Some Quantum Symmetries of Path Algebras

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Outline

- Background
  - Quivers and path algebras
  - Hopf algebras and their actions
- Parametrization of $U_q(\mathfrak{b})$ actions on $\Bbbk Q$
- Bimodules in $\mathrm{rep}(U_q(\mathfrak{b}))$
  - Equivalence w/ category of representations of certain quivers
- Taft algebras
- $U_q(\mathfrak{sl}_2)$

Background

Fix a fix a field $\Bbbk$. We assume $\Bbbk$ contains any necessary roots of unity.

Def: A <u>quiver</u> $Q=(Q_0,Q_1,s,t)$ consists of a finite set of vertices Q_0, a finite set of arrows Q_1, and maps $s,t: Q_1 \longrightarrow Q_0$ giving the <u>source</u> and <u>target</u> of each arrow resp $\underset{s(a)}{\bullet} \xrightarrow{\;a\;} \underset{t(a)}{\bullet}$

The <u>path algebra</u> $\Bbbk Q$ is the associative $\Bbbk$-algebra w/ basis consisting of all paths and multiplication of paths given by concatenation when possible (left to right). $\Bbbk Q$ contains paths of length zero at each vertex i, denoted e_i (these are idempotents)

Note, $\Bbbk Q_0$ is a semisimple $\Bbbk$-algebra and $\Bbbk Q_1$ is a $\Bbbk Q_0$-bimodule $\Rightarrow \Bbbk Q \cong T_{\Bbbk Q_0}(\Bbbk Q_1)$

$$a_1 e_i \otimes a_2 = a_1 \otimes e_i a_2 = 0 \text{ unless } t(a_1)=i=s(a_2) \quad \text{for } a_1, a_2 \in Q_1,\ i \in Q_0$$

A <u>representation</u> V of a quiver Q is an assignment of a fin dim vector space V_i to each vertex $i \in Q_0$ along with a linear map $V_a: V_{ta} \longrightarrow V_{sa}$ to each arrow $a \in Q_1$. (The linear maps go in the opposite dir of the arrows so maps compose right to left and arrows multiply left to right)

Background

Def: A Hopf algebra H is a $\Bbbk$-vector space with

- a multiplication map $m: H\otimes H \longrightarrow H$
- a unit map $u: \Bbbk \longrightarrow H$

algebra structure

- a comultiplication map $\Delta: H \longrightarrow H\otimes H$
- a counit map $\varepsilon: H \longrightarrow \Bbbk$

coalgebra structure

- an antipode $S: H \longrightarrow H$

where ε and Δ are algebra homomorphisms (ie H is a bialgebra) and the following diagrams commute

$$\begin{tikzcd}
 & H\otimes H\otimes H \arrow[dl, "m\otimes id"'] \arrow[dr, "id\otimes m"] & \\
H\otimes H \arrow[dr, "m"'] & & H\otimes H \arrow[dl, "m"] \\
 & H &
\end{tikzcd}$$

associativity

$$\begin{tikzcd}
 & H\otimes H\otimes H & \\
H\otimes H \arrow[ur, "\Delta\otimes id"] & & H\otimes H \arrow[ul, "id\otimes\Delta"'] \\
 & H \arrow[ul, "\Delta"] \arrow[ur, "\Delta"'] &
\end{tikzcd}$$

coassociativity

etc

Background

Def: Given a Hopf algebra H and an algebra A, a (left Hopf) action of H on A consists of

a left H-module structure on A so that

(a) $h\cdot(pq)=\sum_i (h_{1,i}\cdot p)(h_{2,i}\cdot q) \quad \forall\ p,q\in A,\ h\in H$ where $\Delta(h)=\sum_i h_{1,i}\otimes h_{2,i}$

(b) $h\cdot 1_A=\varepsilon(h)1_A \quad \forall\ h\in H$

We say A is a left H-module algebra ie. A is an algebra in the tensor category rep(H)

The category rep(H) is an example of a tensor category (abelian category w/ a tensor product, etc)

We can define the notion of an algebra and a bimodule in a tensor category $\mathcal{C}$

eg. an algebra is an object $A\in\mathcal{C}$ along w/ a multiplication map $m\in \mathrm{Hom}_{\mathcal{C}}(A\otimes A, A)$ st certain diagrams commute

In a recent paper by Etingof, Kinser, and Walton (EKW) they develop the notion of tensor algebras $T_S(E)$

where S is an algebra in $\mathcal{C}$ and E is a bimodule in $\mathcal{C}$

Background

Examples: (We omit the counit b/c checking the relevant conditions is straightforward.)

Fix $q \in \Bbbk^{\times} \setminus \{\pm 1\}$. We will use the following Hopf algebras.

1) The quantized enveloping algebra of the Lie algebra $\mathfrak{b}$, $U_q(\mathfrak{b})$ is given by

generators	$x, \; g, \; g^{-1}$
with relations	$gg^{-1} = 1 = g^{-1}g, \qquad gx = qxg$
and comultication	$\Delta(g) = g \otimes g, \qquad \Delta(x) = 1 \otimes x + x \otimes g$

If q is a primitive r^{th} root of unity and $n \in \mathbb{Z}^+$ st $r \mid n$,

then <u>the generalized Taft algebra</u> is the Hopf quotient $T(r,n) = U_q(\mathfrak{b}) / \langle g^n - 1, x^r \rangle$

If $r = n$, we get the classical Taft algebra $T(n)$

Background

Examples: 2) The quantized enveloping algebra of the Lie algebra $\mathfrak{sl}_2$ is given by

generators $E, \; F, \; K, \; K^{-1}$

with relations $KK^{-1} = 1 = K^{-1}K \qquad KE = q^2 EK$

$$[E,F] = \frac{K-K^{-1}}{q-q^{-1}} \qquad KF = q^{-2}FK$$

and comultiplication $\Delta(E) = 1\otimes E + E\otimes K \qquad \Delta(K) = K\otimes K$

$$\Delta(F) = K^{-1}\otimes F + F\otimes 1$$

If q is a primitive n^{th} root of unity with $n>2$ and n odd,

we have the small quantum group or Frobenius–Lusztig kernel

$$u_q(\mathfrak{sl}_2) = U_q(\mathfrak{sl}_2) \big/ \langle K^n - 1, E^n, F^n \rangle$$

Note: We have isomorphisms of Hopf algebras

$$\left.\begin{array}{l} \langle E,K\rangle \cong U_{q^2}(\mathfrak{b}) \quad \text{where} \quad K \leftrightarrow g, \; E \leftrightarrow x \\ \langle F,K\rangle \cong U_{q^{-2}}(\mathfrak{b}) \quad \text{where} \quad K \leftrightarrow g, \; F \leftrightarrow g^{-1}x \end{array}\right\} \text{analogues of Borel subalgebras}$$

$u_q(\mathfrak{sl}_2)$ has two subalgebras isomorphic to Taft algebras

Parametrization of Actions of $U_q(\mathfrak{b})$ and $U_q(\mathfrak{sl}_2)$ on $\Bbbk Q$

Note, we assume all actions of Hopf algebras on path algebras preserve the filtration by path length

$G = \langle g \rangle$, fix $q \in \Bbbk^\times \setminus \{\pm 1\}$

Theorem: The following data determines a Hopf action of $U_q(\mathfrak{b})$ on $\Bbbk Q$ and all such actions have this form.

(i) A Hopf of $U_q(\mathfrak{b})$ on $\Bbbk Q_0$ determined by

- A permutation action of G on Q_0
- A collection of scalars $(\gamma_i)_{i \in Q_0}$ so that $\gamma_{g \cdot i} = q^{-1} \gamma_i$

  where $x \cdot e_i = \gamma_i e_i - \gamma_{g \cdot i} e_{g \cdot i}$ $\forall$ $i \in Q_0$

(ii) A representation of G on $\Bbbk Q_1$ st $s(g \cdot a) = g \cdot sa$ and $t(g \cdot a) = ta$ $\forall$ $a \in Q_1$

(iii) A $\Bbbk$-linear endomorphism $\sigma : \Bbbk Q_0 \oplus \Bbbk Q_1 \longrightarrow \Bbbk Q_0 \oplus \Bbbk Q_1$ so that

($\sigma 1$) $\sigma(\Bbbk Q_0) = 0$

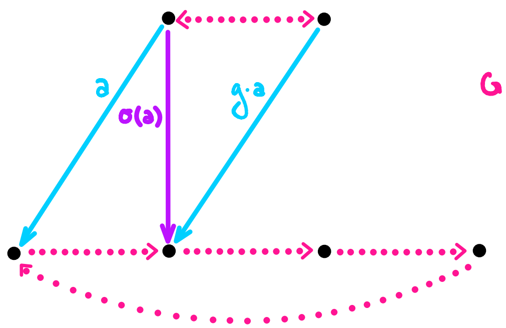
($\sigma 2$) $\sigma(a) = e_{sa} \sigma(a) e_{g \cdot ta}$ $\forall$ $a \in Q_1$

($\sigma 3$) $\sigma(g \cdot a) = q^{-1} g \cdot \sigma(a)$ $\forall$ $a \in Q_1$

where $x \cdot a = \gamma_{ta} a - \gamma_{g \cdot sa}(g \cdot a) + \sigma(a)$

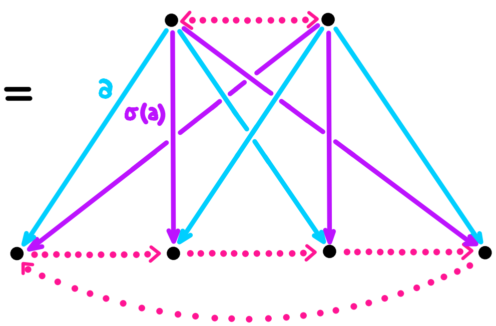
Parametrization of Actions of $U_q(\mathfrak{b})$ and $U_q(\mathfrak{sl}_2)$ on $\Bbbk 2$

Example:



G-action on vertices

$x \cdot a$ is a linear combination of the above

$2 =$ 

G-action on vertices

G-orbit of a

G-orbit of $\sigma(a)$

Let $V = \Bbbk 2$. Then the <u>arrow space</u> $V^i_j := e_i V e_j$ is the subspace of V spanned by arrows w/ source i and target j.

We have the decomposition $V = \bigoplus_{i,j} V^i_j$ In the diagram above, the arrows represent arrow spaces.

Note, we have $\mathrm{lcm}(2,4) = 4$ copies of each arrow space

and $\gcd(2,4) = 2$ isomorphism classes of arrow spaces

Parametrization of Actions of $U_q(\mathfrak{b})$ and $U_q(\mathfrak{sl}_2)$ on $\Bbbk Q$

Corollary: If q is a primitive r^{th} root of unity, then an action of $U_q(\mathfrak{b})$ on $\Bbbk Q$ factors through $T(r,n)$ $\Leftrightarrow$

(1) The action on $\Bbbk Q_0$ factors through $T(r,n)$

(2) $\forall\ a \in Q_1$, $\quad \gamma_{sa}\, g^r \cdot a - \gamma_{ta}\, a = \sigma^r(a)$

(3) g^n acts as identity on all of $\Bbbk Q$

Parametrization of Actions of $U_q(\mathfrak{b})$ and $U_q(\mathfrak{sl}_2)$ on $\Bbbk\mathcal{Q}$

$G = \langle K \rangle$

Theorem: Assuming $\#(G\cdot i) > 2 \ \forall \ i \in \mathcal{Q}_0$, the following data determines a filtered Hopf action of $U_q(\mathfrak{sl}_2)$ on $\Bbbk\mathcal{Q}$ and all such actions have this form.

(i) A Hopf action of $U_q(\mathfrak{sl}_2)$ on $\Bbbk\mathcal{Q}_0$

- A permutation action of G on $\mathcal{Q}_0$.
- 2 collections of scalars $(\gamma_i^E)_{i\in\mathcal{Q}_0}$ and $(\gamma_i^F)_{i\in\mathcal{Q}_0}$

  as before w/ additional relation $\gamma_i^E\gamma_i^F = \dfrac{-q}{(1-q^2)^2} \ \forall \, i \in \mathcal{Q}_0$

(ii) A representation of G on $\Bbbk\mathcal{Q}_1$ st $s(K\cdot a) = K\cdot sa$ and $t(K\cdot a) = K\cdot ta$

(iii) A pair of linear endomorphisms σ^E and σ^F as before so that $\sigma^F\sigma^E = q^2\sigma^E\sigma^F$

Bimodules in $\mathrm{rep}(U_q(\mathfrak{b}))$

Let $S = \Bbbk 2_0$ and $V = \Bbbk 2_1$ which is an S-bimodule st $\Bbbk 2 \cong T_S(V)$

With a graded $U_q(\mathfrak{b})$-action, S is an algebra in $\mathcal{C} := \mathrm{rep}(U_q(\mathfrak{b}))$ and V is an S-bimodule in $\mathcal{C}$.

Then, as in EKW, we call $T_S(V)$ a $\mathcal{C}$-tensor algebra

Def: $T_S(V)$ is a <u>minimal</u>, <u>faithful</u> $\mathcal{C}$-tensor algebra if

V is an indecomposable S-bimodule in $\mathcal{C}$ and

no two-sided ideal of S in $\mathcal{C}$ acts by 0 on V.

These are the "building blocks" of $\mathcal{C}$-tensor algebras.

Bimodules in $\mathrm{rep}(U_q(\mathfrak{b}))$

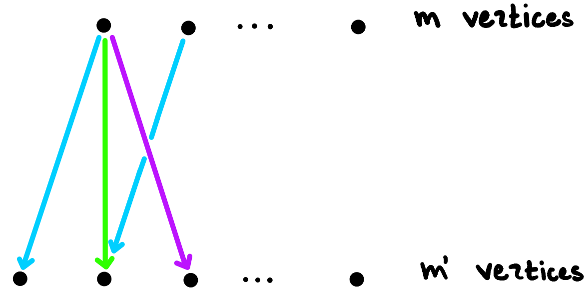
To study minimal, faithful $\mathcal{C}$-tensor algebras, it is sufficient to consider the case where S has two indecomposable summands S_1 and S_2.
The S indecomposable case is addressed by taking $S_1 = S_2$.
We examine S_1-S_2-bimodules in $\mathcal{C}$.

For a positive integer m, let $S_m = \mathbb{k}^m$ be the vector space with coordinate-wise multiplication.

The set of standard basis vectors $\{e_0, e_1, \ldots, e_{m-1}\}$ gives a system of primitive orthogonal idempotents
Define a G-action on S_m by $g \cdot e_i = e_{i+1 \pmod{m}}$
Identify S_m with the path algebra of a quiver w/ m vertices and no arrows.

Bimodules in rep($U_q(\mathfrak{b})$)

Fix positive integers m and m'. An S_m–$S_{m'}$–bimodule V in rep($U_q(\mathfrak{b})$) is the path algebra of a quiver of the following form with a $U_q(\mathfrak{b})$ action.



The action of g gives $\ell = \mathrm{lcm}(m,m')$ isomorphic copies of each arrow space, and there are $d = \gcd(m,m')$ isomorphism types of arrow spaces.

Each arrow space V_j^i is a representation of $\langle g^\ell \rangle \leq G$

$$\Rightarrow V_j^i = \bigoplus_{\lambda \in \Bbbk^\times} V_j^i(\lambda) \quad \text{where} \quad V_j^i(\lambda) = \{ v \in V_j^i \mid (\lambda 1 - g^\ell)^M \cdot v = 0 \text{ for } M \gg 0 \}$$

The action of $U_q(\mathfrak{b})$ gives a map σ w/ property (σ3) $\sigma g = q^{-1} g \sigma$

$$\Rightarrow \quad \sigma(V_j^i(\lambda)) \subseteq V_{j+1}^i(q^\ell \lambda)$$

Let $\mathcal{B} :=$ category of S_m–$S_{m'}$–bimodules in rep($U_q(\mathfrak{b})$)

Bimodules in $\mathrm{rep}(U_q(\mathfrak{b}))$

Define a quiver $\mathcal{Q}(q^\ell,d)$ w/ vertex set $\Bbbk^\times \times \mathbb{Z}/d\mathbb{Z}$ and arrows $(q^\ell\lambda, j+1) \longrightarrow (\lambda, j)$ (a-type)

and a loop at each vertex (λ, j) (b-type)

The connected components of $\mathcal{Q}(q^\ell,d)$ have the form

$q \neq$ root of unity:

$$S_\infty = \begin{tikzcd} \cdots \arrow[r, "a"] & (q^\ell\lambda, k+1) \arrow[loop above, "b"] \arrow[r, "a"] & (\lambda, k) \arrow[loop above, "b"] \arrow[r, "a"] & (q^{-\ell}\lambda, k-1) \arrow[loop above, "b"] \arrow[r, "a"] & \cdots \end{tikzcd}$$

let $p := \mathrm{lcm}(|q^\ell|, d)$

$q =$ root of unity:

$$S_p = \begin{tikzcd} (q^{\ell(p-1)}\lambda, p-1) \arrow[loop above, "b"] \arrow[r, "a"] & (q^{\ell(p-2)}\lambda, p-2) \arrow[loop above, "b"] \arrow[r, "a"] & \cdots \arrow[r, "a"] & (\lambda, 0) \arrow[loop above, "b"] \arrow[lll, bend left, "a"] \end{tikzcd}$$

$\Gamma(q^\ell,d) =$ the quotient of $\Bbbk\mathcal{Q}(q^\ell,d)$ by all relations of the form $ba = q^\ell ab$

$\mathcal{N} =$ category of fin dim reps of $\Gamma(q^\ell,d)$ st the maps associated to b-type loops are nilpotent

Denote $W \in \mathcal{N}$ by $\Big(\underset{\uparrow\ \text{vs assigned to }(\lambda,j)}{W_{\lambda,j}},\ \overset{\text{map assigned to }a_{\lambda,j}\ \downarrow}{A_{\lambda,j}},\ \underset{\uparrow\ \text{map assigned to }b_{\lambda,j}}{B_{\lambda,j}}\Big)$

Bimodules in $\operatorname{rep}(U_q(\mathfrak{b}))$

Theorem: The categories $\mathcal{B}$ and $\mathcal{N}$ are equivalent.

~proof: We construct mutually quasi-inverse functors.

$\mathcal{B} \longrightarrow \mathcal{N}$: Given $V \in \mathcal{B}$, let $W_{\lambda,j} = V^0_{\tau(\lambda,j)}(\lambda)$

$$A_{\lambda,j} = \sigma\big|_{W_{\lambda,j}} : W_{\lambda,j} \longrightarrow W_{q^\ell\lambda, j+1}$$

$$B_{\lambda,j} = (g^\ell - \lambda 1)\big|_{W_{\lambda,j}} : W_{\lambda,j} \longrightarrow W_{\lambda,j} \text{ is nilpotent}$$

$\mathcal{N} \longrightarrow \mathcal{B}$: Given $(W_{\lambda,j}, A_{\lambda,j}, B_{\lambda,j}) \in \mathcal{N}$

Let $\widetilde{W}^0_j = \bigoplus_{\tau(\lambda,k)=j} W_{\lambda,k}$ and $\widetilde{W} = \bigoplus_{j=0}^{m'-1} \widetilde{W}^0_j$

Let $H \leq \Bbbk G$ be the subalgebra generated by g^ℓ

Then $\Bbbk G \otimes_H \widetilde{W} \in \mathcal{B}$

given a bimodule structure $e_0 \widetilde{W} e_j = \widetilde{W}^0_j$

and actions $g^\ell \cdot w = \lambda w + B_{\lambda,j}(w)$ for $w \in W_{\lambda,j}$

$$x \cdot (g^t \otimes w) = q^{-t}\left(\gamma'_j g^t \otimes w - \gamma_0 q^{-1} g^{t+1} \otimes w + g^{t+\varepsilon} \otimes A_{\lambda,j}(w)\right) \text{ for } w \in W_{\lambda,j}$$

$\blacksquare$

Taft Algebras

$q =$ primitive r^{th} root of unity

$T =$ quiver w/ vertices (ζ, i)

where ζ is an $(n/\ell)^{th}$ root of unity and $i \in \mathbb{Z}/d\mathbb{Z}$

w/ a-type arrows and connected components of the form

$$\begin{tikzcd}
\underset{(\zeta,d-1)}{\bullet} \arrow[r, "a"] & \underset{(\zeta,d-2)}{\bullet} \arrow[r, "a"] & \cdots \arrow[r, "a"] & \underset{(\zeta,0)}{\bullet} \arrow[lll, bend left, "a"]
\end{tikzcd}$$

as long as $m = r$ or $m' = r$

relations $\begin{cases} a^r = 0 & m \neq r, m' \neq r \\ a^k = \text{constant} & m = r \text{ or } m' = r \end{cases}$

$U_q(\mathfrak{sl}_2)$

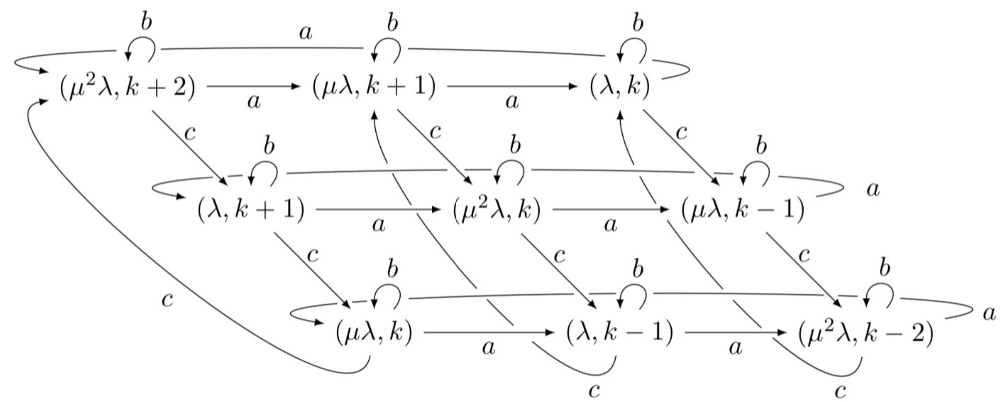
assume $m, m' > 2$ and q = primitive n^{th} root of unity w/ $n>2$ and n odd

Actions parametrized by data for the Borel subalgebras $U_{q^2}(\mathfrak{b})$, $U_{q^{-2}}(\mathfrak{b})$

$\Rightarrow$ we have two collections of scalars $\{\gamma_i^E\}$ and $\{\gamma_i^F\}$ and maps σ^E and σ^F so that $q^2\sigma^E\sigma^F = \sigma^F\sigma^E$

connected components of the quiver have the form:

with relations $ab = q^{-2\ell}ba$ $\quad cb = q^{2\ell}bc$ $\quad q^2ac = ca$

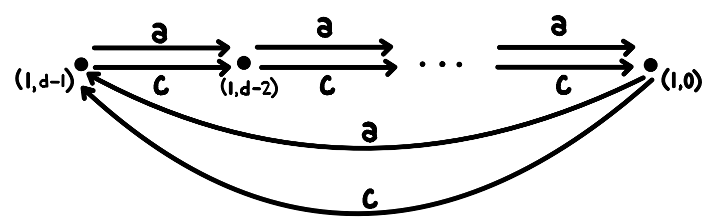


example with $\mathrm{lcm}(|q^{2\ell}|, d) = 3$

actions that factor through $u_q(\mathfrak{sl}_2)$:

with relations $q^2ac = ca$, a^d = constant,

c^d = constant



Thank you!

Questions?