The Structure of Koszul Algebras Defined by Four Quadrics

Matthew Mastroeni

(joint with Paolo Mantero)

Oklahoma State University

Notation

We'll assume the following notation unless otherwise stated:

- k a field
- $S = k[x_1, \ldots, x_n]$ a polynomial ring over k
- $I \subseteq S$ an ideal
- $R = S/I$

Outline

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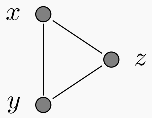
Edge Ideals

To every (simple) graph G, we can associate a square-free quadratic monomial ideal. If G has vertex set $\{v_1, \ldots, v_n\}$, the **edge ideal** of G is the ideal of $S = k[x_1, \ldots, x_n]$ given by:

$$I(G) = (x_i x_j \mid v_i v_j \text{ is an edge of } G)$$

Example

The edge ideal of the graph below is $I = (xy, xz, yz) \subseteq S = k[x, y, z]$.



The Taylor Resolution

If $I = (m_1, \ldots, m_g)$ is a monomial ideal, then for each $J \subseteq \{1, \ldots, g\}$ we set

$$m_J = \operatorname{lcm}(m_j \mid j \in J)$$

The **Taylor resolution** of R is the free resolution $F_\bullet$ given by

$$F_i = \bigoplus_{|J|=i} S e_J \qquad \partial(e_J) = \sum_{p=1}^{i} (-1)^{p+1} \frac{m_J}{m_{J \setminus \{j_p\}}} e_{J \setminus \{j_p\}}$$

where $j_1 < j_2 < \cdots < j_i$.

Running Example

For $I = (xy, xz, yz)$ in $S = k[x, y, z]$, the Taylor resolution of R is:

$$Se_{\{xy,xz,yz\}} \xrightarrow{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}} \begin{matrix} Se_{\{xz,yz\}} \\ \oplus \\ Se_{\{xy,yz\}} \\ \oplus \\ Se_{\{xy,xz\}} \end{matrix} \xrightarrow{\begin{pmatrix} 0 & -z & -z \\ -y & 0 & y \\ x & x & 0 \end{pmatrix}} \begin{matrix} Se_{xy} \\ \oplus \\ Se_{xz} \\ \oplus \\ Se_{yz} \end{matrix} \xrightarrow{\begin{pmatrix} xy & xz & yz \end{pmatrix}} Se_1$$

Minimal Free Resolutions

When the ideal I is graded, $R = S/I$ has a unique up to isomorphism **minimal free resolution**:

- The matrices in the resolution have homogeneous entries of positive degree.
- We keep track of the degrees of the entries by grading the free modules in the resolution.
- $S(-j)^r$ denotes the free module S^r with basis vectors in degree j.

Graded Betti Numbers

For a quotient ring $R = S/I$ with minimal free resolution

$$0 \longrightarrow F_p \xrightarrow{\varphi_p} F_{p-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\varphi_1} F_0$$

we can write each free module $F_i = \bigoplus_{j \in \mathbb{Z}} S(-j)^{\beta_{i,j}}$ with $\beta_{i,j} \geq 0$.

The ranks $\beta_{i,j} = \beta_{i,j}^S(R)$ are called the **graded Betti numbers** of R over S.

This information is often displayed in a table, called the **Betti table** of R, whose entry in the i-th column and j-th row is $\beta_{i,i+j}^S(R)$.

Running Example

Unfortunately, the Taylor resolution of $R = k[x,y,z]/(xy,xz,yz)$ is not minimal:

$$S(-3) \xrightarrow{\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}} S(-3)^3 \xrightarrow{\begin{pmatrix} 0 & -z & -z \\ -y & 0 & y \\ x & x & 0 \end{pmatrix}} S(-2)^3 \xrightarrow{\begin{pmatrix} xy & xz & yz \end{pmatrix}} S$$

Running Example

The minimal free resolution of $R = k[x,y,z]/(xy,xz,yz)$ is:

$$0 \longrightarrow S(-3)^2 \xrightarrow{\begin{pmatrix} 0 & -z \\ -y & 0 \\ x & x \end{pmatrix}} S(-2)^3 \xrightarrow{\begin{pmatrix} xy & xz & yz \end{pmatrix}} S$$

	0	1	2
0	1	–	–
1	–	3	2

Running Example

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	0	1	2
0	1	–	–
1	–	3	2

The 2-minors of the matrix of syzygies recover the generators of I. Such resolutions are called **Hilbert-Burch resolutions**.

Outline

Hilbert Functions

The **Hilbert function** of R is $\mathrm{HF}(R,d) = \dim_k R_d$.

- We can compute the Hilbert function of R from its graded Betti numbers over S:

$$\cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow R \longrightarrow 0$$

$$\begin{aligned}\mathrm{HF}(R,d) &= \sum_{i,j} (-1)^i \dim_k [F_i]_d \\ &= \sum_{i,j} (-1)^i \beta_{i,j}^S(R) \binom{n+d-j-1}{n-1}\end{aligned}$$

Hilbert Series

- The generating series $\mathrm{H}_R(t) = \sum_d \mathrm{HF}(R,d)t^d \in \mathbb{Z}[[t]]$ is a rational function:

$$\mathrm{H}_R(t) = \frac{h_R(t)}{(1-t)^{n-c}}$$

for a unique polynomial $h_R(t) \in \mathbb{Z}[t]$ with $h_R(1) > 0$ and positive integer c.

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  for a unique polynomial $h_R(t) \in \mathbb{Z}[t]$ with $h_R(1) > 0$ and positive integer c.

- We call $\mathrm{ht}\, I = c$ the **height** (or **codimension**) of I.

- We call $e(R) = h_R(1)$ the **multiplicity** (or **degree**) of I.

Running Example

If $I = (xy, xz, yz) \subseteq S = k[x, y, z]$, the Hilbert series of $R = S/I$ is:

	0	1	2
0	1	–	–
1	–	3	2

$$\mathrm{H}_R(t) = \frac{1}{(1-t)^3} - \frac{3t^2}{(1-t)^3} + \frac{2t^3}{(1-t)^3}$$

Running Example

If $I = (xy, xz, yz) \subseteq S = k[x, y, z]$, the Hilbert series of $R = S/I$ is:

	0	1	2
0	1	–	–
1	–	3	2

$$\mathrm{H}_R(t) = \frac{1}{(1-t)^3} - \frac{3t^2}{(1-t)^3} + \frac{2t^3}{(1-t)^3} = \frac{1+2t}{1-t}$$

- Since $(1-t)$ divides the numerator twice, $\operatorname{ht} I = 2$.
- $e(R) = 1 + 2 = 3$

Outline

Koszul Algebras

Let $R_+ = \bigoplus_{d>0} R_d$.

R is a **Koszul algebra** if the minimal free resolution of $R/R_+ \cong k$ over R has the form

$$\cdots \longrightarrow R(-3)^{\beta_3} \longrightarrow R(-2)^{\beta_2} \longrightarrow R(-1)^{\beta_1} \longrightarrow R$$

Example

Let $R = k[x]/(x^2)$. Then the minimal free resolution of k is:

$$\cdots \longrightarrow R(-3) \xrightarrow{x} R(-2) \xrightarrow{x} R(-1) \xrightarrow{x} R$$

So R is Koszul.

Koszul Algebras

- Koszul algebras were introduced by Priddy in 1970 as a way of unifying constructions of resolutions over Steenrod algebras from algebraic topology and universal enveloping algebras of Lie algebras.
- If $R = S/I$ is a Koszul algebra, then I is generated by quadrics (homogeneous polynomials of degree 2).
- There is strong relationship between a Koszul algebra R and its **quadratic dual** $R^!$ (although $R^!$ is non-commutative).

Examples of Koszul Algebras

- Polynomial rings (and exterior algebras)
- Coordinate rings of Grassmannians and suitably general smooth curves
- Many types of toric rings
- All high Veronese subrings of any standard graded algebra
- Quotients by quadratic monomial ideals

Bounds on Betti Numbers

Question (Avramov-Conca-Iyengar '10)

If R is Koszul and I is minimally generated by g elements, does the following inequality hold for all i?

$$\beta_i^S(R) \leq \binom{g}{i}$$

In particular, is $\operatorname{pd}_S R \leq g$?

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In particular, is $\operatorname{pd}_S R \leq g$?

Motivating philosophy: This is true for quadratic monomial ideals. Reasonable properties of quadratic monomial ideals should hold for general Koszul algebras.

Known Cases

- R is **G-quadratic**: after a suitable linear change of coordinates $\varphi : S \to S$, the ideal $\varphi(I)$ has a quadratic Gröbner basis.

  If I has a quadratic initial ideal J with g generators, then I is also generated by g quadrics and

  $$\beta_i^S(R) \leq \beta_i^S(S/J) \leq \binom{g}{i}$$

- R is **LG-quadratic**: R is a quotient of a G-quadratic algebra A by an A-sequence of linear forms.

A Cautionary Example (Conca '13)

The ring

$$R = \frac{k[x,y,z,w]}{(xy, xw, (x-y)z, z^2, x^2+zw)}$$

is Koszul but not LG-quadratic.

A Cautionary Example (Conca '13)

The ring

$$R = \frac{k[x,y,z,w]}{(xy, xw, (x-y)z, z^2, x^2+zw)}$$

is Koszul but not LG-quadratic. Its Betti table is

	0	1	2	3	4
0	1	–	–	–	–
1	–	5	4	–	–
2	–	–	4	6	2

Known Cases

The preceding question has an affirmative answer if:

- $\operatorname{ht} I = g$, so I is a quadratic complete intersection.
- $\operatorname{ht} I = 1$, so $I = zJ$ for a linear form z and J a linear complete intersection.
- $g = 3$ (Boocher-Hassanzadeh-Iyengar '17)

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- $g = 3$ (Boocher-Hassanzadeh-Iyengar '17)
- $\operatorname{ht} I = g - 1$, so I is an almost complete intersection (M '18)

Known Cases

In fact, BHI gave a complete classification of the possible Betti tables of Koszul algebras defined by 3 quadrics. They are:

$$
\begin{array}{c|cccc}
 & 0 & 1 & 2 & 3 \\ \hline
0 & 1 & - & - & - \\
1 & - & 3 & 3 & 1
\end{array}
\qquad
\underbrace{
\begin{array}{c|cccc}
 & 0 & 1 & 2 & 3 \\ \hline
0 & 1 & - & - & - \\
1 & - & 3 & 1 & - \\
2 & - & - & 2 & 1
\end{array}
\qquad
\begin{array}{c|ccc}
 & 0 & 1 & 2 \\ \hline
0 & 1 & - & - \\
1 & - & 3 & 2
\end{array}
}_{\text{ht } I=2}
\qquad
\begin{array}{c|cccc}
 & 0 & 1 & 2 & 3 \\ \hline
0 & 1 & - & - & - \\
1 & - & 3 & - & - \\
2 & - & - & 3 & - \\
3 & - & - & - & 1
\end{array}
$$

Koszul Almost Complete Intersections

Theorem (M '18)

Let $R = S/I$ be a Koszul almost complete intersection with I minimally generated by g quadrics for some $g \geq 2$. Then $\beta^S_{2,3}(R) \leq 2$, and:

(a) *If $\beta^S_{2,3}(R) = 1$, then $I = (xz, zw, q_3, \dots, q_g)$ for some linear forms x, z, and w and some regular sequence of quadrics $q_3, \dots, q_g$ on $S/(xz, zw)$.*

(b) *If $\beta^S_{2,3}(R) = 2$, then $I = I_2(M) + (q_4, \dots, q_g)$ for some 3×2 matrix of linear forms M with $\operatorname{ht} I_2(M) = 2$ and some regular sequence of quadrics $q_4, \dots, q_g$ on $S/I_2(M)$.*

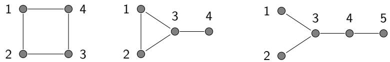
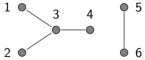
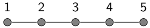
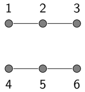
In particular, R satisfies $\beta^S_i(R) \leq \binom{g}{i}$ for all i.

Outline

Buying Into Edge Ideals

For $g = 4$, it is enough to prove the Betti number bound when $\operatorname{ht} I = 2$.

Based on edge ideals of graphs with 4 edges, we expect R to have one of the Betti tables:

Case	$\beta^S(R)$	Graphs
(i)	$\begin{array}{c\|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 1 & - & - & - \\ 1 & - & 4 & 4 & 1 \end{array}$	
(ii)	$\begin{array}{c\|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & 1 & - & - & - & - \\ 1 & - & 4 & 3 & 1 & - \\ 2 & - & - & 3 & 3 & 1 \end{array}$	
(iii)	$\begin{array}{c\|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 1 & - & - & - \\ 1 & - & 4 & 3 & - \\ 2 & - & - & 1 & 1 \end{array}$	
(iv)	$\begin{array}{c\|ccccc} & 0 & 1 & 2 & 3 & 4 \\ \hline 0 & 1 & - & - & - & - \\ 1 & - & 4 & 2 & - & - \\ 2 & - & - & 4 & 4 & 1 \end{array}$	

Koszul Algebras Defined by 4 Quadrics

Theorem (Mantero-M '18)

If $R = S/I$ is a Koszul algebra with $\operatorname{ht} I = 2$ and I minimally generated by $g = 4$, then the Betti table of R is one of the four possibilities realized by edge ideals. In particular, $\beta_i^S(R) \leq \binom{4}{i}$ for all i.

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Even better: We completely describe the structure of the possible defining ideals when $k = \overline{k}$.

A Bound on the Multiplicity

Proposition (Mantero-M '18)

If $R = S/I$ is defined by $g \geq 4$ quadrics and $\operatorname{ht} I = 2$, then $e(R) \leq 2$.

- In general, $e(R) \leq 3$ as long as I is not a complete intersection (Huneke-Mantero-McCullough-Seceleanu '13).
- A linkage argument shows $e(R) = 3$ if and only if the unmixed part of I is $I_2(M)$ for some 3×2 matrix of linear forms.

TOOL: Linkage

Two ideals $I, J \subseteq S$ of height c are **directly linked** if there is a complete intersection ideal $L \subseteq I \cap J$ of height c such that $(L : I) = J$ and $(L : J) = I$, where:

$$(L : I) = \{f \in S \mid fI \subseteq L\}$$

- Linked ideals are unmixed, so the unmixed part of I is directly linked to $(L : I)$ for any complete intersection $L \subseteq I$ of two quadrics.
- If $\operatorname{ht} I = 2$, then $e(S/J) = e(S/L) - e(S/I) = 4 - 3 = 1$, so J is generated by linear forms.

TOOL: Linkage

Theorem (Avramov-Kustin-Miller '88)

An ideal I is directly linked to a complete intersection of height c if and only if there is a $c \times c$ matrix X and a $1 \times c$ matrix Y such that $I = I_1(YX) + I_c(X)$. Such an ideal is called a **Northcott ideal**.

- Explicitly, if I is linked to a complete intersection $J = (f_1, \ldots, f_c)$ by the complete intersection $L = (h_1, \ldots, h_c) \subseteq J$, then
$$Y = \begin{pmatrix} f_1 & \cdots & f_c \end{pmatrix} \qquad X = (a_{i,j})$$
where $h_j = \sum_i a_{i,j} f_i$.
- For a height 2 ideal generated by quadrics, we see that $I = I_2(M)$ for some 3×2 matrix of linear forms.

Outline

Unmixed Parts

Recall that the **unmixed part** I^{unm} of I is the intersection of all primary components J of I with $\operatorname{ht} J = \operatorname{ht} I$.

Proposition (Engheta '07)

If $e(R) = \operatorname{ht} I = 2$, then I^{unm} has one of the following forms:

(i) $(x, y) \cap (z, w)$ *for independent linear forms x, y, z, and w.*

(ii) $(x, y)^2 + (xy + zw)$ *for independent linear forms x, y and forms z, w such that $\operatorname{ht}(x, y, z, w) = 4$.*

(iii) (x, q) *for some linear form x and quadric q.*

Cases (i) and (ii)

Theorem (Mantero-M '18)

Let $R = S/I$ be a ring defined by $g \geq 4$ quadrics with $\operatorname{ht} I = e(R) = 2$. Then I has one of the following forms:

(i$_A$) $(x, y) \cap (z, w)$ *or* $(x, y)^2 + (xz + yw)$ *for independent linear forms* x, y, z *and* w, *in which case we must have* $g = 4$.

(i$_B$) $(a_1 x, \ldots, a_{g-1} x, q)$ *for independent linear forms* $a_1, \ldots, a_{g-1}$ *and some linear form* x *and quadric* $q \in (a_1, \ldots, a_{g-1}) \setminus (x)$.

(ii) $(a_1 x, \ldots, a_{g-1} x, q)$ *for independent linear forms* $a_1, \ldots, a_{g-1}$ *and some linear form* x *and quadric* q *which is a nonzerodivisor modulo* $(a_1 x, \ldots, a_{g-1} x)$.

Cases (i) and (ii)

Corollary

If $R = S/I$ is a ring defined by $g \geq 4$ quadrics with $\operatorname{ht} I = e(R) = 2$, then R is LG-quadratic so that $\beta_i^S(R) \leq \binom{g}{i}$ for all i.

When $g = 4$, the Betti table of R is one of:

	0	1	2	3
0	1	–	–	–
1	–	4	4	1

	0	1	2	3	4
0	1	–	–	–	–
1	–	4	3	1	–
2	–	–	3	3	1

Outline

A Bound on the Projective Dimension

Theorem (Huneke-Mantero-McCullough-Seceleanu '15)

The projective dimension of rings $R = S/I$ defined by 4 quadrics is at most 6, and this bound is realized by $I = (x^2, y^2, a_3x + b_3y, a_4x + b_4y)$ with $\mathrm{ht}(x, y, a_3, a_4, b_3, b_4) = 6$.

A Koszul algebra with $\mathrm{ht}\, I \leq g - 2$ has at least 2 linear syzygies on I.

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A Koszul algebra with $\operatorname{ht} I \leq g - 2$ has at least 2 linear syzygies on I.

Theorem (Mantero-M '18)

If I is a height 2 ideal minimally generated by four quadrics with at least 2 linear syzygies, then $\operatorname{pd}_S R \leq 4$.

Representation by Minors

When $\operatorname{ht} I = 2$ and $e(R) = 1$, the ideal $I = (q_1, \dots q_g)$ is contained in a unique height two minimal prime (x, y) generated by linear forms.

Writing $q_i = a_i x + b_i y$ for some linear forms a_i and b_i, we say that I is **represented by minors** by the matrix

$$M = \begin{pmatrix} y & a_1 & \cdots & a_g \\ -x & b_1 & \cdots & b_g \end{pmatrix}$$

Representation by Minors

Theorem (Huneke-Mantero-McCullough-Seceleanu '13)

After a suitable change of generators for I and (x, y), there are only 5 possible forms for M:

(1) M *is 1-generic*

(2) $M = \begin{pmatrix} y & 0 & a_2 & \cdots & a_g \\ -x & b_1 & b_2 & \cdots & b_g \end{pmatrix}$ *where* $D = \begin{pmatrix} y & a_2 & \cdots & a_g \\ -x & b_2 & \cdots & b_g \end{pmatrix}$ *is 1-generic*

(3) $M = \begin{pmatrix} y & 0 & 0 & a_3 & \cdots & a_g \\ -x & b_1 & b_2 & b_3 & \cdots & b_g \end{pmatrix}$

(4) $M = \begin{pmatrix} y & 0 & a_2 & a_3 & \cdots & a_g \\ -x & b_1 & 0 & b_3 & \cdots & b_g \end{pmatrix}$ *where* $D = \begin{pmatrix} y & a_3 & \cdots & a_g \\ -x & b_b & \cdots & b_g \end{pmatrix}$ *is 1-generic*

(5) $M = \begin{pmatrix} y & 0 & a_2 & a_3 & \cdots & a_g \\ -x & b_1 & 0 & \lambda a_3 & \cdots & b_g \end{pmatrix}$ *for some* $\lambda \in k$

TOOL: 1-Generic Matrices

A $r \times s$ matrix M of linear forms in S is **1-generic** if whenever we have $w^{\mathsf{T}} M v = 0$ for $w \in k^r$ and $v \in k^s$, we have either $w = 0$ or $v = 0$.

- The exact sequence $0 \to S/(I:y)(-1) \xrightarrow{y} S/I \to S/(I,y) \to 0$ induces an exact sequence:

$$\begin{tikzcd} \mathrm{Tor}_2(S/(I:y),k)_3 \arrow[r] & \mathrm{Tor}_2^S(S/I,k)_3 \arrow[r] & \mathrm{Tor}_2^S(S/(I,y),k)_3 \arrow[dll, rounded corners] \\ \mathrm{Tor}_1^S(S/(I:y),k)_2 \arrow[r] & 0 & \end{tikzcd}$$

- If M is 1-generic and $k = \overline{k}$, then $(I:y) = I_2(M)$ is a prime ideal generated by quadrics of expected height.

TOOL: 1-Generic Matrices

- In that case, $S/I_2(M)$ has an Eagon-Northcott resolution:

	0	1	2	$\cdots$	g
0	1	–	–	–	–
1	–	$\binom{g+1}{2}$	$2\binom{g+1}{3}$	$\cdots$	$g\binom{g+1}{g+1}$

- The Betti table of $S/(I,y) = S(y, a_1x, \ldots, a_gx)$ is:

	0	1	2	3	$\cdots$	$g+1$
0	1	1	–	–	–	–
1	–	g	$\binom{g+1}{2}$	$\binom{g+1}{3}$	$\cdots$	1

So, S/I has no linear syzygies if M is 1-generic!

Betti Tables of Koszul Algebras Defined by 4 Quadrics

It suffices to find the possible Betti tables when $\operatorname{ht} I = 2$ and $e(R) = 1$.

- Being Koszul together with the bound on the projective dimension greatly restricts the shape of the Betti table of R:
  - $\beta^S_{i,j}(R) = 0$ for all i and $j > 2i$. (Backelin '88, Kempf '90)
  - $\beta^S_{i,2i}(R) = 0$ for $i > \operatorname{ht} I$. (Avramov-Conca-Iyengar '10)
  - $\beta^S_{g,g+1}(R) = 0$ if $\operatorname{ht} I \geq 2$. (consequence of Koh '99)

Betti Tables of Koszul Algebras Defined by 4 Quadrics

It suffices to find the possible Betti tables when $\operatorname{ht} I = 2$ and $e(R) = 1$.

- There are only 2 possible shapes for the Betti table of R:

	0	1	2	3
0	1	–	–	–
1	–	4	a	c
2	–	–	b	d

	0	1	2	3	4
0	1	–	–	–	–
1	–	4	a	c	–
2	–	–	b	d	e
3	–	–	–	–	f

- Computing the Hilbert series using that $\operatorname{ht} I = 2$ and $e(R) = 1$ reduces this to an integer programming problem.

Betti Tables of Koszul Algebras Defined by 4 Quadrics

It suffices to deduce the possible Betti tables when $e(R) = 1$.

- There are only 2 possible shapes for the Betti table of R:

	0	1	2	3
0	1	–	–	–
1	–	4	3	–
2	–	–	1	1

	0	1	2	3	4
0	1	–	–	–	–
1	–	4	2	–	–
2	–	–	4	4	1
3	–	–	–	–	–

- Computing the Hilbert series using that $\operatorname{ht} I = 2$ and $e(R) = 1$ reduces this to an integer programming problem.

Case (iii)

Theorem (Mantero-M '19)

The ring $R = S/I$ has Betti table

	0	1	2	3
0	1	–	–	–
1	–	4	3	–
2	–	–	1	1

if and only if $I = (xz, yz, a_3x + b_3y, a_4x + b_4y)$ *for some linear forms* x, y, z, a_3, a_4, b_3, b_4 *such that* $\mathrm{ht}(a_3x + b_3y, a_4x + b_4y, a_3b_4 - a_4b_3, z) = 3$ *and* $\mathrm{ht}(x, y) = 2$. *In particular,* R *is LG-quadratic.*

TOOL: Annihilators of Cohomology

The dual of the last differential in the resolution of R yields a presentation:

$$S(3)^3 \oplus S(4) \xrightarrow{\varphi_3^*} S(5) \longrightarrow \operatorname{Ext}^3_S(R,S) \longrightarrow 0$$

- For $\mathfrak{a}_i = \operatorname{Ann}_S \operatorname{Ext}^i_S(R,S)$, we have $\prod_i \mathfrak{a}_i \subseteq I$.
  (Eisenbud-Evans ??, Schenzel '79)
- $\mathfrak{a}_2 = \operatorname{Ann}_S \operatorname{Ext}^2_S(R,S) = I^{\mathrm{unm}} = (x,y)$.
  (Eisenbud-Huneke-Vasconcelos '92)

TOOL: Annihilators of Cohomology

- If z is the linear form in the last differential of the resolution of R, then $z \neq 0$ and $z(x,y) \subseteq \mathfrak{a}_2\mathfrak{a}_3 \subseteq I$.

Theorem (Buchsbaum-Eisenbud Acyclicity Criterion)

A complex of finitely generated free S-modules

$$0 \longrightarrow F_s \xrightarrow{\varphi_s} F_{s-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\varphi_1} F_0 \longrightarrow 0$$

is acyclic if and only if $\operatorname{ht} I_{r_i}(\varphi_i) \geq i$ *for all* $i \geq 1$, *where* $r_i = \sum_{j \geq i} (-1)^{j-i} \operatorname{rank} F_j$.

Case (iv)

Surprisingly, having the Betti table below does not determine whether R is Koszul!

	0	1	2	3	4
0	1	–	–	–	–
1	–	4	2	–	–
2	–	–	4	4	1

$(*)$

Case (iv)

Theorem (Mantero-M '19)

The ring $R = S/I$ has Betti table $(*)$ if and only if for some linear forms satisfying specific height conditions, I has one of the the following forms:

(a) $(x^2, b_3x, a_3x + b_3y, a_4x + b_4y)$

(b) $(xy, a_2x, b_3y, a_4x + b_4y)$

(c) $(b_3x, b_4x, a_3x + b_3y, a_4x + b_4y)$

(d) (a_1x, a_2x, b_3y, b_4y) *with* (a_1x, a_2x) *and* (b_3y, b_4y) *transversal*

TOOL: Buchsbaum-Rim Complexes

We can view the four quadric generators of $I = (q_1, q_2, q_3, q_4)$ as syzygies on its 4×2 matrix of linear syzgies ℓ:

$$\begin{pmatrix} q_1 & q_2 & q_3 & q_4 \end{pmatrix} \ell = 0 \quad \Longrightarrow \quad \ell^{\mathsf{T}} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix} = 0$$

If I has Betti table $(*)$, then ℓ cannot be 1-generic!

TOOL: Buchsbaum-Rim Complexes

If ℓ were 1-generic:

- $I_2(\ell)$ is a prime ideal generated by quadrics of expected height.

TOOL: Buchsbaum-Rim Complexes

If ℓ were 1-generic:

- $I_2(\ell)$ is a prime ideal generated by quadrics of expected height.
- $\mathrm{Coker}\,\ell^{\mathsf{T}}$ is resolved by a Buchsbaum-Rim complex:

$$S(-4)^2 \xrightarrow{-\ell} S(-3)^4 \xrightarrow{Q} S(-1)^4 \xrightarrow{\ell^{\mathsf{T}}} S^2$$

$$Q = \begin{pmatrix} 0 & -\Delta_{3,4} & \Delta_{2,4} & -\Delta_{2,3} \\ \Delta_{3,4} & 0 & -\Delta_{1,4} & \Delta_{1,3} \\ -\Delta_{2,4} & \Delta_{1,4} & 0 & -\Delta_{1,2} \\ \Delta_{2,3} & -\Delta_{1,3} & \Delta_{1,2} & 0 \end{pmatrix}$$

where $\Delta_{i,j}$ is the minor involving rows i and j of ℓ.

TOOL: Buchsbaum-Rim Complexes

If ℓ were 1-generic:

- This shows that $I \subseteq I_2(\ell)$.
- Of the representations by minors of height 2 ideals of multiplicity 1 described by Huneke-Mantero-McCullough-Seceleanu, we know I must contain a reducible quadric if it has 2 independent linear syzygies.

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So, ℓ is not 1-generic. Considering how many other zeros can appear in ℓ gives the 4 possible forms of the ideal.

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Case (iv)

Theorem (Mantero-M '19)

The ring $R = S/I$ has Betti table $(*)$ if and only if for some linear forms satisfying specific height conditions, I has one of the the following forms:

(a) $(x^2, b_3x, a_3x + b_3y, a_4x + b_4y)$ **not Koszul**

(b) $(xy, a_2x, b_3y, a_4x + b_4y)$ **not Koszul**

(c) $(b_3x, b_4x, a_3x + b_3y, a_4x + b_4y)$ **not Koszul**

(d) (a_1x, a_2x, b_3y, b_4y) *with* (a_1x, a_2x) *and* (b_3y, b_4y) *transversal* **Koszul**

Detecting Non-Koszulness

Theorem (M '18, Avramov-Conca-Iyengar '10)

If R is Koszul, then $\mathrm{Syz}_1^S(I)$ is generated by linear and Koszul syzygies.

For example, if $I = (xy, a_2x, b_3y, a_4x + b_4y)$ and we set $q = a_4x + b_4y$:

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For example, if $I = (xy, a_2x, b_3y, a_4x + b_4y)$ and we set $q = a_4x + b_4y$:

$$\mathrm{Syz}_1^S(I) = \mathrm{Im}\begin{pmatrix} -a_2 & -b_3 & q & 0 & 0 & 0 \\ y & 0 & 0 & q & 0 & a_4b_3 \\ 0 & x & 0 & 0 & q & a_2b_4 \\ 0 & 0 & -xy & -a_2x & -b_3y & -a_2b_3 \end{pmatrix}$$

where the last column is not generated by the linear and Koszul syzygies.

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where the last column is not generated by the linear and Koszul syzygies.

Sadly, this argument fails if $I = (b_3x, b_4x, a_3x + b_3y, a_4x + b_4y)$.

Another Cautionary Example (Roos '93)

For each integer $n \geq 2$, the resolution of $\mathbb{Q}$ over the ring

$$R_n = \frac{\mathbb{Q}[x, y, z, u, v, w]}{(x, y)^2 + (v, w)^2 + L + (z, u)^2}$$

where

$$L = ((x + nw)z - wu, wz + (x + (n-2)w)u, yz, vu)$$

is linear for n steps but fails to be linear at the $(n+1)$-th step.

Passing Koszulness Around

Proposition (Conca-De Negri-Rossi '13)

Let S be a standard graded k-algebra and R be a quotient ring of S.

(a) *If S is Koszul and $\operatorname{reg}_S(R) \leq 1$, then R is Koszul.*

(b) *If R is Koszul and $\operatorname{reg}_S(R)$ is finite, then S is Koszul.*

Here:

$$\operatorname{reg}_S(R) = \sup\{j \mid \beta^S_{i,i+j}(R) \neq 0\}$$

In particular, Koszul-ness passes to and from quotients by a regular sequence of quadrics.

TOOL: Symmetric Algebras

It is enough to check the ring below is not Koszul.

$$R = \frac{k[x, y, a, b]}{(x^2 - y^2, xy, bx, ax - by)}$$

Given a module M with t generators over a ring R', a presentation of the symmetric algebra $\mathrm{Sym}_{R'}(M)$ is given by:

$$\mathrm{Sym}_{R'}(M) = \frac{R'[u_1, \ldots, u_t]}{(\sum_i f_i u_i \mid (f_1, \ldots, f_t) \in \mathrm{Syz}_1^{R'}(M))}$$

TOOL: Symmetric Algebras

It is enough to check the ring below is not Koszul.

$$R = \frac{k[x, y, a, b]}{(x^2 - y^2, xy, bx, ax - by)} \cong \mathrm{Sym}_{R'}(M)$$

where $R' = k[x, y]/(x^2 - y^2, xy)$ and M has a periodic resolution

$$\cdots \longrightarrow R'(-2)^2 \xrightarrow{\begin{pmatrix} y & 0 \\ x & y \end{pmatrix}} R'(-1)^2 \xrightarrow{\begin{pmatrix} x & 0 \\ -y & x \end{pmatrix}} R'^2 \longrightarrow M \longrightarrow 0$$

TOOL: Symmetric Algebras

Theorem (Herzog-Hibi-Ohsugi '00)

Suppose $\varphi : R \to R'$ is an algebra retract of standard graded k-algebras. Then R is Koszul if and only if R' is Koszul and R' has a linear resolution as an R-module (via φ).

TOOL: Symmetric Algebras

Theorem (Herzog-Hibi-Ohsugi '00)

Suppose $\varphi : R \to R'$ is an algebra retract of standard graded k-algebras. Then R is Koszul if and only if R' is Koszul and R' has a linear resolution as an R-module (via φ).

- This reduces the number of syzygies we need to compute from about 340 for k to about 80 for R'.
- The resolutions of R' and k over R fail to be linear 6 steps back if $\mathrm{char}(k) \neq 2$ and 5 steps back if $\mathrm{char}(k) = 2$.

Consequences of the Structure Theorem

Theorem (Mantero-M '19)

All Koszul algebras defined by $g \leq 4$ quadrics are LG-quadratic.

- Conca's example of a Koszul algebra that is not LG-quadratic is minimal in terms of height, multiplicity, and number of generators.
- We can explicitly describe the defining ideal of any Koszul algebra defined by $g \leq 4$ quadrics (for $k = \overline{k}$).
- We are able to determine when such Koszul algebras have the Backelin-Roos and absolutely Koszul properties.

Outline

Further Questions

1. What about Koszul algebras defined by $g \geq 5$ quadrics?
   - What do Koszul algebras $R = S/I$ defined by $g = 5$ quadrics with $\operatorname{ht} I = 2$ and $e(R) = 1$ look like?

2. Is there a *method* for producing other examples of Koszul algebras which are not LG-quadratic?

3. Can we remove the $k = \overline{k}$ assumption from the structure theorem?
   - Is there a structure theorem for nondegenerate prime ideals P with $\operatorname{ht} I = e(S/P) = 2$?
   - Is there some ring which is not LG-quadratic but becomes LG-quadratic after field extension?

Further Questions

4. What other nice properties of edge ideals carry over to general Koszul algebras?
   - Is $\operatorname{reg} R \leq \operatorname{ht} I$? (It's known that $\operatorname{reg} R \leq \operatorname{pd}_S R$.)
5. Can we characterize when $\operatorname{Sym}_R(M)$ is Koszul?