Hochschild cohomology of general twisted tensor products

Pablo Ocal

joint with T. Karadag, D. McPhate, T Oke, S. Witherspoon.

arxiv: 1909.02181

Algebra and Geometry Seminar - ISU

Roadmap

1. Introduction
2. Hochschild cohomology
3. Twisted tensor products
4. Results
5. Applications

1. Introduction.

A unital associative algebras over k field.

$HH^*(A)$: The Hochschild cohomology encodes infinitesimal information about A.

$HH^0(A) \cong Z(A)$ the center of A.

$HH^1(A) \cong \mathrm{OutDer}(A)$ the outer derivations of A

$HH^2(A)$ the "important" infinitesimal deformations of A

$A \otimes_\tau B$: Čap, Schichl, Vanžura whenever an algebra has v.s. structure is given by a tensor product of two subalgebras, then it isomorphic to a twisted tensor product.

Nowadays this has applications in: operator algebras, algebraic topology, quantum symmetries.

2. Hochschild cohomology

Def: The Hochschild cohomology of A is: $HH^n(A) := \mathrm{Ext}^n_{A^e}(A,A)$ $\qquad HH^*(A) := \bigoplus_{n \in \mathbb{N}} \mathrm{Ext}^n_{A^e}(A,A)$

$A^e := A \otimes A^{op}$

Def: Bar resolution Consider $A^{\otimes(n+2)}$ as A^e-mod for $n \in \mathbb{N}$. Then:

$$\cdots \xrightarrow{d_3} A^{\otimes 4} \xrightarrow{d_2} A^{\otimes 3} \xrightarrow{d_1} A\otimes A \xrightarrow{m_A} A \longrightarrow 0$$

with:

$$d_n(a_0\otimes \cdots \otimes a_{n+1}) = \sum_{i=0}^{n} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}$$

for $a_i \in A$. is the augmented bar resolution.

Operations:

Cup product. $\quad \cup : HH^n(A) \times HH^m(A) \longrightarrow HH^{n+m}(A)$

Gerstenhaber bracket: $\quad [-,-] : HH^n(A) \times HH^m(A) \longrightarrow HH^{n+m-1}(A)$

Structures:

$(HH^*(A), \cup)$ — Graded commutative algebra

$(HH^*(A), [-,-])$ — Graded Lie algebra

$(HH^*(A), \cup, [-,-])$ — Gerstenhaber algebra

$x \in HH^n(A)$
$y \in HH^m(A)$
$x \cup y = (-1)^{n\cdot m}\, y \cup x$

3. Twisted tensor products.

Def: Given A, B, a twisting map $\tau : B\otimes A \longrightarrow A \otimes B$ is a bijective linear map satisfying:

$\tau(1_B \otimes a) = a \otimes 1_B$, $\tau(b\otimes 1_A) = 1_A \otimes b$ for all $a\in A$, $b \in B$, and

$$\begin{tikzcd}
B\otimes B\otimes A\otimes A \arrow[r, "1\otimes\tau\otimes 1"] \arrow[d, "m_B\otimes m_A"'] & B\otimes A\otimes B\otimes A \arrow[r, "\tau\otimes\tau"] \arrow[d, phantom, "\circlearrowright"] & A\otimes B\otimes A\otimes B \arrow[d, "1\otimes\tau\otimes 1"] \\
B\otimes A \arrow[r, "\tau"] & A\otimes B & A\otimes A\otimes B\otimes B \arrow[l, "m_A\otimes m_B"']
\end{tikzcd}$$

multiply
twisting
twisting
multiplying

The twisted tensor algebra $A\otimes_\tau B$ is $A\otimes B$ with multiplication:

$$m_{A\otimes_\tau B} : A\otimes B\otimes A\otimes B \xrightarrow{1\otimes\tau\otimes 1} A\otimes A\otimes B\otimes B \xrightarrow{m_A\otimes m_B} A\otimes B$$

We want to understand $HH^*(A\otimes_\tau B)$.

Def: Let M, N be A-bim, B-bim respectively such that their module actions are compatible with τ. Then $M \otimes N$ is an $(A \otimes_\tau B)$-bimodule.

There is a similar notion of compatibility of resolutions: $P_\cdot \longrightarrow M$
$Q_\cdot \longrightarrow N$

Thm: $\boxed{\text{[Shepler–Witherspoon]}}$ Under these compatibility conditions, given $P_\cdot \longrightarrow A$, $Q_\cdot \longrightarrow B$, we can construct: $P_\cdot \otimes_\tau Q_\cdot \longrightarrow A \otimes_\tau B$.

Examples:

1. The bar resolution is always compatible.
2. The Koszul resolutions for strongly graded τ are compatible.

4. Results.

Thm: $\boxed{\text{[KMOOW]}}$ Let $P_\cdot \longrightarrow A$, $Q \longrightarrow B$ such that:

(i) $P_\cdot \otimes_\tau Q \longrightarrow A \otimes_\tau B$ is "nice".

(ii) $\sigma: (P_\cdot \otimes_\tau Q_\cdot) \otimes_{A \otimes_\tau B} (P_\cdot \otimes_\tau Q_\cdot) \longrightarrow (P \otimes_A P_\cdot) \otimes_\tau (Q_\cdot \otimes_B Q_\cdot)$

is "nice".

Then we give an explicit formula for the Gerstenhaber bracket.

Prop: $\boxed{\text{[KMOOW]}}$ This applies to:

1. Bar resolution.
2. Koszul resolutions for strongly graded τ.

Our formulas are applicable in full generality.

5. Applications

The Jordan plane: $\dfrac{k\langle x, y\rangle}{(yx - xy - x^2)} \cong k[x] \otimes_\tau k[y]$ $\quad \tau(y \otimes x) = x \otimes y + x^2 \otimes 1$.

[Lopes–Solotar]

[Negron-Witherspoon] [Grimley-Nguyen-Witherspoon]

[Volkov]

Thm: $\boxed{\text{[KMOOW]}}$ We provide an explicit Gerstenhaber algebra structure on:

$$HH^*\left(k\langle x,y\rangle / (yx - xy - x^2)\right)$$ using elementary techniques.

Thank you!

$HH^*(A \otimes_\tau B) \overset{?}{\cong} HH^*(A) \otimes_\tau HH^*(B)$

$HH(A \otimes B) \boxed{\cong} HH^*(A) \otimes HH^*(B)$

[Le-Zhou: 2010] ?
2014