilon) = \\
&= F_m(it^{n+k}) + F_n(it^{m+k}) = \\
&= i^2(n+k)t^{m+n+k}\epsilon + i^2(m+k)t^{n+m+k}\epsilon = \\
&= 2(-k-(m+n)/2)t^{m+n+k}\epsilon = \\
&= 2L_{m+n}(t^k).
\end{aligned}$$

The proof is finished. □

References

[1] Feigin, B.L., Fuchs, D.B.: Funkts. Anal. Prilozh. **16**, 47-63 (1982)

[2] Kac, V.G.: Proceedings of the International Congress of Mathematicians, Helsinki (1978)

[3] Kac, V.G.: Group theoretical methods in physics. In: Lecture Notes in Physics. Beiglböck, W., Böhm, A., Takasugi, E. (eds.), Vol. 94, p. 441. Berlin, Heidelberg, New York: Springer 1979

[4] Kac, V.G., Raina, A.K.: Bombay Lectures On Highest Weight Representations of Infinite Dimensional Lie Algebras. In: Advanced Series in Mathematical Physics. Phong, D.H., Yau, S-T. (eds.), Vol. 2. Singapore, New Jersey, Hong Kong: World Scientific 1987

[5] Shapovalov, N.N.: Funct. Anal. Appl. **6**, 307-312 (1972)

Master's Theses in Mathematical Sciences 2002:E16
ISSN 1404-6342

LUNFMA-3014-2002

Mathematics
Centre for Mathematical Sciences
Lund University
Box 118, SE-221 00 Lund, Sweden

http://www.maths.lth.se/