The Koszul Property of Algebras Associated to Matroids

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(joint work with Matt Mastroeni and Irena Peeva)

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Iowa State University

Outline

1. Koszul Algebras
2. Matroids and Lattices
3. Orlik-Solomon Algebras
4. Graded Möbius Algebras
5. Chow Rings of Matroids

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Notation

- $\mathbb{K}$ a field
- $V = \langle x_1, \ldots, x_n \rangle$ an n-dimensional $\mathbb{K}$-vector space
- $T = \bigoplus_{i=0}^{\infty} V^{\otimes i}$ the tensor algebra V over $\mathbb{K}$
- $I \subseteq T$ an ideal generated by g quadrics
- $R = T/I$: such $\mathbb{K}$-algebras are called **quadratic**

Koszul Algebras

Let $R_+ = \bigoplus_{i>0} R_i$.

R is a **Koszul algebra** if $R/R_+ \cong \mathbb{K}$ has a linear free resolution over R.

Koszul Algebras

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Example

Let $R = \mathbb{K}[x,y]/(x,y)^2$. Then the minimal free resolution of $\mathbb{K}$ is:

$$\cdots \xrightarrow{\begin{pmatrix} x & y & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & x & y & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & x & y & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & x & y \end{pmatrix}} R(-2)^4 \xrightarrow{\begin{pmatrix} x & y & 0 & 0 \\ 0 & 0 & x & y \end{pmatrix}} R(-1)^2 \xrightarrow{\begin{pmatrix} x & y \end{pmatrix}} R$$

Properties of Koszul Algebras

- Every Koszul algebra R corresponds to a **quadratic dual** algebra $R^!$ which is also Koszul.
- $\mathrm{P}_R(t) = \sum_i \beta_i^R(\mathbb{K}) t^i = \sum_i (\dim_{\mathbb{K}} R_i^!) t^i = \mathrm{H}_{R^!}(t)$
- $\mathrm{P}_R(t)\,\mathrm{H}_R(-t) = 1$

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Example

For $R = \mathbb{K}[x, y]/(x, y)^2$, we have $\mathrm{H}_R(t) = 1 + 2t$ so that:

$$\mathrm{P}_R(t) = \frac{1}{1-2t} = 1 + 2t + 4t^2 + 8t^3 + \cdots$$

Examples of Koszul Algebras

- Polynomial rings and exterior algebras
- Quotients by quadratic monomial ideals (Fröberg 1975)
- All high degree Veronese subrings of any standard graded algebra (Backelin 1986)
- Quadratic Gorenstein rings of regularity 2 (Conca, Rossi, Valla 2001)

How to Detect Koszulness

We say that R or I is **G-quadratic** if, after a suitable linear change of coordinates, the ideal $\varphi(I)$ has a Gröbner basis consisting of quadrics.

$$\text{G-quadratic} \Longrightarrow \text{Koszul} \Longrightarrow \text{quadratic}$$

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Example

Neither converse is true:

$$K[x, y, z]/(x^2 - yz, y^2 - xz, z^2 - xy)$$

is Koszul but not G-quadratic.

$$K[x, y, z, w]/(x^2, y^2, xz + yw)$$

is quadratic but not Koszul.

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Matroids

A **matroid** M is a pair $(E, \mathcal{I})$ where E is a finite set E and $\mathcal{I} \subseteq \mathcal{P}(E)$ satisfying:

- $\varnothing \in \mathcal{I}$.
- If $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$,
- If $A, B \in \mathcal{I}$, $|B| > |A|$, then $\exists b \in B \setminus A$ with $A \cup b \in \mathcal{I}$.

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Elements of $\mathcal{I}$ are called **independent sets**.
Subsets $A \subseteq E$ with $A \notin \mathcal{I}$ are called **dependent**.
Minimal dependent sets are called **circuits**.

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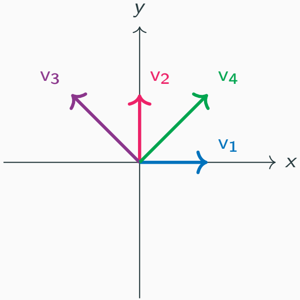
The **uniform matroid** $U_{r,n}$ is the $(r-1)$-skeleton of an $(n-1)$-simplex.

How to Make a Matroid: 1A. Sets of Vectors

A set of vectors $A = \{v_1, v_2, \ldots, v_n\}$ defines a **representable matroid** $M(A)$ whose independent sets are subsets of linearly independent vectors in A.

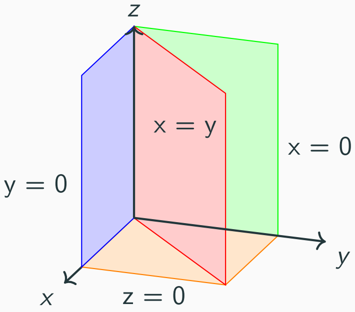


$$A = \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

$$M(A) \cong U_{2,4}$$

How to Make a Matroid: 1B. Hyperplane arrangements

Dually, an arrangement of hyperplanes $\mathcal{A} = \{H_1, \ldots, H_n\}$ in $\mathbb{C}^d$ defines a representable matroid $M(\mathcal{A})$.
We assume $\mathcal{A}$ is central and essential, i.e. $\bigcap_i H_i = \{0\}$.

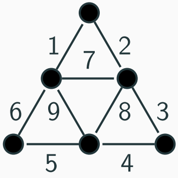


$$A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$M(\mathcal{A}) = M(A)$$

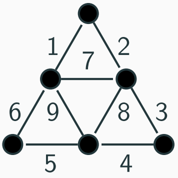
How to Make a Matroid: 2. Graphs

Let $G = (V, E)$ be a graph. Define a subset $A \subset E$ to be independent if it contains no cycles. Then $M(G) = (E, \mathcal{I})$ is a matroid.



Circuits of $M(G) =$
$\{\{1,2,7\},\{3,4,8\},\{5,6,9\},\{7,8,9\},\{1,2,8,9\},\{3,4,7,9\},$
$\{5,6,7,8\},\{1,2,5,6,8\},\{1,2,3,4,9\},\{3,4,5,6,7\},\{1,2,3,4,5,6\}\}.$

All graphic matroids are representable



$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

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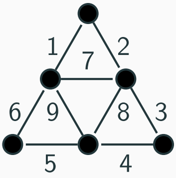

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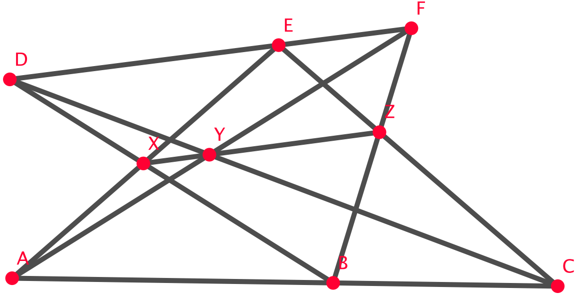
$M(G) = M(A)$ is representable.

$U_{2,4}$ is representable but not graphic.

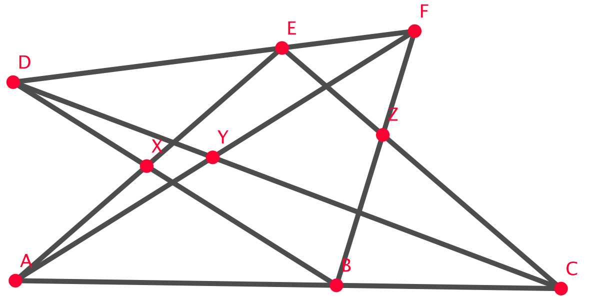
Are all matroids representable?

Consider the configuration of points in the projective plane from Pappus' Hexagon Theorem. This is a representation of the **Pappus matroid**.



Are all matroids representable?

No. The **Non-Pappus matroid** is not representable. (Any representable matroid would be subject to Pappus's Theorem, so X,Y,Z would be collinear.)



Most matroids are not representable

0% of all matroids are representable. (Nelson 2018)
Thus matroids are much more general than graphs, vector configuration, or hyperplane arrangements.

Rank and Flats

If M is a matroid with ground set E and $F \subseteq E$:

- The **rank** of F is

$$\mathrm{rk}(F) = \max\{|X| \mid X \subseteq F, X \text{ independent}\}.$$

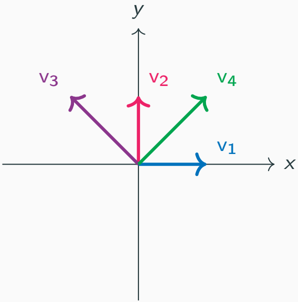
- The **closure** of F is

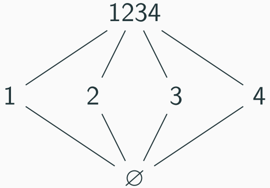
$$\mathrm{cl}(F) = \{x \in E \mid \mathrm{rk}(F \cup \{x\}) = \mathrm{rk}(F)\}.$$

- F is a **flat** if $\mathrm{cl}(F) = F$.

The Lattice of Flats - Running Example 1

The set of all flats ordered by inclusion forms a lattice $\mathcal{L}(M)$ graded by the ranks of flats.

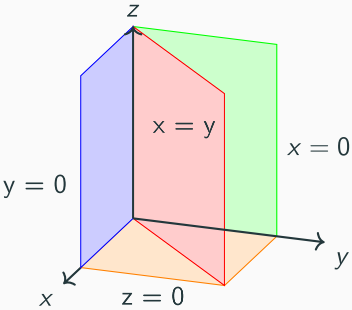


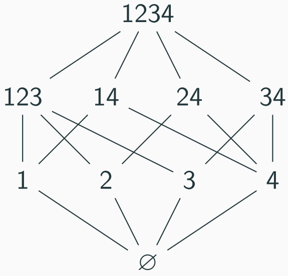


$\mathcal{L}(U_{2,4})$

The Lattice of Flats - Running Example 2

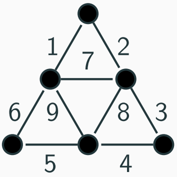
For $\mathcal{A} = \{H_1 = V(x), H_2 = V(y), H_3 = V(x-y), H_4 = V(z)\}$ in $\mathbb{C}^3$ below, its lattice of flats $\mathcal{L}(\mathcal{A})$ is



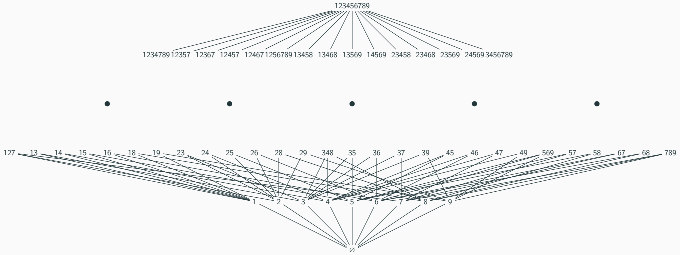


$\mathcal{L}(\mathcal{A})$

The Lattice of Flats - Running Example 3



$\mathcal{L}(M(G))$:



Geometric Lattices

A **geometric** lattice $\mathcal{L}$ is

- **graded** : There is a function $\mathrm{rk} : \mathcal{L} \to \mathbb{N}$ such that $x > y \Rightarrow \mathrm{rk}(x) > \mathrm{rk}(y)$ and if x covers y then $\mathrm{rk}(x) = \mathrm{rk}(y) + 1$.
- **semimodular** : $\mathrm{rk}(x) + \mathrm{rk}(y) \geq \mathrm{rk}(x \vee y) + \mathrm{rk}(x \wedge y)$ for all $x, y \in \mathcal{L}$.
- **atomic** : Every $x \in \mathcal{L}$ is a join of atoms (covers of $\hat{0}$).

Write $\mathcal{L}_i = \{x \in \mathcal{L} \mid \mathrm{rk}(x) = i\}$.

Geometric Lattices vs. Matroids

Theorem (Garrett Birkhoff 1935)

There is a bijection between the set of finite geometric lattices and the set of simple matroids.

A matroid is **simple** if it has no circuits of size 1 (loops) or 2 (parallel elements).

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Orlik-Solomon Algebras - Algebraic Definition

Let $M = (E, \mathcal{I})$ be a matroid and $\mathcal{L} = \mathcal{L}(M)$ is lattice of flats, where $E = \{1, \ldots, n\}$. The **Orlik-Solomon Algebra** of M is

$$\mathrm{OS}(M) = \frac{\bigwedge_{\mathbb{K}} \langle e_1, \ldots, e_n \rangle}{(\partial(e_C) \mid C \subseteq E \text{ is a circuit})},$$

where $e_C = \prod_{i \in C} e_i$ and

$$\partial \left(\prod_{i=1}^{t} e_{j_i} \right) = \sum_{i=1}^{t} (-1)^i e_{j_1} \cdots \widehat{e_{j_i}} \cdots e_{j_t}.$$

Orlik-Solomon Algebras - Geometric Definition

Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be a complex hyperplane arrangement in $\mathbb{C}^d$.
Let $M_{\mathcal{A}} = \mathbb{C}^d \setminus \bigcup_{i=1}^n H_i$ be the complement.

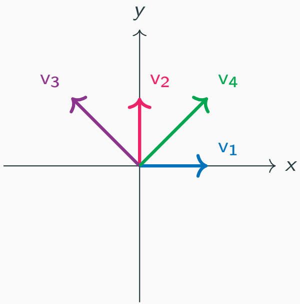
Theorem (Orlik-Solomon 1980)

The cohomology ring of $M_{\mathcal{A}}$ is

$$H^*(M_{\mathcal{A}}; \mathbb{K}) \cong \mathrm{OS}(M(\mathcal{A})),$$

where $M(\mathcal{A})$ is the associated matroid. In particular, $H^*(M_{\mathcal{A}}; \mathbb{K})$ depends only on the intersection lattice.

Orlik-Solomon Algebras - Running Example 1



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$$M(A) \cong U_{2,4}$$

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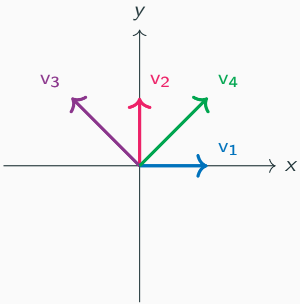
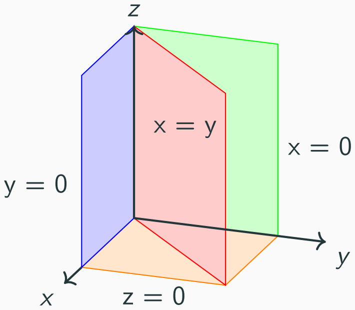


$$A = \begin{pmatrix} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

$$M(A) \cong U_{2,4}$$

$$\begin{aligned} \mathrm{OS}(U_{2,4}) &= \frac{\bigwedge_{\mathbb{K}}\langle e_1, \ldots, e_4\rangle}{(\partial(e_{123}), \partial(e_{124}), \partial(e_{134}), \partial(e_{234}))} \\ &= \frac{\bigwedge_{\mathbb{K}}\langle e_1, \ldots, e_4\rangle}{(e_1e_2 - e_1e_3 + e_2e_3, e_1e_2 - e_1e_4 + e_2e_4, e_1e_3 - e_1e_4 + e_3e_4)}. \end{aligned}$$

Orlik-Solomon Algebras - Running Example 2



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$$M(\mathcal{A}) = M(A)$$

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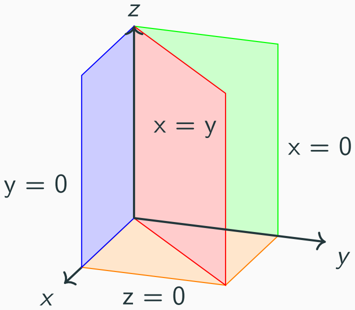

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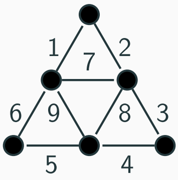
$$M(\mathcal{A}) = M(A)$$

$$\begin{aligned} \mathrm{OS}(\mathcal{A}) &= \frac{\bigwedge_{\mathbb{K}}\langle e_1, \ldots, e_4\rangle}{(\partial(e_{123}))} \\ &= \frac{\bigwedge_{\mathbb{K}}\langle e_1, \ldots, e_4\rangle}{(e_1 e_2 - e_1 e_3 + e_2 e_3)}. \end{aligned}$$

Orlik-Solomon Algebras - Running Example 3



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$$\begin{aligned}\mathrm{OS}(M(G)) &= \frac{\bigwedge_{\mathbb{K}}\langle e_1,\ldots,e_9\rangle}{(\partial(e_{127}),\partial(e_{348}),\ldots,\partial(e_{123456}))}\\ &= \frac{\bigwedge_{\mathbb{K}}\langle e_1,\ldots,e_9\rangle}{(\partial(e_{127}),\partial(e_{348}),\partial(e_{569}),\partial(e_{789}))}.\end{aligned}$$

Gröbner bases of Orlik-Solomon Ideals

An element F in a geometric lattice $\mathcal{L}$ is **modular** if

$$\mathrm{rk}(F) + \mathrm{rk}(G) = \mathrm{rk}(F \vee G) + \mathrm{rk}(F \wedge G)$$

for all $G \in \mathcal{L}$. A lattice is **supersolvable** if there is a saturated chain $\hat{0} \leq F_1 \leq \cdots \leq F_{r-1} \leq \hat{1}$ of modular elements.

Theorem (Björner-Ziegler 1991, Peeva 2003)

Let M be a matroid with Orlik-Solomon algebra $\mathrm{OS}(M) = E/J$. Then the following are equivalent:

- *$\mathcal{L}(M)$ is supersolvable.*
- *J has a quadratic Gröbner basis with respect to some monomial order.*

Orlik-Solomon Ideals - Koszul Property

Corollary

If $\mathcal{L}(M)$ is supersolvable, then $\mathrm{OS}(M)$ is Koszul.

Remark: The converse is open.

Theorem (Papadima-Yuzvinsky 1999)

Let $\mathcal{A}$ be a complex hyperplane arrangement with complement $M_{\mathcal{A}}$. Then $\mathrm{OS}(\mathcal{A})$ is Koszul if and only if $M_{\mathcal{A}}$ is a rational $K(\pi, 1)$-space.

Orlik-Solomon Algebras - Graphic Matroids

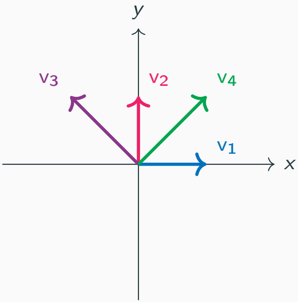
Now let G be a simple graph with graphic matroid $M(G)$.

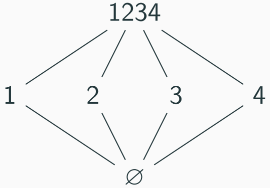
Theorem (Stanley 1972, Schenck-Suciu 2002)

The following are equivalent:

- $\mathrm{OS}(M(G))$ *is quadratic.*
- $\mathrm{OS}(M(G))$ *has a quadratic Gröbner basis.*
- $\mathrm{OS}(M(G))$ *is Koszul.*
- $\mathcal{L}(M(G))$ *is supersolvable.*
- G *is chordal.*

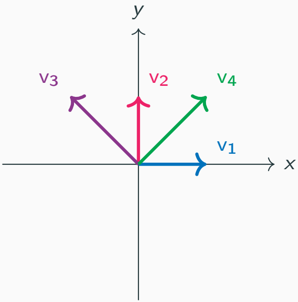
Orlik-Solomon Algebras - Running Example 1

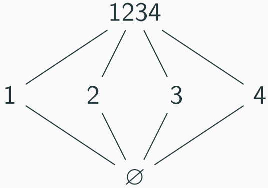




$\mathcal{L}(U_{2,4})$

Orlik-Solomon Algebras - Running Example 1



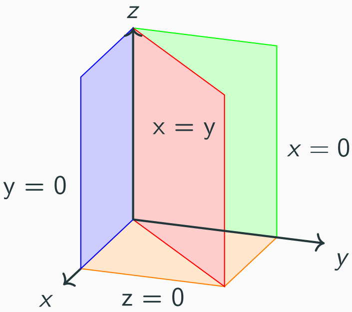


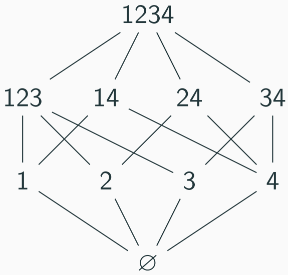
$\mathcal{L}(U_{2,4})$

$$\mathrm{OS}(U_{2,4}) = \frac{\bigwedge_{\mathbb{K}}\langle e_1, \ldots, e_4\rangle}{J},$$

where $J = (e_1e_2 - e_1e_3 + e_2e_3, e_1e_2 - e_1e_4 + e_2e_4, e_1e_3 - e_1e_4 + e_3e_4)$.
$in_<(J) = (e_1e_2, e_1e_3, e_2e_3)$.
So $\mathrm{OS}(U_{2,4})$ is Koszul.

Orlik-Solomon Algebras - Running Example 2





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where $J = (e_1e_2 - e_1e_3 + e_2e_3)$ and $in_<(J) = (e_1e_2)$.
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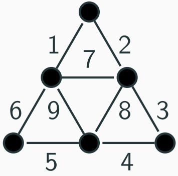

$$\mathrm{OS}(M(G)) = \frac{\bigwedge_{\mathbb{K}}\langle e_1, \ldots, e_9\rangle}{J},$$

where $J = (\partial(e_{127}), \partial(e_{348}), \partial(e_{569}), \partial(e_{789}))$ and
$in_<(J) = (e_1e_2, e_3e_4, e_5e_6, e_7e_8)$ (G is chordal!)
So $\mathrm{OS}(M(G))$ is Koszul.

Outline

1. Koszul Algebras

2. Matroids and Lattices

3. Orlik-Solomon Algebras

4. Graded Möbius Algebras

5. Chow Rings of Matroids

Graded Möbius Algebras - Algebraic Definition

Let M be a simple matroid with finite ground set E.
The **graded Möbius algebra** of M is the commutative ring

$$\mathrm{GM}(M) = \bigoplus_{F \in \mathcal{L}(M)} \mathbb{K} y_F$$

with multiplication

$$y_F y_G = \begin{cases} y_{F \vee G}, & \text{if } \mathrm{rk}(F \vee G) = \mathrm{rk}\, F + \mathrm{rk}\, G \\ 0, & \text{otherwise.} \end{cases}$$

In particular, $HF_{\mathrm{GM}(M)}(i) = \#\mathcal{L}(M)_i$.
$\mathcal{L}(M)$ graded + atomic $\Rightarrow$ $\mathrm{GM}(M)$ is standard graded.

Graded Möbius Algebras - Geometric Definition

Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be a complex hyperplane arrangement in $\mathbb{C}^d$.
Let $\Psi : \mathbb{C}^d \hookrightarrow \prod_{H \in \mathcal{A}} \mathbb{C}^d / H = \prod_{H \in \mathcal{A}} \mathbb{C}^1 \hookrightarrow \prod_{H \in \mathcal{A}} \mathbb{P}^1_{\mathbb{C}} = \prod_{i=1}^{n} \mathbb{P}^1_{\mathbb{C}}$.
Let $S_{\mathcal{A}}$ be the image of Ψ with closure $\overline{S_{\mathcal{A}}}$ called the **Schubert variety** of $\mathcal{A}$.

Theorem (Huh, Wang 2017)

The cohomology ring of $\overline{S_{\mathcal{A}}}$ *is*

$$H^{2*}(\overline{S_{\mathcal{A}}}; \mathbb{K}) \cong \mathrm{GM}(M(\mathcal{A})).$$

Graded Möbius Algebras - Presentation

Proposition (Mastroeni-M-Peeva 2023)

Let M be a simple matroid. Let L be the ideal generated by all binomials $y_{C \smallsetminus i} - y_{C \smallsetminus j}$ for all circuits C of M and all $i, j \in C$. Then:

$$Q = (y_i^2 \mid i \in E) + (y_{C \smallsetminus i} - y_{C \smallsetminus j} \mid C \text{ is a circuit of } M, i, j \in C),$$

and the generators of the latter ideal are a Gröbner basis for Q with respect to every lex ordering for any ordering of the elements of E. (If $C = \{i_1, \dots, i_t\}$, then $y_{C \smallsetminus i_k} = y_{i_1} y_{i_2} \cdots \widehat{y_{i_k}} \cdots y_{i_t}$.)

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Let M be a simple matroid. Let L be the ideal generated by all binomials $y_{C\smallsetminus i} - y_{C\smallsetminus j}$ for all circuits C of M and all $i, j \in C$. Then:

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See related result of Maeno-Numata 2011.

Let M be a simple matroid. M is:

- **C-chordal** if for every circuit C of M of size at least four there is an element $e \in E$ and circuits A, B of M such that $A \cap B = \{e\}$ and $C = (A \setminus e) \sqcup (B \setminus e)$.
- **T-chordal** if for every circuit C of M of size at least four there is an element $w \in E \setminus C$ and elements $u, v \in C$ such that $\{u, v, w\}$ is a circuit.

Graded Möbius Algebras - Quadracity

Theorem (Mastroeni-M-Peeva 2023)

Let M be a simple matroid. Then

$$M \text{ is C-chordal } \Rightarrow \mathrm{GM}(M) \text{ is quadratic } \Rightarrow M \text{ is T-chordal.}$$

Graded Möbius Algebras - Quadracity

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Remark: Neither converse is true.

Graded Möbius Algebras - Quadracity

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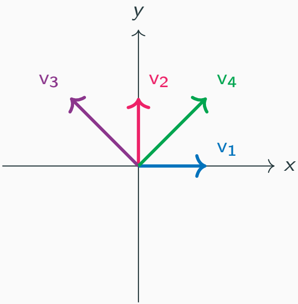
Remark: Neither converse is true.

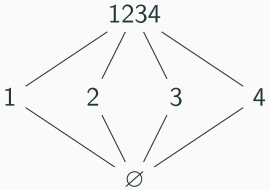
Corollary (Mastroeni-M-Peeva 2023)

Let G be a simple graph. Then

$$G \text{ is chordal } \iff \mathrm{GM}(M(G)) \text{ is quadratic }.$$

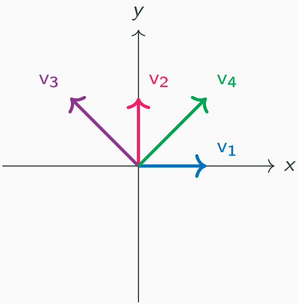
Graded Möbius Algebras - Running Example 1

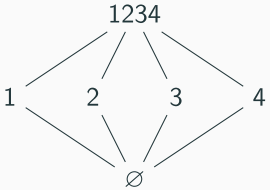




$\mathcal{L}(U_{2,4})$

Graded Möbius Algebras - Running Example 1



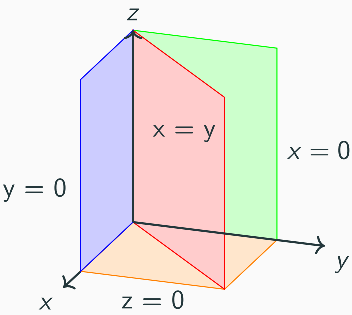


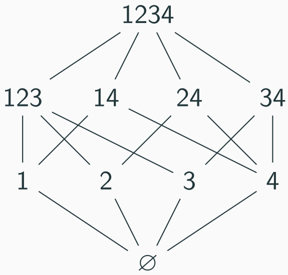
$\mathcal{L}(U_{2,4})$

$$\mathrm{GM}(U_{2,4}) = \frac{\mathbb{K}[y_1, \ldots, y_4]}{J},$$

where $J = (y_1^2, y_2^2, y_3^2, y_4^2, y_1 y_2 - y_1 y_3, y_1 y_3 - y_2 y_3, \ldots, y_2 y_4 - y_3 y_4)$.
$in_<(J) = (y_1^2, y_2^2, y_3^2, y_4^2, y_1 y_2, y_1 y_3, y_1 y_4, y_2 y_3, y_2 y_4)$.
So $\mathrm{GM}(U_{2,4})$ is Koszul.

Graded Möbius Algebras - Running Example 2



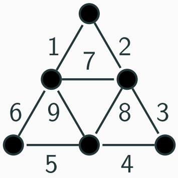


$\mathcal{L}(\mathcal{A})$

$$\mathrm{GM}(\mathcal{A}) = \frac{\mathbb{K}[y_1, \ldots, y_4]}{J},$$

where $J = (y_1^2, y_2^2, y_3^2, y_4^2, y_1y_2 - y_1y_3, y_1y_3 - y_2y_3)$ and
$in_<(J) = (y_1^2, y_2^2, y_3^2, y_4^2, y_1y_2, y_1y_3)$.
So $\mathrm{GM}(\mathcal{A})$ is Koszul.

Graded Möbius Algebras - Running Example 3



$$\mathrm{GM}(M(G)) = \frac{\mathbb{K}[y_1, \ldots, y_9]}{J}$$

$$J = (y_1^2, \ldots, y_9^2, y_1y_2 - y_1y_7, y_1y_7 - y_2y_7, \ldots, y_7y_8 - y_7y_9, y_7y_9 - y_8y_9).$$

Betti table of $\mathbb{K}$ over $\mathrm{GM}(M(G))$:

	0	1	2	3	4
0	1	9	53	260	1,156
1	–	–	–	–	1

$\mathrm{GM}(M(G))$ is quadratic (G is chordal) but not Koszul.

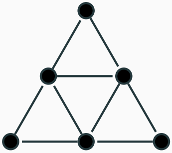
Graded Möbius Algebras - Graphic Matroids

A graph G is **strongly chordal** if G is chordal and every cycle of even length $n \geq 6$ has an odd chord.

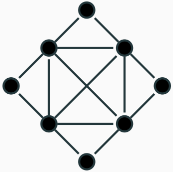
Theorem (Farber 1983)

The following are equivalent:

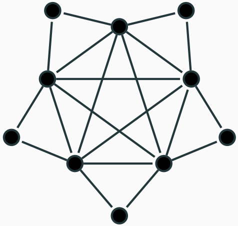
- *G is strongly chordal.*
- *G is chordal and has no induced n-trampoline.*



3-trampoline



4-trampoline



5-trampoline

Graded Möbius Algebras - Graphic Matroids

Theorem (Mastroeni-M-Peeva 2023)

Let G be a graph. If $\mathrm{GM}(M(G))$ is Koszul, then G is strongly chordal.

We conjecture that the converse holds.
It reduces to the following purely combinatorial statement.

Conjecture

Is a graph G strongly chordal if and only if there is a total order $\prec$ on the edges of G with the property that for every cycle C of size at least four in G and every edge $e \in C \setminus \min_{\prec} C$, there is a chord c of C and edges $a, b \in C \setminus e$ such that $T = \{a, b, c\}$ is a 3-cycle with $\min_{\prec} T \neq c$?

Outline

1. Koszul Algebras

2. Matroids and Lattices

3. Orlik-Solomon Algebras

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5. Chow Rings of Matroids

Chow Rings of Matroids - Algebraic Definition

The **Chow ring of a matroid** M is the (commutative) ring:

$$\underline{\mathrm{CH}}(M) = \frac{\mathbb{K}[x_F \mid F \in \mathcal{L} \setminus \{\varnothing\}]}{(x_F x_{F'} \mid F, F \text{ incomp}) + (\sum_{G \supseteq F} x_G \mid \mathrm{rk}\, F = 1)}$$

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Remark: $\underline{\text{CH}}(M)$ is clearly standard graded and quadratic.

Chow Rings of Matroids - Geometric Definition

Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be a complex hyperplane arrangement in $\mathbb{C}^d$.
Let $\mathbb{P}\mathcal{A} \subseteq \mathbb{P}^{d-1}_{\mathbb{C}}$ be the projectivization with complement $M_{\mathbb{P}\mathcal{A}} = \mathbb{P}^{d-1}_{\mathbb{C}} \setminus \bigcup_{i=1}^{n} \mathbb{P}H_i$.
Let

$$\Phi : M_{\mathbb{P}\mathcal{A}} \to \mathbb{P}^{d-1}_{\mathbb{C}} \times \prod_{F \in \mathcal{L}(\mathcal{A}) \setminus \mathbb{C}^d} \mathbb{P}(\mathbb{C}^d / F)$$

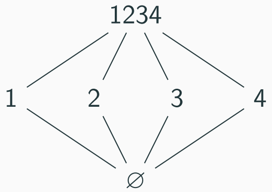
be the natural map with image $Y_{\mathbb{P}\mathcal{A}}$ with closure $\overline{Y_{\mathbb{P}\mathcal{A}}}$.
$\overline{Y_{\mathcal{A}}}$ is the (projectivized) **wonderful compactification** à la de Concini & Procesi (1996).

Theorem (Feichtner-Yuzvinksky 2003)

$$H^{2*}(\overline{Y_{\mathcal{A}}}; \mathbb{K}) \cong \underline{\mathrm{CH}}(M(\mathcal{A})).$$

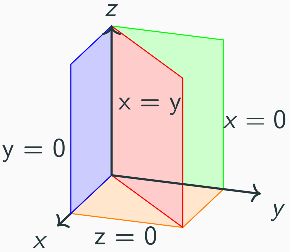
Chow Rings of Matroids - Running Example 1

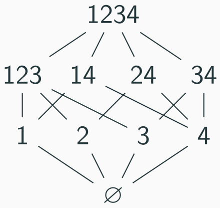
$\mathcal{L}(U_{2,4})$:



$$\underline{\mathrm{CH}}(U_{2,4}) = \frac{\mathbb{K}[x_1, x_2, x_3, x_4, x_{1234}]}{(x_1x_2, x_1x_3, x_1x_4, x_2x_3, x_2x_4, x_3x_4) + (x_1 + x_{1234}, x_2 + x_{1234}, x_3 + x_{1234}, x_4 + x_{1234})}$$

Hilbert series of $\underline{\mathrm{CH}}(U_{2,4})$ is $1 + t$.

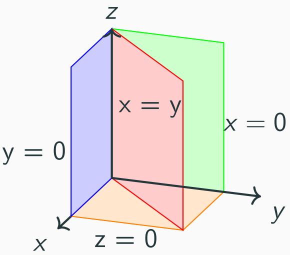


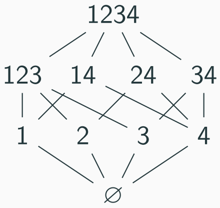


$\mathcal{L}(\mathcal{A})$

$$\underline{\mathrm{CH}}(M(\mathcal{A})) = \frac{\mathbb{K}[x_1, x_2, x_3, x_4, x_{123}, x_{14}, x_{24}, x_{34}, x_{1234}]}{J},$$

Chow Rings of Matroids - Running Example 2





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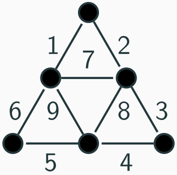
$$\underline{\mathrm{CH}}(M(\mathcal{A})) = \frac{\mathbb{K}[x_1, x_2, x_3, x_4, x_{123}, x_{14}, x_{24}, x_{34}, x_{1234}]}{J},$$

$$J = (x_1x_2, x_1x_3, x_1x_4, x_2x_3, x_2x_4, x_3x_4, x_1x_{24}, x_1x_{34}, x_2x_{14}, x_2x_{34}, x_3x_{14},$$
$$x_3x_{24}, x_4x_{123}, x_{123}x_{14}, x_{123}x_{24}, x_{123}x_{34}, x_{14}x_{24}, x_{14}, x_{34}, x_{24}x_{34},$$
$$x_1 + x_{123} + x_{14} + x_{1234}, x_2 + x_{123} + x_{24} + x_{1234},$$
$$x_3 + x_{123} + x_{34} + x_{1234}, x_4 + x_{14} + x_{24} + x_{34} + x_{1234}).$$

Hilbert series of $\underline{\mathrm{CH}}(M(\mathcal{A}))$ is $1 + 5t + t^2$.

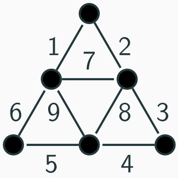
Chow Rings of Matroids - Running Example 3



$$\mathrm{GM}(M(G)) = \frac{\mathbb{K}[x_1, \ldots, x_9, x_{127}, x_{13}, \ldots, x_{123456789}]}{J}$$

$J =$ (3119 quadratic monomials) $+$ (9 linear forms).

Chow Rings of Matroids - Running Example 3



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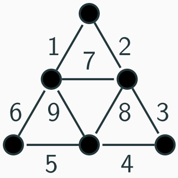
Hilbert series of $\mathrm{GM}(M(G))$ is $1 + 79t + 255t^2 + 79t^3 + t^4$.

Chow Rings of Matroids - Running Example 3



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$$J = (3119 \text{ quadratic monomials}) + (9 \text{ linear forms}).$$

Hilbert series of $\mathrm{GM}(M(G))$ is $1 + 79t + 255t^2 + 79t^3 + t^4$.
In general, $\mathrm{reg}(\mathrm{GM}(M(G))) = \mathrm{rk}\, M(G) - 1 = \#V - 2$.

Augmented Chow Rings of Matroids - Algebraic Definition

The **augmented Chow ring** of a simple matroid M is the ring

$$\mathrm{CH}(M) := \mathbb{K}[y_i, x_F \mid i \in E, F \in \mathcal{L}(M) \smallsetminus \{E\}]/(I_M + J_M),$$

where

$$I_M = (y_i - \textstyle\sum_{i \notin F} x_F \mid i \in E),$$

$$J_M = (x_F x_G \mid F, G \text{ incomparable }) + (y_i x_F \mid i \in E, i \notin F).$$

Proposition (Braden, Huh, Matherne, Proudfoot, Wang 2020)

- $\mathrm{GM}(M) \subset \mathrm{CH}(M)$,
- $\mathrm{CH}(M) = \underline{\mathrm{CH}}(M) \otimes_{\mathrm{GM}(M)} \mathbb{K}$.

Applications of Chow Rings of Matroids

Conjecture (Top Heavy Conjecture of Dowling and Wilson 1974)

For any geometric lattice $\mathcal{L}$ of rank r and if $j \leq \frac{r}{2}$, then

$$|\mathcal{L}_j| \leq |\mathcal{L}_{r-j}|.$$

Theorem (Braden, Huh, Matherne, Proudfoot, Wang 2020)

The Top Heavy conjecture is true.

Proof relies on "Hodge theory" of matroids, which is a collection of properties of $\underline{\mathrm{CH}}(M)$ and $\mathrm{CH}(M)$ collectively known as the **Kähler package**.

The Kähler Package

A commutative graded Artinian K-algebra $A = \bigoplus_{i=0}^{d} A_i$ is said to have the **Kähler package** provided:

- **Poincare Duality**: There is a nondegenerate, bilinear pairing
$$P : A_i \times A_{d-i} \to K$$
- **Lefschetz Property**: There is a linear form $L \in A_1$ such that
$$\begin{aligned} A_i &\to A_{d-i} \\ x &\mapsto L^{d-2i}x \end{aligned}$$
is an isomorphism.
- **Hodge-Riemann Relations**: The symmetric bilinear form on A_i:
$$(x, y) \mapsto (-1)^i P(x, L^{d-2i}y)$$
is positive definite on the kernel of L^{d-2i+1}.

Algebraic properties of $\underline{\mathrm{CH}}(M)$

Theorem (Adiprasito, Huh, Katz 2018, BHMPW 2020)

For any matroid M, $\underline{\mathrm{CH}}(M)$ and $\mathrm{CH}(M)$ have the Kähler package.

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For any matroid M, $\underline{\mathrm{CH}}(M)$ and $\mathrm{CH}(M)$ have the Kähler package.

So $\underline{\mathrm{CH}}(M)$ and $\mathrm{CH}(M)$ are nice commutative, graded, Artinian rings.

- They are **Gorenstein**: $\dim_{\mathbb{K}}(\underline{\mathrm{CH}}(M)_i) = \dim_{\mathbb{K}}(\underline{\mathrm{CH}}(M)_{d-i})$
- They are **quadratic**:
  $\underline{\mathrm{CH}}(M) \cong \mathbb{K}[z_1, \ldots, z_n]/(\text{homogeneous quadrics})$.

Algebraic properties of $\underline{\mathrm{CH}}(M)$

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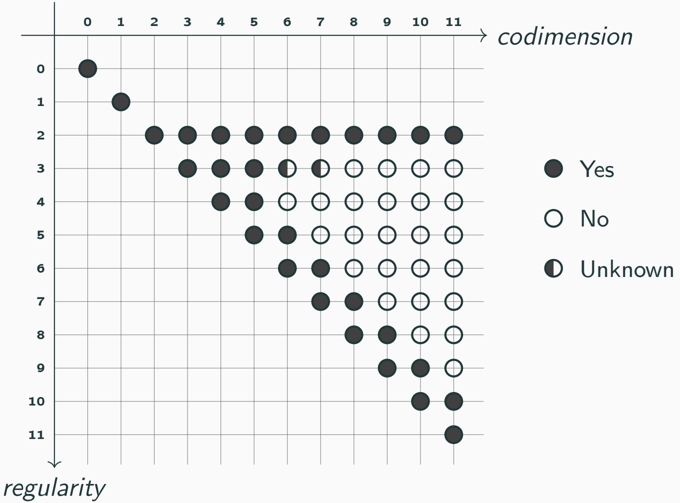
Question

Are $\underline{\mathrm{CH}}(M)$ and $\mathrm{CH}(M)$ Koszul?

Are quadratic Gorenstein rings always Koszul?



Mastroeni, Schenck, Stillman (2021)
(3,8) dot: M, Seceleanu (2021)

Koszulness of Chow Rings

Conjecture (Dotsenko 2020)

The Chow ring of any matroid is Koszul.

Koszulness of Chow Rings

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Theorem (Mastroeni, M 2021)

$\underline{\mathrm{CH}}(M)$ *and* $\mathrm{CH}(M)$ *are Koszul for any matroid* M*.*

Koszulness of Chow Rings

Conjecture (Dotsenko 2020)

The Chow ring of any matroid is Koszul.

Theorem (Mastroeni, M 2021)

$\underline{\mathrm{CH}}(M)$ *and* $\mathrm{CH}(M)$ *are Koszul for any matroid* M.

Proof uses a Koszul filtration and the known (non-quadratic) Gröbner basis of Feichtner-Yuzvinsky.

Idea of the Proof

- If $\operatorname{rk} M = 1$, then $\underline{\mathrm{CH}}(M) \cong \mathbb{Q}$.
- If $\operatorname{rk} M = 2$, then $\underline{\mathrm{CH}}(M) \cong \mathbb{Q}[x_E]/(x_E^2)$.
- In both cases, $\underline{\mathrm{CH}}(M)$ has a Koszul filtration.
- Use natural matroid operations (restriction and truncation) to inductively lift the filtration.

Rationality of Poincaré Series

Recall that R Koszul $\Longrightarrow P^R_K(t) = \frac{1}{H_R(-t)}$.

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Corollary

For any matroid $\underline{\mathrm{CH}}(M)$ *has a rational Poincaré series.*

Rationality of Poincaré Series

Recall that R Koszul $\Longrightarrow P^R_K(t) = \frac{1}{H_R(-t)}$.

Corollary

For any matroid $\underline{\mathrm{CH}}(M)$ *has a rational Poincaré series.*

Example (Bøgvad / Anick)

$R = \mathbb{K}[x_1, x_2, x_3, x_4, x_5]/(x_1^2, x_2^2, x_3^2, x_4^2, x_5^2, x_1x_2, x_4x_5, x_1x_3+x_3x_4+x_2x_5)$.
Let $\widetilde{R} = R \ltimes \omega_R(-3)$ is **quadratic**, Artinian, and Gorenstein, but $P^R_K(t)$ is irrational.

Roots of the h-polynomial

Theorem (Reiner-Welker (2005))

If R is Koszul and Artinian, then $H_R(t)$ has a real root.

Roots of the h-polynomial

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Conjecture (Huh)

$H_{\underline{CH}(M)}(t)$ and $H_{\mathrm{CH}(M)}(t)$ are real rooted.

Roots of the h-polynomial

Theorem (Reiner-Welker (2005))

If R is Koszul and Artinian, then $H_R(t)$ has a real root.

If R is Gorenstein, it has 2 real roots.

Conjecture (Huh)

$H_{\underline{CH}(M)}(t)$ and $H_{\mathrm{CH}(M)}(t)$ are real rooted.

Would imply that the coefficients of $H_{\underline{\mathrm{CH}}(M)}(t) = \sum_i h_i t^i$ are ultra-log concave; i.e $\{h_i/\binom{n}{i}\}$ is log-concave.

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See related work by Ferroni, Matherne, Stevens, and Vecchi (2022).

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