Lattice points in slices of rectangular prisms

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Algebra and Geometry Seminar

 

This talk is based on joint work with Luis Ferroni
"Lattice points in slices of prisms" (arXiv:2202.11808)

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- The theory of alcoved polytopes $\to$ triangulations.
- Much more! (tropical geometry, coding theory, statistics of permutations, etc.)

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$$\mathrm{vol}(\Delta_{k,n}) = \frac{1}{(n-1)!} A(n-1, k-1),$$

where $A(n-1, k-1) = \{\sigma \in \mathfrak{S}_{n-1}$ having $k-1$ descents$\}$.

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It follows from his proof that the hypersimplex admits a certain unimodular triangulation.

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- $a_0 = 1$.
- $a_1, \ldots, a_{d-2}$ can be negative in general. ☹

h^*-polynomials

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Remark (Major problems)

- Find conditions that h^*-polynomials of lattice polytopes must satisfy (inequalities, for example).
- Find combinatorial interpretations of the coefficients of the h^*-polynomial, at least for particular families of polytopes.

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$$[t^m]\operatorname{ehr}(\Delta_{k,n}, t) = \frac{1}{(n-1)!}\sum_{\ell=0}^{k-1} W(\ell, n, m+1) A(n-1, k-\ell-1),$$

which in particular is positive.

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Regarding the h^*-polynomial we have the following combinatorial interpretation.

Theorem (Early '17 - Kim '20)

Consider the hypersimplex $\Delta_{k,n}$. The coefficient of degree m of its h^*-polynomial is given by

$$[x^m]h^*(\Delta_{k,n}, x) = \# \left\{ \begin{array}{c} \textit{decorated ordered set partitions} \\ \textit{of type } (k,n) \textit{ and winding number } m \end{array} \right\},$$

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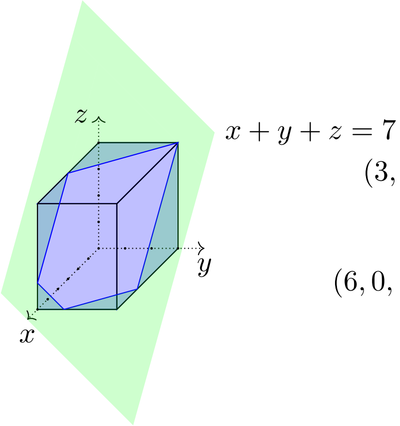
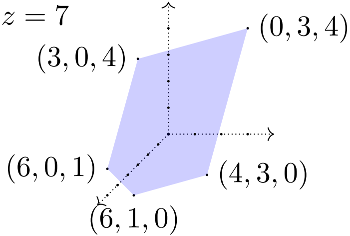
$$\mathscr{R}_{k,\mathbf{c}} = \left\{ x \in \mathscr{R}_{\mathbf{c}} : \sum_{i=1}^{n} x_i = k \right\}.$$

Example (The basic example)

Consider $\mathbf{c} = (1, \ldots, 1) \in \mathbb{Z}^n_{>0}$. The k-th slice of $\mathscr{R}_{\mathbf{c}}$ is precisely the hypersimplex $\Delta_{k,n}$.

Example





If you consider the 3-dimensional rectangular prism of sides 6, 3 and 4 and you intersect it with the hyperplane $x+y+z=7$ you get the polytope on the right.

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$$\mathscr{R}'_{a,b,\mathbf{c}} := \left\{ x \in \mathscr{R}_{\mathbf{c}} : a \leq \sum_{i=1}^{n} x_i \leq b \right\}.$$

We say that this is a "fat slice" of the prism $\mathscr{R}_{\mathbf{c}}$.

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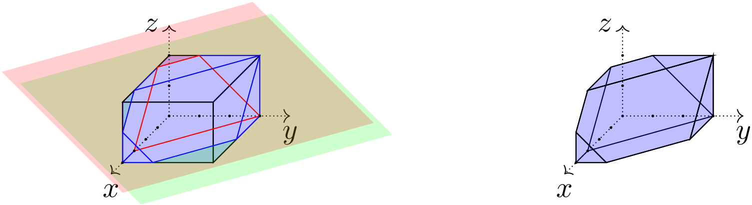

Figure: $\mathscr{R}'_{3,5,(4,3,2)}$

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Proposition

Let $\mathbf{c} = (c_1, \ldots, c_n) \in \mathbb{Z}^n_{>0}$ *and* $0 \leq a < b$. *Then, the fat slice* $\mathscr{R}'_{a,b,\mathbf{c}}$ *has the same Ehrhart polynomial as the thin slice* $\mathscr{R}_{k,\mathbf{c}'}$ *where* $k = b$ *and* $\mathbf{c}' = (\mathbf{c}, b - a) \in \mathbb{Z}^{n+1}_{>0}$.

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Conjecture (F., Jochemko, Schröter '21)

All positroids are Ehrhart positive.

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Definition

Let $\mathbf{c} = (c_1, \ldots, c_n) \in \mathbb{Z}^n_{>0}$ and $k > 0$. The **algebra of Veronese type** $\mathscr{V}(\mathbf{c}, k)$ is defined as the the graded algebra over a field $\mathbb{F}$ generated by all the monomials $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ such that $\alpha_1 + \cdots + \alpha_n = k$ and $\alpha_i \leq c_i$ for all i.

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Theorem (Hibi and De Negri '97)

There is an isomorphism between $\mathscr{V}(\mathbf{c}, k)$ *and the Ehrhart ring of* $\mathscr{R}_{k,\mathbf{c}}$.

A consequence of the above result is that the Hilbert function of $\mathscr{V}(\mathbf{c}, k)$ coincides with $\mathrm{ehr}(\mathscr{R}_{k,\mathbf{c}}, t)$ and moreover, the numerator of the Hilbert series is $h^*(\mathscr{R}_{k,\mathbf{c}}, x)$.

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A **weighted permutation** is a pair (σ, w) where $\sigma \in \mathfrak{S}_n$ and w assigns weight to the cycles of σ. The **total weight** of (σ, w) is the sum of the weights $w(\mathfrak{c})$ of all cycles $\mathfrak{c}$ of σ.

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Then (σ, w) is $\mathbf{c}$-compatible because

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and

$$w((2\ 4)) = 11 < \sum_{i \in (2\ 4)} c_i = 4 + 8 = 12.$$

Also, the total weight is $w(\sigma) = 6 + 11 = 17$

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$$\mathrm{Des}(\sigma, \mathbf{s}) := \{ i \in [n-1] : s_i > s_{i+1} \text{ or } s_i = s_{i+1} \text{ and } \sigma_i > \sigma_{i+1} \}.$$

The **flag descent number** of a $\mathbf{c}$-colored permutation $(\sigma, \mathbf{s}) \in \mathfrak{S}_n^{(\mathbf{c})}$ is defined by

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Let $\mathbf{c} = (c_1, \ldots, c_n) \in \mathbb{Z}^n_{>0}$. A **$\mathbf{c}$-colored permutation on** $[n]$ is a pair $(\sigma, \mathbf{s})$ where $\sigma \in \mathfrak{S}_n$ and $\mathbf{s}$ is a function $\mathbf{s} : [n] \to \mathbb{Z}_{\geq 0}$ such that $s_i := \mathbf{s}(i) \leq c_i - 1$ for each i. The set of all such $\mathbf{c}$-colored permutations is denoted by $\mathfrak{S}_n^{(\mathbf{c})}$

Definition

The set of **descents** of a $\mathbf{c}$-colored permutation $(\sigma, \mathbf{c})$ is given by

$$\mathrm{Des}(\sigma, \mathbf{s}) := \{i \in [n-1] : s_i > s_{i+1} \text{ or } s_i = s_{i+1} \text{ and } \sigma_i > \sigma_{i+1}\}.$$

The **flag descent number** of a $\mathbf{c}$-colored permutation $(\sigma, \mathbf{s}) \in \mathfrak{S}_n^{(\mathbf{c})}$ is defined by

$$\mathrm{fdes}(\sigma, \mathbf{s}) := s_n + \sum_{i \in \mathrm{Des}(\sigma, \mathbf{s})} c_{i+1}.$$

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Remark

The case that $\mathbf{c} = (r, \ldots, r)$, reduces to a result by Han and Josuat-Vergès (2016), and when $r = 1$ we recover Laplace's result on hypersimplices.

Ordered Set Partitions

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$$w(\mathfrak{p}) < \sum_{i \in \mathfrak{p}} c_i$$

for all $\mathfrak{p} \in P(\xi)$.

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$$w(\{4,7,8\}) = 4 < 5+1+1 = 7.$$

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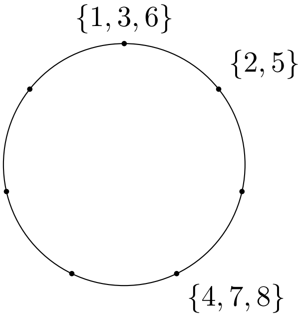
This decorated ordered set partition can be visualized as follows.



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$$[x^m]h^*(\mathscr{R}_{k,\mathbf{c}}, x) = \# \left\{ \begin{matrix} \mathbf{c}\text{-compatible decorated ordered set partitions} \\ \text{of type } (k, n) \text{ and winding number } m \end{matrix} \right\},$$

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$$p_{n,m,\mathbf{c}}(x) = \sum_{\ell=0}^{\infty} W(\ell, n, m+1, \mathbf{c}) x^{\ell}.$$

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The h^**-polynomial of a slice of a prism is always real-rooted. Moreover, if* $\mathbf{c} = (c_1, \dots, c_n)$ *and* $\mathbf{c}' = (c_1, \dots, c_{n-1}, c_n - 1, 1)$*, then*

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$$h^*(\mathscr{R}_{k,\mathbf{c}}, x) \preceq h^*(\mathscr{R}_{k,\mathbf{c}'}, x)$$

namely, these two polynomials interlace.

THANK YOU!