Resurgence via asymptotic resurgence

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Algebra and Geometry Seminar
Iowa State University

Symbolic Powers

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$$I^{(s)} = \left\{ f : \frac{\partial f}{\partial x^\alpha} \in I \text{ for all } |\alpha| \leq s - 1 \right\}$$

Comparing regular and symbolic powers

Given an ideal $I \subset S = \mathbb{K}[x_0, \ldots, x_n]$:

- *Regular powers* I^r are 'easy' to describe algebraically
- *Symbolic powers* $I^{(s)}$ are 'easy' to describe geometrically

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Containment Problem

For which pairs of positive integers (s, r) do we have $I^{(s)} \subset I^r$?

Containment examples

Example 1: Ideal of rank 1 matrices

$$M = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}, P = I_2(M) \subset \mathbb{K}[a, \ldots, i].$$

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- Partial derivatives of $\det(M)$ are 2×2 minors of M
- $\det(M) \in P^{(2)} \setminus P^2$
- So $P^{(2)} \neq P^2$
- Can check $P^{(3)} \subset P^2$.

Containment examples

Example 2: Ideal of 3 points in $\mathbb{P}^2$

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- $(xyz)^{2k} \in I^{(4k)}$ (mixed partials of $(xyz)^{2k}$ of order $4k - 1$ are divisible by xy, xz, or yz)
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So $I^{(4k)} \not\subset I^{3k+1}$ but $I^{(4k)} \subset I^{3k}$

- More precisely: $I^r = I^{(r)} \cap M^{2r}$ where $M = \langle x, y, z \rangle$
- $I^{(s)} \subset I^r$ if and only if $s \geq \frac{4}{3}r$.

Uniform containment

Ein-Lazarsfeld-Smith '01, Hochster-Huneke '02, Ma-Schwede '17:

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$I = (xy, xz, yz) = (x, y) \cap (x, z) \cap (y, z)$

- Uniform containment: $I^{(2r)} \subset I^r$
- Previous slide: $I^{(\lceil 4/3 \cdot r \rceil)} \subset I^r$

Resurgence

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- $I = (xy, xz, yz) = (x, y) \cap (x, z) \cap (y, z)$
- $\rho(I) = 4/3$.

Asymptotic resurgence

Refinement introduced by Guardo, Harbourne, and Van Tuyl '13:

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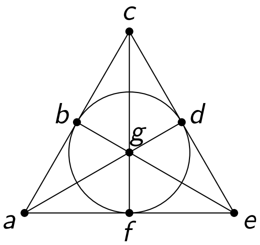
In general $\widehat{\rho}(I) \leq \rho(I)$ (strict inequality may occur [DHSSTG'14], but these examples are rare!).

Asymptotic resurgence $<$ Resurgence [DFMS '19]

Fano plane



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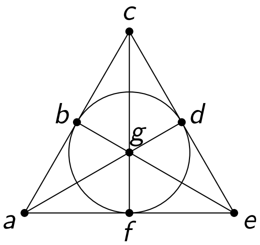

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$$I = \langle abc, adg, aef, bdf, beg, cde, cfg \rangle$$

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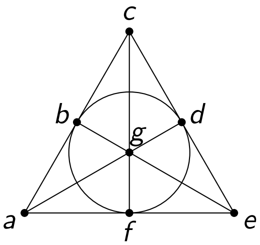

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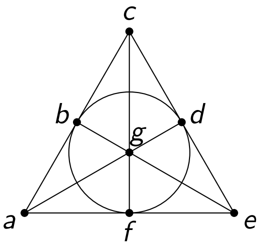

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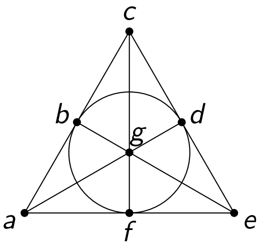

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- $\widehat{\rho}(I) = \sup\{\frac{s}{r} : I^{(st)} \not\subset I^{rt} \text{ for all } t \gg 0\} = 9/7!$

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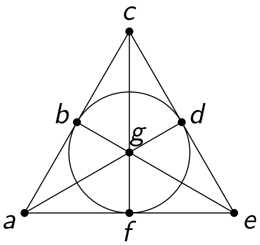

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$I^{(3t)} \subset I^{2t}$ for infinitely many $t \in \mathbb{Z}_{>0}$

Integral closures

Integral closure of $I \subset S = \mathbb{K}[x_0, \ldots, x_n]$ is:

$\overline{I} := \{f : \text{ there is } c \neq 0 \in S \text{ s.t. } cf^k \in I^k \text{ for infinitely many } k \in \mathbb{Z}_{>0}\}$

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- $m = abcdefg$
- $m^3 \in I^7$ (take product of all generators)
- So $m^3 \in I^6 \implies m \in \overline{I^2}$ (but $m \notin I^2$!)

Asymptotic resurgence and integral closures

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- Recall: $\widehat{\rho}(I) = \sup\{\frac{s}{r} : I^{(st)} \not\subset I^{rt} \text{ for all } t \gg 0\}$
- Theorem holds in *analytically unramified* rings (just need finiteness of integral closures)

When asymptotic resurgence is less than resurgence

Theorem A (AR<R) [-D '20]

Suppose I is an ideal of $S = \mathbb{K}[x_0, \ldots, x_n]$ and $\widehat{\rho}(I) < \rho(I)$. Then there are positive integers M, N so that

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- If $\widehat{\rho}(I) < \rho(I)$ then $\rho(I)$ is rational.

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Theorem B [-D '20]

If the symbolic Rees algebra of an ideal is finitely generated, then $\rho(I)$ is rational.

For instance, the resurgence of monomial ideals is rational.

A conjecture of Harbourne refining uniform containment

Uniform containment

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- ideals of general points in $\mathbb{P}^2$ [HH '13] and $\mathbb{P}^3$ [D '15]

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Counterexamples to Harbourne's conjecture

Dumnicki-Szemberg-Tutaj-Gasińska '13

If I is the ideal of the intersection points of a certain line arrangement in $\mathbb{P}^2$ ($c = 2$), then $I^{(4)} \subset I^2$ (uniform containment) but $I^{(2\cdot 2-2+1=3)} \not\subset I^2$.

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Several other examples followed. (For many of these $\widehat{\rho}(I) < \rho(I)$.)

Counterexamples to Harbourne's conjecture

Dumnicki-Szemberg-Tutaj-Gasińska '13

If I is the ideal of the intersection points of a certain line arrangement in $\mathbb{P}^2$ ($c = 2$), then $I^{(4)} \subset I^2$ (uniform containment) but $I^{(2\cdot 2-2+1=3)} \not\subset I^2$.

Several other examples followed. (For many of these $\widehat{\rho}(I) < \rho(I)$.)

Stable Harbourne conjecture: Grifo '20

$I^{(cr-c+1)} \subset I^r$ for all $r \gg 0$.

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$$I = \bigcap_{0 \leq i < j \leq n} \langle x_i, x_j \rangle = \langle \prod_{i \neq j} x_i : j = 0, \ldots, n \rangle$$

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- If $\rho(I) < c$ then I satisfies the stable Harbourne conjecture (and more!)
- Follows quickly from Theorem A (AR<R) that $\rho(I) < c$ is implied by $\widehat{\rho}(I) < c$.

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Suppose $I \subset \mathbb{K}[x_0, \cdots, x_n]$ is radical with $\operatorname{codim}(I) = c$. If $I^{(rc-c)} \subset \overline{I^r}$ for some $r \in \mathbb{Z}_{>0}$ then $\widehat{\rho}(I) \leq c - \frac{1}{r}$. In particular, I has expected resurgence.

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A containment for squarefree monomial ideals [-D '20]

If $I \subset \mathbb{K}[x_0, \ldots, x_n]$ is a squarefree monomial ideal of codimension c, then $I^{(rc-c)} \subset I^r$ for $r \geq n+1$. In particular, squarefree monomial ideals have expected resurgence.

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Question

Suppose I is a radical ideal in $\mathbb{K}[x_0, \ldots, x_n]$. Is $I^{(rc-c)} \subset \overline{I^r}$ satisfied for some $r \gg 0$? If so, can r be chosen *uniformly* for all radical ideals? Can we drop the assumption that I is radical?

Thank you!