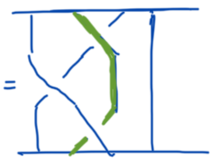
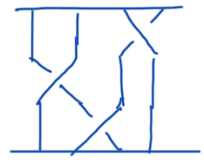
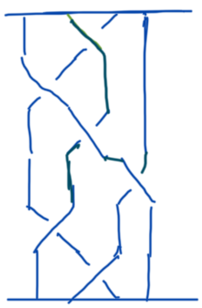
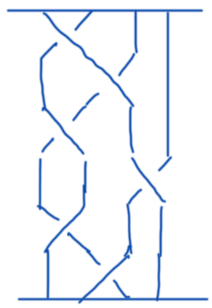
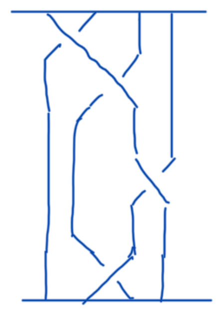
Example. A few braids in 4 strings.

$\sigma_1 =$  $\sigma_2 =$  $\sigma_3 =$  $\sigma_1^{-1} =$  ...

$b_1 =$  $b_2 =$  $b_3 =$ 

$= \sigma_1 \sigma_2 \sigma_1$ $\quad = \sigma_2 \sigma_1 \sigma_2$ $\quad = \sigma_3 \sigma_1^{-1} \sigma_2^{-1}$

Exercise: Verify: $b_1 = b_2$., $\sigma_1 \sigma_3 = \sigma_3 \sigma_1$.

 =  =  = 

$\sigma_1 \sigma_2 \sigma_3 \sigma_2^{-1}$

$b_2 b_3$
$= (\sigma_2 \sigma_1 \sigma_2)(\sigma_3 \sigma_1^{-1} \sigma_2^{-1})$

$b_1 b_3$
$= (\sigma_1 \sigma_2 \sigma_1)(\sigma_3 \sigma_1^{-1} \sigma_2^{-1})$

Def. $Br_n :=$ Braid group on n strings.

Theorem: The braid group has the following presentation:

$$Br_n = \left\langle \sigma_1, \ldots, \sigma_{n-1} \;\middle|\; \begin{array}{l} \sigma_i \sigma_j = \sigma_j \sigma_i \quad \text{if } |i-j| > 1 \\ \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \end{array} \right\rangle$$

Braid group is related to many other important objects

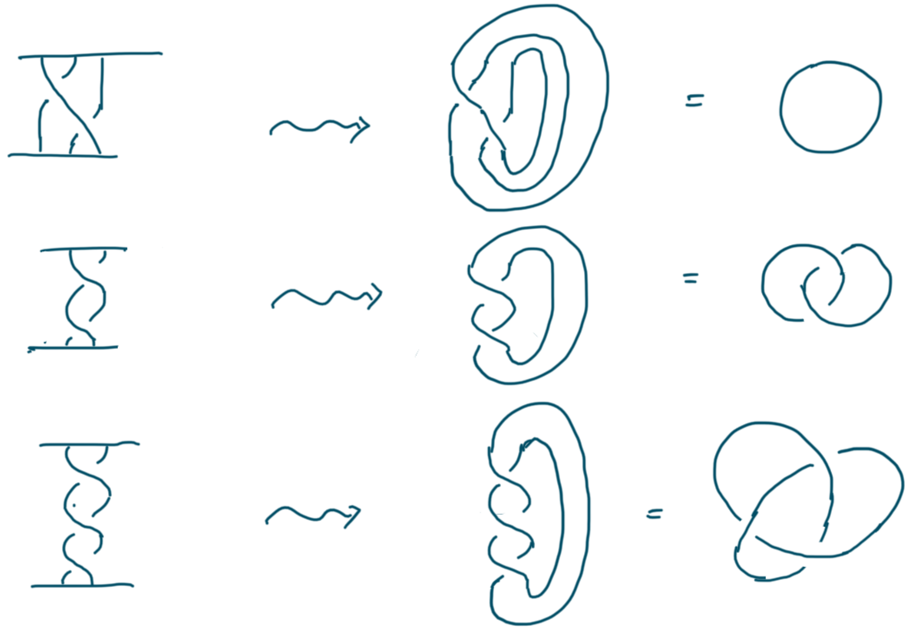
1. Braids $\longrightarrow$ Permutations.

$$\sigma_j \longmapsto s_j = (j, j+1)$$

$$\ker(\pi) \hookrightarrow Br_n \xrightarrow{\pi} S_n \text{ (symmetric gp).}$$

(called pure braid group). $S_n \simeq \dfrac{Br_n}{\langle\langle \sigma_i^2 = 1 \rangle\rangle}$..

2. Braids $\rightsquigarrow$ links or knots



⋮

3. $Br_n \longrightarrow Aut(F_n)$ $\quad F_n$ = free gp. on n-generators. $x_1, \dots, x_n$

$$\sigma_i \longmapsto \sigma_i(\cdot)$$

$$\sigma_i(x_i) = x_i x_{i+1} x_i^{-1}$$
$$\sigma_i(x_{i+1}) = x_i$$
$$\pi_i(x_j) = x_j \text{ if } j \neq i, i+1$$

<u>Exercise</u>. Verify that $\sigma_i(\cdot)$'s satisfy the braid group relations.

Fundamental group.

Def

- A based space (X, x_0) means a space X together with a basepoint. $x_0 \in X$.
- A based map $f:(X,x_0) \longrightarrow (Y, y_0)$ means a continuous map $f: X \to Y$ s.t. $f(x_0) = y_0$.

- Let $f_0, f_1 : (X, x_0) \longrightarrow (Y, y_0)$ be based maps.
  A based homotopy $h: f_0 \sim f_1$ is a map $h: X \times [0,1] \longrightarrow Y$ s.t. each $h_t = h(\cdot, t)$ is a based map, $h_0 = f_0$, $h_1 = f_1$.

- Forget about basepoint to get the def of homotopy.

- Say two spaces X and Y are homotopy equivalent, written $X \simeq Y$ if there exists $X \underset{g}{\overset{f}{\rightleftarrows}} Y$ s.t $g \circ f \simeq id_X$, $f \circ g \simeq id_Y$.
  Say X is contractible if $X \simeq$ a point.

· Big. Q: understand spaces upto homotopy. Invariants help.
Fundamental gp of a space is one of the most important invarian.

Def.
$\pi_1(X, x_0) :=$ set of based homotopy classes of loops in X, based at:
a loop based at x_0 means a based map $\gamma : (S^1, *) \longrightarrow (X, x_0)$

$\pi_1(X, x_0)$ is a group under composition of loops.
$\gamma_1 * \gamma_2 = \gamma_1$ followed by γ_2., γ^{-1} = reverse of γ.

"Theorem". Let $G \curvearrowright X$ be a free, properly discontinuous action on nice contractible space X. Then $\pi_1(X/G) = G$.

Example

1. $\pi_1(S^1) = \pi_1(\mathbb{R}/\mathbb{Z}) \simeq \mathbb{Z}$

2. $\pi_1\Big($  $\Big) = \pi_1(\mathbb{R}^2/\mathbb{Z}^2) = \mathbb{Z}^2$.

3. $\pi_1(\mathbb{C} - \{a, b\}) = \pi_1\Big($  $\Big) = \pi_1\left(\dfrac{\text{infinite 4-valent tree}}{F_2}\right) = F_2$.

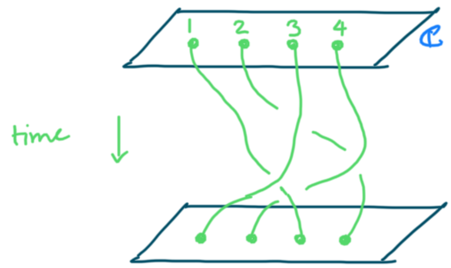
Similarly
$\pi_1(\mathbb{C}$ minus n points$) = F_n$.

Braid group as fundamental group

Def. Let X be a space. Let $\mathcal{H} = \bigcup_{i \neq j} \{(x_1, \ldots, x_n) \in X^n : x_i = x_j\}$.
Let $X^n_o = (X^n - \mathcal{H})$
Let $\mathrm{Conf}_n(X) := X^n_o / S_n$ = space of n distinct points on X

Let $Br_n(X) := \pi_1(\mathrm{Conf}_n(X))$

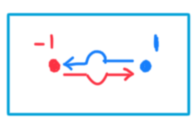
Observation. $Br_n = Br_n(\mathbb{C}) = \pi_1(\mathrm{Conf}_n(\mathbb{C}), *)$



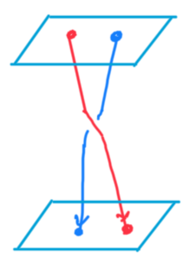
Take $* = \{1, 2, 3, 4\}$.

← a loop in $\mathrm{Conf}_4(\mathbb{C})$

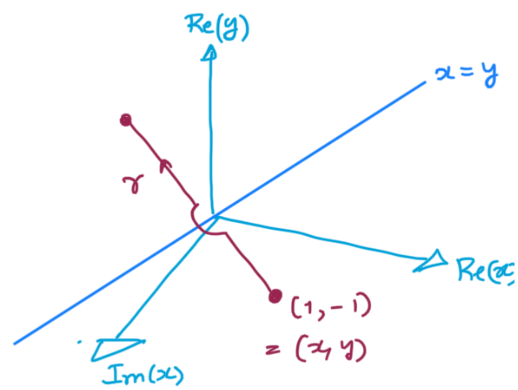
Three views of a braid.



top view



side view



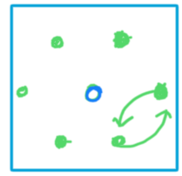
configuration space view.

The path γ lies in plane spanned by the axes $\mathrm{Re}\,x$, $\mathrm{Re}\,y$, except for all small detour near origi

Braid groups of $\mathbb{C}^*$ and S^2.

Consider $Br_n(\mathbb{C}^*)$. basepoint at $\{1, \zeta, \zeta^2, \dots, \zeta^{n-1}\}$, $\zeta = e^{2\pi i/n}$.

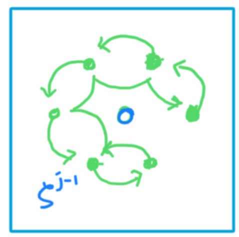
Some elements in $Br_6(\mathbb{C}^*)$.

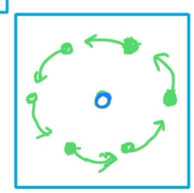
$\rho_0 =$  $\rho_1 =$  $\dots$ $\rho_5 =$ 

Observe:
$$(*)\begin{cases} \rho_i \rho_j \rho_i = \rho_j \rho_i \rho_j & \text{if } j = i \pm 1 \quad (\text{subscripts in } \mathbb{Z}/n) \\ \rho_i \rho_j = \rho_j \rho_i & \text{o.w.} \end{cases}$$

Let $I_j = \rho_j \rho_{j+1} \cdots \rho_{j+n-2}$ $\qquad$ $D_j = \rho_j \rho_{j-1} \cdots \rho_{j-n+2}$

$=$ 

$=$ 

Observe: In $Br_n(\mathbb{C}^* \cup 0)$: $D_j = D_{j+1} = \cdots =$  $= t$

In $Br_n(\mathbb{C}^* \cup \infty)$: $I_j = I_{j+1} = \cdots = t^{-1}$!

Theorem The groups $Br_n(\mathbb{C}^* \cup 0)$, $Br_n(\mathbb{C}^* \cup \infty)$ and $Br_n(S^2)$ are have presentations on generators $\rho_0, \dots, \rho_{n-1}$ and following relat

(a) For $Br_n(\mathbb{C}^* \cup 0)$, relations $(*)$ and $D_j = D_{j+1} = \cdots$

(b) For $Br_n(\mathbb{C}^* \cup \infty)$, relations $(*)$ and $I_j = I_{j+1} = \cdots$

(c) For $Br_n(S^2)$, relations in (a) and (b).

Observe: $t \rho_j t^{-1} = \rho_{j-1}$ for all j in $Br_n(\mathbb{C})$, and $Br_n(S^2)$.

space of points on the sphere.

$$\mathbb{C} \cup \infty = S^2 = \left\{ \mathbb{C}\binom{x}{y} : \binom{x}{y} \in \mathbb{C}^2 - \{0\} \right\}$$

$\tau = \frac{x}{y} \longleftarrow\!\shortmid \mathbb{C}\binom{x}{y}$ $\quad PGl_2(\mathbb{C}) \circlearrowright X$ by linear fractional transformatio

$$g\tau = \frac{a\tau + b}{c\tau + d}, \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

faithful and 3-transitive because $\quad z \longmapsto \frac{(z_1 - z_2)(z - z_0)}{(z_1 - z_0)(z - z_2)}$ take $\begin{Bmatrix} z_0 \\ z_1 \\ z_2 \end{Bmatrix} \to \begin{Bmatrix} 0 \\ 1 \\ \infty \end{Bmatrix}$

<u>Def.</u> $n \geq 3$.

$\mathcal{M}_n := (S^2)^n_0 / S_n \times PGl_2\mathbb{C}$ = space of n points on $\mathbb{P}^1$.

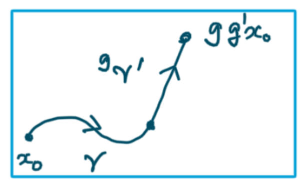
$= \dfrac{\left((S^2)^n_0 / PGl_2(\mathbb{C})\right)}{S_n}$ $\longleftarrow$ manifold since $PGl_2(\mathbb{C})$ action is free

So $\mathcal{M}_n$ = manifold / finite group. ; this is an example of an "orbifo

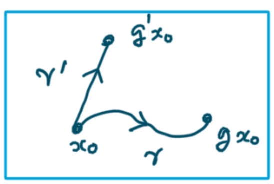
<u>Def.</u> Consider $G \circlearrowright X \ni x_0$.

· $\pi_1^{orb}(X/G, x_0) := \{ (\gamma, g) : g \in G,\ \gamma$ homotopy class of map from x_0 to $gx_0 \}$.

$(\gamma, g) \cdot (\gamma', g') = (\gamma * {}^{g}\gamma', gg') =$



where



.

<u>Fact</u>: The map $Br_n(S^2) \longrightarrow \pi_1^{orb}(\mathcal{M}_n)$ is surjective with kernel generated by center $(Br_n(S^2)) = \mathbb{Z}/2 = \langle t^n \rangle$.

<u>Theorem</u> $\pi_1^{orb}(\mathcal{M}_n)$ has presentation on generators $\rho_0, \dots, \rho_{n-1}$ relations of $Br_n(S^2)$ and $t^n = 1$.

- Braid group as fundamental group of discriminant complement.

$$\Delta = \underset{n\geq 1}{A_n},\ \underset{n\geq 4}{D_n},\ E_6,\ E_7,\ E_8. \quad \text{(simply laced dynkin diagrams)}$$

$$\begin{aligned} V &= \mathrm{Fun}(\Delta, \mathbb{C}) \simeq \mathbb{C}^n \\ &\cup\!\mid \\ L &= \mathrm{Fun}(\Delta, \mathbb{Z}), \qquad \langle e_x, e_y\rangle = \begin{cases} 2 & \text{if } x=y \\ -1 & x \text{ --- } y \\ 0 & \text{o.w.} \end{cases} \quad \text{(root lattice)} \\ &\cup\!\mid \\ \Phi &= \{ r \in \mathbb{Z}\Delta : \langle r, r\rangle = 2\}, \quad \text{(root system)} \end{aligned}$$

(reflection group) $R = \langle R_v : v \in \Phi_\Delta \rangle \subseteq \mathrm{Aut}(\mathbb{C}\Delta)$, $\quad R_v(x) = x - (1-(-1))\dfrac{\langle x, v\rangle}{v^2}$

(mirrors) $\mathcal{H} = \bigcup_{v \in \Phi_\Delta} v^\perp$, $\quad v^\perp = \{ x \in \mathbb{C}\Delta : \langle v, x\rangle = 0\}$

$$\mathbb{A}^n \underset{\text{(Chevalley theorem)}}{\simeq} V/R \supseteq \overset{\text{(discriminant)}}{\mathcal{D}} = \mathcal{H}/R$$

(discriminant complement) $= (V - \mathcal{H})/R = (V/R) - \mathcal{D}$.

Theorem
$R_\Delta = \mathrm{Cox}(\Delta, 2)$.
$:= \langle s_i : i \in \Delta \mid s_i^2 = 1\ \forall i$
$s_i s_j s_i = s_j s_i s_j$ if i — j
$s_i s_j = s_j s_i$ o.w.

Example. $\Delta = A_n$.

$$L \cong \{ x \in \mathbb{Z}^{n+1} : \textstyle\sum x_i = 0\} \supseteq \Phi = \{\pm(e_i - e_j)\}.$$

$$\begin{aligned} &\cap\!\mid \\ V &= \{ x \in \mathbb{C}^{n+1} : \textstyle\sum x_i = 0\} \\ &\cup\!\mid \\ \mathcal{H} &= \bigcup_{i\neq j} \{x_i = x_j\}. \end{aligned}$$

$R_{e_i - e_j}$ = reflection in $x_i = x_j$

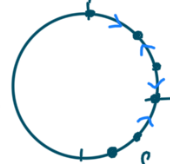
$R \cong S_{n+1}$

$$(V - \mathcal{H})/S_{n+1} \overset{\text{deformation retraction}}{\simeq} \Big(\mathbb{C}^{n+1} - \bigcup_{i\neq j}\{x_i = x_j\}\Big)/S_{n+1}$$

So
$$\pi_1\big((V - \mathcal{H})/R\big) = Br_{n+1} = \mathrm{Cox}(A_n, \infty)$$

<u>Theorem.</u> $\pi_1\big((V_\Delta - \mathcal{H}_\Delta)/R_\Delta\big) = \mathrm{cox}(\Delta, \infty) \leftarrow$ (Artin group of type Δ)

discriminant complement for a complex reflection groups

$\Delta = A_4$ or $\tilde{A}_{11}$ (12 gon) . = 

$\mathcal{E} = \mathbb{Z}[\omega]$, $\omega = e^{2\pi i/3}$

$$\langle e_x, e_y \rangle = \begin{cases} 3 & \text{if } x = y \\ \sqrt{-3} & \text{if } o \longleftarrow o \\ 0 & \text{ow} \end{cases}$$

$$\mathbb{C}^{9,1} \simeq V = \mathbb{C}L \supseteq L = \mathrm{Fun}(\Delta, \mathcal{E}) / \mathrm{radical}\,(\mathrm{Fun}(\Delta, \mathcal{E}))$$

$$\cup\!\!\!\mid$$

$\circlearrowright \mathbb{P}_-(V) = \mathbb{B}^9$ $\quad \Phi = \{ r \in L : \langle r, r \rangle = 3 \}$. $\supseteq s_1, \ldots, s_{12}$ corresponding to vertices of $\tilde{A}_{11}$.

$R = \langle R_v^\omega : v \in \Phi \rangle \subsetneq \mathrm{Aut}(\mathbb{C}^{9,1}) .= U(9,1)$ $\quad$ [$R_v^\omega(x) = x - (1-\omega) \dfrac{\langle x, v \rangle}{v^2} \cdot v$

Thm $= \langle R_1, \ldots, R_2 \mid$ coxeter-rels + ? $\rangle$ $\quad$ [write $R_{s_i} = R_i$

$$\mathbb{B}^9 \supseteq \mathcal{H} = \bigcup_{v \in \Phi} v^\perp, \quad v^\perp = \mathbb{P}_-\{ x \in V : \langle v, x \rangle = 0 \}.$$

Consider : $\quad \mathbb{B}^9 / R \supseteq \mathcal{H}/R = \mathcal{D}$

$$\cup\!\!\!\mid$$

$$(\mathbb{B}^9/R - \mathcal{D}) = (\mathbb{B}^9 - \mathcal{H})/R$$

[$\mathbb{P}_-(A)$ means complex 1 dim subspaces of negative norm

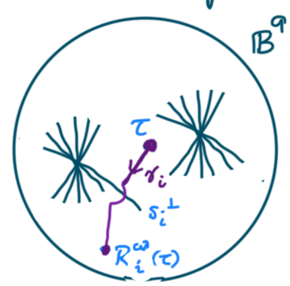
$\mathbb{P}_-(V) = \mathbb{B}^9_{\mathbb{C}} = \mathbb{C}H^9$

Dihedral group $D_{12} \circlearrowright \mathbb{B}^9$. fixing a unique point τ.

The 12 mirrors corresponding to the $\tilde{A}_{11}$ diagram are exactly the mirrors closest to it.

By construction $\pi_1^{orb}(\mathbb{P}_-(V - \mathcal{H})/R, \tau)$ contains generators of monodromy $g_1, \ldots, g_{12}$

$$g_i = (\gamma_i, R_i)$$



Theorem [D-M, T]. Let $\mathcal{M}_{12}^{st} = \dfrac{\{ (x_1 \ldots x_{12}) \in (\mathbb{P}^1)^{12} : \text{each } x_j \text{ has multiplicity} \leq 5 \}}{S_n \times PGl_2(\mathbb{C})}$.

One has $\quad \mathcal{M}_{12}^{st} \simeq \mathbb{B}^9/R$, and $\quad \mathcal{M}_{12} \simeq (\mathbb{B}^9 - \mathcal{H})/R$

In particular $\quad \pi_1^{orb}((\mathbb{B}^9 - \mathcal{H})/R, \tau) \simeq \pi_1^{orb}(\mathcal{M}_{12}, a)$

$$g_i \longmapsto \rho_i$$