Research Statement - Jonas T. Hartwig

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My main area of research is representation theory, particularly of associative algebras related to Lie theory. I am also interested in connections to geometry, combinatorics and mathematical physics.

1. Principal flag orders and their spherical subalgebras

1.1. History

In two short papers by Gelfand and Tsetlin from 1950, bases and matrix coefficients are given for finite-dimensional irreducible representations of $\mathfrak{g}_n$, where $\mathfrak{g}_n$ is the general linear Lie algebra $\mathfrak{gl}_n$, respectively the orthogonal Lie algebra $\mathfrak{so}_n$. Crucially, in the chain $\mathfrak{g}_1 \subset \mathfrak{g}_2 \subset \cdots \subset \mathfrak{g}_n$, the branching rule $\mathfrak{g}_k \downarrow \mathfrak{g}_{k-1}$ is multiplicity-free. The bases are parametrized by triangular arrays of integers subject to interleaving conditions. Of particular importance is the *Gelfand-Tsetlin subalgebra* Γ of the universal enveloping algebra $U(\mathfrak{g}_n)$, defined as the subalgebra generated by the centers $Z(U(\mathfrak{g}_k))$ for $k = 1, 2, \ldots, n$. Two remarkable facts are that

- Γ is maximal commutative in $U(\mathfrak{g}_n)$ [2][10][27]; and
- $U(\mathfrak{g}_n)$ is free as a left Γ-module [30][1].

By construction, Γ acts diagonally in the Gelfand-Tsetlin bases. Moreover, the action of the root vectors of the reductive Lie algebra $\mathfrak{g}_n$ on the basis vectors can be expressed using shift operators (acting on the entries of the integer arrays) and rational function in the entries of the arrays.

In [2] these formulas were used to construct new infinite-dimensional irreducible representations of $\mathfrak{gl}_n$. Futorny and Ovsienko introduced in 2010 the notion of a *Galois order* [9] as certain subalgebra of the invariants of a skew group algebra, and proved that the enveloping algebra $U(\mathfrak{gl}_n)$ is an example of a Galois order. Any Galois order has an analog of the Gelfand-Tsetlin subalgebra Γ. It is expected, although still unproven, that $U(\mathfrak{so}_n)$ will also be an example. In [8] it was shown that finite W-algebras of type A are examples of Galois orders.

Knowing that an algebra U is a Galois order has valuable consequences for its representation theory, in particular for the category $\mathcal{GZ}$ of U-modules upon which Γ acts locally finitely. In the case of $U(\mathfrak{gl}_n$, this category contains all weight modules hence all of category $\mathcal{O}$. Understanding the category $\mathcal{GZ}$ is very much an active area of current research (see e.g. [5] and references therein). A completely separate application is to noncommutative birational equivalence: Computing the division ring of fractions of a Galois order is equivalent to solving a corresponding noncommutative invariant theory problem [8].

In [17] I devised a new method to help proving that a given algebra is a Galois order. Previous methods relied on proving that the candidate subalgebra Γ is maximal commutative, i.e. not properly contained in any other commutative subalgebra. For the finite W-algebras, this was done by hand in [8], which meant somewhat tedious leading term considerations. Worse is that in the trigonometric (=quantum) case, this method does not work. Thus it was unknown for a long time whether the quantum group $U_q(\mathfrak{gl}_n)$ is a Galois order. In my paper [17] I define a class of algebras called *principal Galois orders* and prove that they are indeed examples of Galois orders. With this method it suffices to show that an algebra is a principal Galois order, which turns out to be easier (even though they technically form a proper subclass). Interestingly, all examples in the literature so far are actually principal Galois orders, which provides an alternative way to study them. Using this method I was able to prove that $U_q(\mathfrak{gl}_n)$ is a principal Galois order. In particular this implies that its Gelfand-Tsetlin subalgebra is maximal commutative, which also had been an open problem for quite some time. Another application of principal Galois orders is a uniform construction of a family of simple modules parametrized by the maximal spectrum of Γ, [17]. This generalizes previous results for $U(\mathfrak{gl}_n)$ and other algebras, see [3].

In 2019, Webster introduced the notion of a *principal flag order*. On the one hand, these algebras are special cases of principal Galois orders, of a particularly simple form. On the other hand, any principal Galois order U appears as the "spherical subalgebra" of a principal flag order F. And in a nice case (all examples in the literature fall into this case), F and U are Morita equivalent. Thus the representation theory of U can be much simplified by instead studying the representation theory of F.

Let us be more precise by defining a special case of principal flag orders and their spherical subalgebras.

1.2. Definition (rational case)

Let V be a complex vector space, $W \leq \mathrm{GL}(V)$ be a complex reflection group, and $P \subset V$ be a W-invariant lattice acting on V by translations. Let $\mathbb{C}[V]$ be the algebra of polynomial functions on V, and $\mathbb{C}(V)$ its field of fractions. The semidirect product $P \rtimes W$ acts on $\mathbb{C}(V)$. Let $\mathcal{F} = \mathbb{C}(V)\#\mathbb{C}P \rtimes W$ be the corresponding smash product (skew group algebra). Notice that $\mathcal{F}$ acts naturally on $\mathbb{C}(V)$.

A *rational flag order* is a subalgebra F of $\mathcal{F}$ such that

1. $\mathbb{C}[V] \subseteq F$,
2. $\mathbb{C}(V)F = \mathcal{F}$,
3. $X(p) \in \mathbb{C}[V]$ for all $p \in \mathbb{C}[V]$ and all $X \in F$.

The subalgebra $U = eFe$ where $e = \frac{1}{|W|}\sum_{w \in W} w$ is the symmetrizing idempotent, is called the *spherical subalgebra* of F.

Replacing $\mathbb{C}[V]$ by $\mathbb{C}[T]$, the algebra of functions on a complex torus $T = (\mathbb{C}^*)^n$, one obtains a trigonometric analog of these notions, relevant to the quantum algebra (q-analog) examples. In the general case of a principal flag/Galois order, $\mathbb{C}[V]$ is replaced by an arbitrary integrally closed noetherian domain, see [32][17] for details.

1.3. Examples

Examples of rational flag orders include:

- Trigonometric Cherednik algebras associated to $W = G(\ell, p, n)$, [32][25].
- The nilHecke algebra [32]
- Any Braverman-Finkelberg-Nakajima Coulomb branch [32].

Examples of rational Galois orders include:

- The spherical subalgebra of the previous examples. Thus in particular:
- The universal enveloping algebra $U(\mathfrak{gl}_n)$, [9].
- Any orthogonal Gelfand-Tsetlin (OGZ) algebra [26][32].
- Any truncated (=level p) Yangian $Y_p(\mathfrak{gl}_n)$ [8].
- Any finite W-algebra of type A [8].
- "Parabolic" subalgebras of the above examples [17].

1.4. The quantum group $U_q(\mathfrak{gl}_n)$ and generalizations

In [6] we proved that the quantized enveloping algebra $U_q(\mathfrak{gl}_n)$ is a "Galois ring" which is weaker than Galois order. In [17] I proved that $U_q(\mathfrak{gl}_n)$ is a Galois order.

In [14] I introduced a new family of algebras called *quantum OGZ algebras*. They are quantizations of the orthogonal Gelfand-Tsetlin (OGZ) algebras introduced by Mazorchuk [26]. Special cases include $U_q(\mathfrak{gl}_n)$ and quantized Heisenberg algebras. I showed that quantum OGZ algebras are Galois rings, with symmetry group being a direct product of complex reflection groups $G(m,p,r_k)$. Later, in [17], I proved that the quantum OGZ algebras are also examples of Galois orders.

1.5. Category of Gelfand-Tsetlin modules

Let F be a rational flag order. A finitely generated F-module V is a *Gelfand-Tsetlin module* if $\mathbb{C}[V]$ acts locally finite on V. Equivalently, V splits as a direct sum of generalized weight spaces with respect to $\mathbb{C}[V]$.

The full subcategory of Gelfand-Tsetlin modules is one of the primary objects of study in the representation theory of a rational (or more generally, principal) flag order. In particular a crucial problem is to understand the simple objects.

The *fiber* over $\lambda \in V$, denoted $\Phi_F(\lambda)$, is the set of isomorphism classes of simple Gelfand-Tsetlin F-modules V such that $V_\lambda \neq 0$, where

$$V_\lambda = \{v \in V \mid \forall p \in \mathbb{C}[V]\, \exists N > 0 : (p - p(\lambda)1)^N v = 0\}.$$

The main theorem of Futorny-Ovsienko [10] is the following (which we formulate only in the present special case of rational flag orders).

- **Theorem** The set $\Phi_F(\lambda)$ is non-empty and finite for any $\lambda \in V$.

The cardinality can be explicitly bound in terms of orders of stabilizer subgroups, provided it is known that F is flat as a left $\mathbb{C}[V]$-module. This theorem can be regarded as a rough classification of simple Gelfand-Tsetlin modules.

1.6. Ongoing and future work

This subsection contains some topics related to Galois order that I am working on.

1.6.1. The orthogonal Lie algebra

An obvious question is how much of the Gelfand-Tsetlin theory for the general linear Lie algebra can be carried over to the orthogonal Lie algebra. Evidence suggest most of it should carry over. According to Mazorchuk [27], the Gelfand-Tsetlin subalgebra Γ is maximal commutative. Moreover the corresponding classical integrable system associated to $\mathfrak{so}_n$ has been studied by Evans and Colarusso [1]. Their work indicates that $U(\mathfrak{so}_n)$ is free as a left and right Γ-module. However, it is still unknown whether $U(\mathfrak{so}_n)$ is a Galois order. The category of Gelfand-Tsetlin modules has not been studied systematically either; only partial results are known.

1.6.2. Residue description of principal flag/Galois orders

Given a principal flag order F with an embedding $F \to \mathcal{F}$, how can we describe the image in a global way (not just with a generating set)? For any ideal I of the algebra Λ one can construct a principal flag order by considering rational shift operators which preserve Λ and I. For example, Ginzburg-Kapranov-Vasserot [11] proved that the affine Hecke algebra (and its degenerate version) can be obtained in this way. In that case I is the principal ideal generated by the Weyl denominator (generator of the relative invariants $\mathbb{C}[T]^W_{\mathrm{sgn}}$ as a $\mathbb{C}[T]^W$-module). It is expected that when such a global description is possible, it would be much easier to describe simple Gelfand-Tsetlin modules, because it allows the direct calculation of relevant localizations.

1.6.3. Flatness and deformation quantization

A crucial property of the known examples of principal flag and Galois orders is that they are flat (in fact, free) over their Gelfand-Tsetlin subalgebras. Since flatness means locally free, this implies that upon localization and completion, the cardinality of isoclasses of simple Gelfand-Tsetlin modules in a given fiber can be bounded from above with a precise bound [10][32]. In this context I am interested in the connections to poisson geometry and deformation quantization. In particular, I would like to solve the following problem:

- Derive sufficient conditions for a principal flag order to be flat over its Gelfand-Tsetlin subalgebra.

1.6.4. Noncommutative deformations of Kleinian singularities

These noncommutative algebras, introduced by Rosenberg (Type A) and Crawley-Boevey and Holland (Types ADE), are deformations of the algebra of functions on a Kleinian singularity $\mathbb{C}^2/\Gamma$ where Γ is a finite subgroup of $\mathrm{SL}(2,\mathbb{C})$. They are the simples examples of symplectic reflection algebras which are not rational Cherednik algebras [4]. In ongoing work I have shown that noncommutative Kleinian singulariteis of Type D are examples of Galois orders. Previously, via generalized Weyl algebras, it is known that the Type A singularities are also Galois orders. I am interested in finding a uniform realization of all types ADE noncommutative Kleinian singularities as Galois orders, and to study the consequences for representation theory.

1.6.5. Integrable systems on Slodowy slices (with M. Colarusso)

The goal of this project is to generalize the classical Kostant-Wallach integrable system related to the Lie algebra $\mathfrak{gl}_n$, to the case of an arbitrary Slodowy slice. This is motivated by the connection to Gelfand-Tsetlin theory for finite W-algebras [8], which have been shown to be quantizations of Slodowy slices by Gan and Ginzburg.

1.6.6. Cherednik algebras and Hopf Galois orders

Work of Webster [32] implies that rational Cherednik algebras associated to some complex reflection groups are examples of principal flag orders. In ongoing work I have shown the same for trigonometric Cherednik algebras, and affine and degenerate affine Hecke algebras. Via Dunkl-Opdam differential-reflection representations one can also realize these algebras using a differential analog of principal flag orders. I am also developing a generalized framework which include both group algebra and differential analogs of Galois order and which makes contact with the theory of Hopf Galois extensions.

1.6.7. Crystal bases

In [18] we gave explicit crystal data for the crystal bases of finite-dimensional irreducible $\mathfrak{sl}_n$-modules in terms of Gelfand-Tsetlin patterns. We conjecture that this method may lead to crystal bases for other examples of Galois orders, such as OGZ algebras.

2. Twisted generalized Weyl algebras

Introduced by Mazorchuk and Turowska [29] in 1999, these are certain $\mathbb{Z}^n$-graded algebras given by generators and relations involving automorphisms σ_i and distinguished elements t_i of a base ring R. On the one hand this family contains many interesting noncommutative algebras such as multiparameter quantized Weyl algebras, primitive quotients of enveloping algebras, quantum spheres; their tensor products and finite order invariant subalgebras. On the other hand their representation theory is reminiscent of that of weight modules over semi-simple Lie algebras.

2.1. Definition

Given an algebra R with n commuting automorphisms σ_i and n central non-zerodivisors t_i of R, let $\tilde{A}$ be the algebra obtained from R by adjoining $2n$ generators $X_i^{\pm}$ to R subject to relations:

$$\sigma_i^{\pm 1}(r)X_i^{\pm} = X_i^{\pm}\sigma^{\mp 1}(r) \qquad \forall r \in R,$$
$$X_i^{\pm}X_i^{\mp} = \sigma_i^{\pm 1}(t_i),$$
$$[X_i^{\pm}, X_j^{\mp}] \qquad i \neq j.$$

The associated *twisted generalized Weyl algebra* A is defined as $\tilde{A}/I$ where I can be defined as

$$I = \{a \in \tilde{A} \mid \exists r \in Z(R) \cap R_{\text{reg}} : ra = 0\}$$

The original definition of I is less explicit [29]. I showed the equivalence to this simpler definition in [15].

2.2. Examples

- If $\sigma_i(t_j) = t_j$ for all $i \neq j$, then A is a generalized Weyl algebra (GWA) as introduced by Bavula in the early 1990's.
- If $R = \mathbb{C}[t_1, t_2, \ldots, t_n]$ is a polynomial algebra in the t_i's and if $\sigma_i(t_j) = t_j - \delta_{ij}$, then A is isomorphic to the n:th Weyl algebra.
- If $t_i = 1$ for all i then A is isomorphic to the skew group algebra $R\#\mathbb{Z}^n$.
- If $R = \mathbb{C}[u_1, u_2, \ldots, u_{n+1}]$ and $t_i = u_i(u_{i+1} - 1)$ and $\sigma_i(u_j) = u_j - \delta_{ij} + \delta_{i+1,j}$ then A is isomorphic to a quotient of $U(\mathfrak{gl}_{n+1})$ [31][22].

2.3. Consistency equations and their solutions

A critical point about TGWAs is that the automorphisms σ_i and elements t_i must satisfy *(binary and ternary) consistency relations* in order to ensure that A is non-trivial:

$$\sigma_i(t_j)\sigma_j(t_i) = \sigma_i^{-1}(t_j)\sigma_j^{-1}(t_i) \quad \forall i \neq j$$

$$\sigma_i\sigma_j(t_k)\sigma_i^{-1}\sigma_j^{-1}(t_k) = \sigma_i\sigma_j^{-1}(t_k)\sigma_i^{-1}\sigma_j(t_k) \quad \forall i \neq j \neq k \neq i$$

The binary relations were pointed out in [29]. We discovered the ternary relations in [7] and proved the binary and ternary relations together were sufficient for consistency. Thus the natural question arises to describe all solutions to these TGWA consistency equations.

Some solutions were given in [29] and [31]. In [13], I found solutions attached to any symmetric generalized Cartan matrix. In [22] we generalized these by finding solutions attached to multiquivers. In the two papers [19][20], we obtained a complete classification of solutions in the case when R is a polynomial ring and the σ_i are additive shift automorphisms.

2.3.1. *Future work*

I am interested in studying solutions to the binary consistency equation when R is replaced by the algebra of holomorphic functions on an affine complex variety.

2.4. Relation to enveloping algebras of Lie (super) algebras

One of the motivations to introduce TGWAs was to find a class which contains Bavula's generalized Weyl algebras as well as enveloping algebras of simple Lie algebras [29]. Although the class of TGWAs is not the answer to this question, there are many interesting connections. In [29] the authors constructed some modules reminiscent of Lie algebra modules. Another connection is that the *Mickelsson-Zhelobenko step algebras* $Z(\mathfrak{gl}_{n+1}, \mathfrak{gl}_n \times \mathfrak{gl}_1)$ are examples of TGWAs [28].

In [13], I found that Kac-Moody Serre relations (which appear in presentations of simple Lie algebras and Kac-Moody algebras) and generalizations of such relations hold among the generators $X_i^{\pm}$, under reasonable assumptions in any TGWA. In [22] we proved that a primitive quotient U/J of the enveloping algebra U of a finite-dimensional simple Lie algebra is graded isomorphic to a TGWA if and only if J is the annihilator of a completely pointed simple weight module. We studied the super analog of these questions in [23].

2.5. The category of weight modules

Assume from now on that R is commutative. A finitely generated A-module is a *weight module* with respect to the subalgebra R if

$$V = \bigoplus_{\mathfrak{m}\in\mathrm{MaxSpec(R)}} V_{\mathfrak{m}}, \quad V_{\mathfrak{m}} = \{v \in V \mid \mathfrak{m}v = 0\}$$

Simple weight modules over TGWAs have been studied in [29][28][12][15][16]. There is a further important notion, that of weight modules with *no inner breaks*, first introduced in [29], which I generalized in [12] and reinterpreted in [15]. A weight module V has *no inner breaks* if $J_{\mathfrak{m}}V_{\mathfrak{m}} = 0$ for every $\mathfrak{m} \in \mathrm{MaxSpec}(R)$ where $J_{\mathfrak{m}}$ is the radical of the gradation form a certain graded subquotient $C_{\mathfrak{m}}$ of A.

In [12], I classified all simple weight modules without inner breaks over an arbitrary TGWA.

In [15] I found a connection between the representation theory of a class of TGWAs and two-dimensional lattice models. The algebras depend on two polynomials that must satisfy the binary consistency equation, which I re-interpreted as a quantization of the ice rule (local current conservation) in statistical mechanics. Solutions can accordingly be expressed in terms of multisets of higher spin vertex configurations on a twisted cylinder. I obtained a complete classification of all simple weight modules by proving that none of them have any inner breaks, and described their support geometrically. I also described the center of these TGWAs, which involves a remarkable analog of the Casimir operator.

In the paper [16], we provide a combinatorial way to compute the signature of the unique indefinite inner product on any simple weight modules over the TGWAs from [15]. In particular this gives a concrete way to determine which simple weight modules are unitarizable.

2.5.1. Future work

The classification of simple weight modules is still open, in the general case. As the rank two case showed, it is most interesting in cases where the support has discrete geometric interpretation. Therefore I am interested in the category of weight modules primarily in the cases when one expects a nice description of simple modules to be possible. For example when R is replaced by the ring of entire functions in the complex plane. I believe the Grothendieck ring of the category (see below) will be especially interesting in this case and it should relate even more intimately to the combinatorial properties of the six vertex model.

2.6. Tensor products and Grothendieck rings

Fix an algebra R and an n-tuple $\sigma = (\sigma_1, \dots, \sigma_n)$ of commuting automorphism of R. For each n-tuple $t = (t_1, \dots, t_n)$ of central regular elements of R, let $A(t)$ be the TGWA associated with (R, σ, t). In the paper [21] we proved that there are algebra maps $A(t) \otimes A(t') \to A(tt')$ of TGWAs which can be assembled into a bialgebra, analogous to Zelevinsky's Hopf algebras and in more recent work by Khovanov [24] and many others. This equips the complexified Grothendieck group of the category of weight modules with the structure of an algebra. We computed presentations of this algebra in serveral cases. In particular, indecomposable weight modules can be broken down into a tensor product of modules over simpler algebras (the Weyl algebra). For example, any indecomposable object of category $\mathcal{O}$ of $\mathfrak{sl}_2$ can be written as a tensor product of at most two Weyl algebra modules.

2.6.1. Future work

I plan to apply the tensor product structure to describe indecomposable modules more generally, and investigate other situations where this approach may be used.

3. Noncommutative invariant theory

We say two (noncommutative) Ore domains are *birationally equivalent* if they have isomorphic division rings of fractions.

Originally formulated in a special case by B. Feigin at RIMS in 1992, an Ore domain A is said to *satisfy the quantum Gelfand-Kirillov conjecture* if it is birationally equivalent to a quantum Weyl algebra over a purely transcendental field extension of the base field. This is a q-analog of the classical property of some enveloping algebras conjectured by Gelfand and Kirillov in the 1960's.

A related notion, the *q-difference Noether problem* for a group G acting on a quantum Weyl algebra (or a quantum plane) A asks whether the subring of G-invariants, A^G, is birationally equivalent to A (possibly with different deformation parameters).

A quirk of the quantum algebra world is that the quantum plane $\mathbb{C}_q[x,y] = \mathbb{C}\langle x,y\rangle/\langle xy - qyx\rangle$ and quantum Weyl algebra $A_1^q(\mathbb{C}) = \mathbb{C}\langle x,\partial\rangle/\langle \partial x - qx\partial - 1\rangle$ are actually birationally equivalent (in the $q \to 1$ limit this is not true: one is commutative and the other is not!). Consequently any theorem in the q-deformed setting simultaneously generalize the "noncommutative" setting (dealing with the Weyl algebra) and the "commutative" setting (dealing with polynomial algebras).

In [6] we proved that the Drinfeld-Jimbo quantum group $U_q(\mathfrak{gl}_n)$ satisfies the quantum Gelfand-Kirillov conjecture. The method was based on our realization of this algebra as a Galois ring, and proving that the q-difference Noether problem for all classical Weyl groups has a positive solution, generalizing results on multisymmetric functions of Mattuck and Miyata in the case $q = 1$, and q-deforming the noncommutative Noether problem for the symmetric group.

Generalizing this, in [14] I proved that the quantum OGZ algebras satisfy the quantum Gelfand-Kirillov conjecture. The proof relies on proving these are Galois rings, and that any complex reflection group $G = G(m, p, n)$ has a positive solution to the q-difference Noether problem.

3.1. Open problem

It is known that finite W-algebras of type A satisfy the usual Gelfand-Kirillov conjecture [8]. An open problem is to construct a q-analog of finite W-algebras of type A and to determine whether they satisfy the quantum Gelfand-Kirillov conjecture.

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1. M. Colarusso and S. Evens, *The Complex Orthogonal Gelfand–Zeitlin System*, International Mathematics Research Notices, 2019.
2. Yu.A. Drozd, V.M. Futorny, S.A. Ovsienko, *Harish-Chandra subalgebras and Gelfand-Zetlin modules*. In: Finite dimensional algebras and related topics (pp. 79-93). Springer, Dordrecht.
3. N. Early, V. Mazorchuk, E. Vishnyakova, *Canonical Gelfand-Zeitlin modules over orthogonal Gelfand-Zeitlin algebras*, International Mathematics Research Notices, rnz002, 2019.
4. P. Etingof and V. Ginzburg, *Symplectic reflection algebras, Calogero-Moser space, and deformed Harish-Chandra homomorphism*, Invent. Math. 147, 243–348 (2002).
5. V. Futorny, D. Grantcharov, L.E. Ramirez, P. Zadunaisky, *Gelfand-Tsetlin theory for rational Galois algebras*, Preprint arXiv:1801.09316.
6. V. Futorny and J.T. Hartwig, *Solution of a q-difference Noether problem and the quantum*

Gelfand-Kirillov conjecture for gl_N , Math. Z. 276 Issue 1 (2014) 1-37. arXiv:1111.6044
7. V. Futorny and J.T. Hartwig, *On the consistency of twisted generalized Weyl algebra*, Proc. Math. Soc. 2012, arXiv:1103.4374.
8. V. Futorny, S. Ovsienko, A. Molev, *The Gelfand–Kirillov conjecture and Gelfand–Tsetlin modules for finite W-algebras*, Advances in Mathematics 223 (2010) 773–796.
9. V. Futorny and S. Ovsienko, *Galois orders in skew monoid rings*, Journal of Algebra 324 (2010) 598–630.
10. V. Futorny and S. Ovsienko, *Fibers of characters in Gelfand-Tsetlin categories*, Transactions of the American Mathematical Society, Volume 366 no. 8 (2014) 4173–4208.
11. V. Ginzburg, M. Kapranov, E. Vasserot, *Residue Construction of Hecke Algebras*, Advances in Mathematics 128, 119 (1997).
12. J.T. Hartwig, *Locally finite simple weight modules over twisted generalized Weyl algebras*, Journal of Algebra 2006. arXiv:math/0502352.
13. J.T. Hartwig, *Twisted generalized Weyl algebras, polynomial Cartan matrices and Serre-type relations*, Communications in Algebra, 38 Issue 12 (2010), 4375-4389. arXiv:0908.2054.
14. J.T. Hartwig, *The q-difference Noether problem for complex reflection groups and quantum OGZ algebras*, Commun. Algebra 45 Issue 3 (2017) 1166-1176. arXiv:1512.09234.
15. J.T. Hartwig, *Noncommutative singularities and lattice models*, Journal of Algebra 2018. arXiv:1612.08125.
16. J.T. Hartwig, *Unitarizability of weight modules over noncommutative Kleinian fiber products*, Preprint. arXiv:1702.08168.
17. J.T. Hartwig, *Principal Galois orders and Gelfand-Zeitlin modules*, Advances in Mathematics 359 Article no. 106806 (2020). arXiv:1710.04186.
18. J.T. Hartwig and O. Kingston,*Gelfand-Tsetlin crystals*, Preprint arXiv:2005.06639.
19. D. Rosso and J.T. Hartwig, *Cylindrical Dyck paths and the Mazorchuk-Turowska equation*, Journal of Algebraic Combinatorics, 2016, arXiv:1507.04414.
20. D. Rosso and J.T. Hartwig, *Classification of twisted generalized Weyl algebras over polynomial rings*, Journal of Algebra, 2020, arXiv:1903.12105).
21. J.T. Hartwig and D. Rosso, *Grothendieck rings of towers of twisted generalized Weyl algebras*, Preprint. arXiv:2003.00957).
22. J.T. Hartwig and V. Serganova, *Twisted generalized Weyl algebras and primitive quotients of enveloping algebras*, Algebras and Representation Theory 2016. arXiv:1504.05361.
23. J.T. Hartwig and V. Serganova, *Clifford and Weyl superalgebras and spinor representations*, Transformation Groups 2018. arXiv:1807.04220.
24. M. Khovanov, *Heisenberg algebra and a graphical calculus*, Fundamenta Mathematicae 225 (2014) 169-210.

25. E. LePage and B. Webster, *Rational Cherednik algebras of $G(\ell, p, n)$ from the Coulomb perspective*, Preprint arXiv:1912.00046.
26. V. Mazorchuk, *Orthogonal Gelfand-Zetlin Algebras I.*, Contributions to Algebra and Geometry 40, no. 2 (1999) 399-415.
27. V. Mazorchuk, *On Gelfand-Zetlin modules over orthogonal Lie algebras*, Algebra Colloquium 8 no. 3 (2001) 345-360.
28. V. Mazorchuk, M. Ponomarenko, L. Turowska, *Some associative algebras associated to $U(\mathfrak{g})$ and twisted generalized Weyl algebras*, Math. Scand. 92 (2003), 5–30.
29. V. Mazorchuk and L. Turowska, *Simple weight modules over twisted generalized Weyl algebras*, Comm. Alg. 27 (1999) 2613-2625.
30. S. Ovsienko, *Strongly nilpotent matrices and Gelfand–Zetlin modules*, Linear algebra and its applications 365 (2003) 349-367.
31. A. Sergeev, *Enveloping algebra $U(gl(3))$ and orthogonal polynomials in several discrete indeterminates*, in "Noncommutative structures in mathematics and physics", Proc. NATO Advanced Research Workshop, Kiev, 2000, S. Duplij and J. Wess, eds., Kluwer, Feb. 2001, pp. 113–124, arXiv:math/0202182.
32. B. Webster, *Gelfand-Tsetlin modules in the Coulomb context*, Preprint 2019, arXiv:1904.05415.