Exact Factorizations of Finite Tensor Categories

Issac Odegard

Iowa State University

iodegard@iastate.edu

September 29, 2020

Overview

Introduction

Exact factorizations of fusion categories were studied by Shlomo Gelaki in [3], in which exact factorizations were related to exact sequences of fusion categories defined in [1]. We explore the generalization of exact factorizations to finite tensor categories.

$\mathbb{k}$-linear Abelian Categories

By a $\mathbb{k}$-**linear abelian category**, we mean a category $\mathcal{C}$ such that

- $\mathcal{C}$ is additive (Hom sets are $\mathbb{k}$-vector spaces), and
- every morphism $f : X \to Y$ has a canonical decomposition

$$K \to X \to I \to Y \to C$$

We recall the usual definitions of subobject, quotient object, subquotient object, simple object, semisimple object, indecomposable object, exact sequence.

Jordan-Hölder Series

An object X has **finite length** if there is a filtration

$$0 = X_0 \subset X_1 \subset \cdots \subset X_{n-1} \subset X_n = X$$

where X_i/X_{i-1} is simple for all i.

- If Y is simple, then the **multiplicity** of Y in X, denoted $[X : Y]$, is the number of such subquotients isomorphic to Y.

Projective Objects

Definition

Let $\mathcal{C}$ be an abelian category. An object $P \in \mathcal{C}$ is **projective** if $\mathrm{Hom}_{\mathcal{C}}(P, -)$ is exact. We have

$$
\begin{tikzcd}
 & X \arrow[d, two heads, "f"] \\
P \arrow[ur, dotted, "h"] \arrow[r, "g"'] & Y
\end{tikzcd}
$$

Projective Covers

Definition

Let $\mathcal{C}$ be an abelian category. Let $X \in \mathcal{C}$. Then a **projective cover** of X is a pair $(P_{\mathcal{C}}(X), \phi_{\mathcal{C},X})$ where $P_{\mathcal{C}}(X)$ is projective in $\mathcal{C}$ and $\phi_{\mathcal{C},X} \in \mathrm{Hom}_{\mathcal{C}}(P_{\mathcal{C}}(X), X)$ is an epi, and we have

$$\begin{tikzcd} P_{\mathcal{C}}(X) \arrow[r, two heads, "\phi_{\mathcal{C},X}"] & X \\ P \arrow[u, dotted, two heads, "h"] \arrow[ur, two heads, "g"'] & \end{tikzcd}$$

Finite Abelian Categories

A $\Bbbk$-linear abelian category $\mathcal{C}$ is **finite** if

1. $\mathcal{C}$ has finite dimensional spaces of morphisms,
2. every object of $\mathcal{C}$ has finite length,
3. every object of $\mathcal{C}$ has a projective cover, and
4. there are finitely many isomorphism classes of simple objects.

Example: The category of finite dimensional representations of a finite dimensional algebra A

Finite Abelian Categories

Let $\mathcal{C}$ be a finite $\Bbbk$-linear abelian category. For any Y and simple X, we have

$$\dim \operatorname{Hom}_{\mathcal{C}}(P(X), Y) = [Y : X]$$

Deligne's Tensor product

Let $\mathcal{C}, \mathcal{D}$ be finite abelian categories. Deligne's tensor product $\mathcal{C} \boxtimes \mathcal{D}$ is an abelian $\Bbbk$-linear category which is universal for the functor assigning to every $\Bbbk$-linear abelian category $\mathcal{A}$, the category of right exact in both variables bilinear bifunctors $\mathcal{C} \times \mathcal{D} \to \mathcal{A}$.

Monoidal Categories

A **monoidal category** is a quintuple $(\mathcal{C}, \otimes, a, \mathbb{1}, \iota)$ where

- $\mathcal{C}$ is a category,
- $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a bifunctor,
- $a_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$ is a natural isomorphism (the associativity constraint),
- $\mathbb{1} \in \mathcal{C}$ is the unit object, and
- $\iota : \mathbb{1} \otimes \mathbb{1} \to \mathbb{1}$ is an isomorphism,

such that the functors $L_{\mathbb{1}} : X \mapsto \mathbb{1} \otimes X$ and $R_{\mathbb{1}} : X \mapsto X \otimes \mathbb{1}$ are equivalences, and such that

Monoidal Categories

$$
\begin{tikzcd}
& ((W \otimes X) \otimes Y) \otimes Z \arrow[dl, "a_{W,X,Y} \otimes \mathrm{Id}_Z"'] \arrow[dr, "a_{W\otimes X,Y,Z}"] & \\
(W \otimes (X \otimes Y)) \otimes Z \arrow[d, "a_{W,X\otimes Y,Z}"] & & (W \otimes X) \otimes (Y \otimes Z) \arrow[d, "a_{W,X,Y\otimes Z}"] \\
W \otimes ((X \otimes Y) \otimes Z) \arrow[rr, "\mathrm{Id}_W \otimes a_{X,Y,Z}"] & & W \otimes (X \otimes (Y \otimes Z))
\end{tikzcd}
$$

Dual Objects

An object $X^* \in \mathcal{C}$ is a **left dual** of X if there exist morphisms $\mathrm{ev}_X : X^* \otimes X \to \mathbb{1}$ and $\mathrm{coev}_X : \mathbb{1} \to X \otimes X^*$ such that

$$X = \mathbb{1} \otimes X \xrightarrow{\mathrm{coev}_X \otimes \mathrm{Id}} (X \otimes X^*) \otimes X \xrightarrow{a_{X,X^*,X}} X \otimes (X^* \otimes X) \xrightarrow{\mathrm{Id} \otimes \mathrm{ev}_X} X \otimes \mathbb{1} = X$$

and

$$X^* \to X^* \otimes (X \otimes X^*) \to (X^* \otimes X) \otimes X^* \to X^*$$

are the identity morphisms. One defines right duals similarly.

- We say $\mathcal{C}$ is **rigid** if left and right duals exist.

Tensor Categories

Let $\Bbbk$ be an algebraically closed field. Let $\mathcal{C}$ be a finite $\Bbbk$-linear abelian rigid monoidal category.

- $\mathcal{C}$ is a **tensor category** if $\otimes$ is bilinear on morphisms and $\mathrm{End}_{\mathcal{C}}(\mathbb{1}) \cong \Bbbk$
- A **tensor subcategory** $\mathcal{C} \subset \mathcal{D}$ of a tensor category $\mathcal{D}$ is a full subcategory closed under subquotients, tensor products, and duality
- $\mathcal{C}$ is a **fusion category** if it is a semisimple finite tensor category.

Example: Rep(H) where H is a finite dimensional Hopf algebra e.g. $H = \Bbbk[G]$.

Grothendieck Ring

Let $\mathcal{C}$ be a finite tensor category. The **Grothendieck ring** of $\mathcal{C}$, denoted $\mathrm{Gr}(\mathcal{C})$, is the free abelian group generated by isomorphism classes $[X_i]$ of simple objects in $\mathcal{C}$ where $[X] + [Y] = [X \oplus Y]$ and $[X][Y] = [X \otimes Y]$. For any X, we write

$$[X] = \sum_i [X : X_i] X_i$$

Define $K_0(\mathcal{C})$ to be the free abelian group generated by isomorphism classes of indecomposable projective objects of $\mathcal{C}$.

Frobenius-Perron Dimension

The **Frobenius-Perron dimension** of an object X, denoted $\mathrm{FPdim}(X)$, is the maximal non-negative eigenvalue of the matrix of left multiplication by X in $\mathrm{Gr}(\mathcal{C})$.

- Example: Let $\mathcal{C} = \mathrm{Rep}(H)$ for some finite dimensional Hopf algebra H e.g. $H = \Bbbk[G]$. Then $\mathrm{FPdim}(X) = \dim_{\Bbbk}(X)$ for every $X \in \mathcal{C}$.

Properties of Projective Objects

Let $\mathcal{C}$ be a finite tensor category.

- $\{X_i\}_{i \in I}$ be the simples with $X_0 = \mathbb{1}$
- $X_{i^*} = X_i^*$, $X_{^*i} = {}^*X_i$
- $P_i = P(X_i)$

Properties of Projective Objects

The dual of a projective object is projective, so there is a map $D : I \to I$ such that $P_i^* = P_{D(i)}$. Then

$$\dim \operatorname{Hom}_{\mathcal{C}}(P_i^*, X_j) = \delta_{D(i),j}$$

We will also make use of the object $\mathbb{1}^D := X_{D(0)}$.

Properties of Projective Objects

The object $R_{\mathcal{C}} = \sum_i \mathrm{FPdim}(X_i) P_i \in K_0(\mathcal{C}) \otimes_{\mathbb{Z}} \mathbb{R}$ satisfies

$$ZR_{\mathcal{C}} = R_{\mathcal{C}} Z = \mathrm{FPdim}(Z) R_{\mathcal{C}}$$

for all $Z \in \mathcal{C}$.

- $\mathrm{FPdim}(\mathcal{C}) := \mathrm{FPdim}(R_{\mathcal{C}})$

Module Categories

Let $(\mathcal{C}, \otimes, a, \mathbb{1}, \iota)$ be a tensor category. A **left module category over** $\mathcal{C}$ is a $\Bbbk$-linear abelian category $\mathcal{M}$ equipped with an action bifunctor $\otimes : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$ and a natural isomorphism

$$m_{X,Y,M} : (X \otimes Y) \otimes M \to X \otimes (Y \otimes M)$$

such that $M \mapsto \mathbb{1} \otimes M$ is an equivalence and a diagram is satisfied.

- $\mathcal{M}$ is **exact** if $P \otimes M$ is projective whenever P is projective.

Exact Factorizations of Fusion Categories

Let $\mathcal{B}$ be a fusion category, and let $\mathcal{A}, \mathcal{C} \subseteq \mathcal{B}$ be fusion subcategories of $\mathcal{B}$. Let $\mathcal{AC}$ be the full abelian subcategory of $\mathcal{B}$ spanned by direct summands in $X \otimes Y$, where $X \in \mathcal{A}$ and $Y \in \mathcal{C}$. We say $\mathcal{B}$ factorizes into a product of $\mathcal{A}$ and $\mathcal{C}$ if $\mathcal{B} = \mathcal{AC}$, and this factorization is exact if $\mathcal{A} \cap \mathcal{C} = \mathrm{Vec}$, and denote it by $\mathcal{B} = \mathcal{A} \bullet \mathcal{C}$.

- **Example**: If $\mathcal{B} = \mathrm{Vec}(G_1) \bullet \mathrm{Vec}(G_2)$, then $\mathcal{B} = \mathrm{Vec}(G, \omega)$ where $G = G_1 G_2$ and $\omega \in H^3(G, \Bbbk^\times)$ is trivial on G_1 and G_2.

Research Problem

It was shown in [3] that the following are equivalent:

1. Every simple object of $\mathcal{B}$ can be uniquely expressed in the form $X \otimes Y$, where $X \in \mathcal{A}$ and $Y \in \mathcal{C}$ are simple objects
2. $\mathcal{B} = \mathcal{A} \bullet \mathcal{C}$ (i.e. $\mathcal{B} = \mathcal{A}\mathcal{C}$ and $\mathcal{A} \cap \mathcal{C} = \mathrm{Vec}$)
3. $\mathrm{FPdim}(\mathcal{B}) = \mathrm{FPdim}(\mathcal{A})\mathrm{FPdim}(\mathcal{C})$ and $\mathcal{A} \cap \mathcal{C} = \mathrm{Vec}$

What can be said about exact factorizations of finite tensor categories which are not fusion?

Exact Factorization of Finite Tensor Categories

Let $\mathcal{B}$ be a finite tensor category, and let $\mathcal{A}, \mathcal{C} \subseteq \mathcal{B}$ be tensor subcategories of $\mathcal{B}$. We note that $\mathcal{B}$ is naturally a left module category over $\mathcal{A} \boxtimes \mathcal{C}^{op}$.

- Let $\mathcal{AC}$ be the indecomposable component of $\mathbb{1}$
- We say $\mathcal{B}$ factorizes as $\mathcal{AC}$ if $\mathcal{B} = \mathcal{AC}$ is exact over $\mathcal{A} \boxtimes \mathcal{C}^{op}$
- Further, if $\mathcal{A} \cap \mathcal{C} = \text{Vec}$, then we say the factorization is exact and write $\mathcal{B} = \mathcal{A} \bullet \mathcal{C}$

Exact Factorizations of Finite Tensor Categories

Lemma

Let $\mathcal{A},\mathcal{C}$ be tensor subcategories of a finite tensor category $\mathcal{B}$. Suppose that $\mathcal{A}\cap\mathcal{C} = Vec$. Let $X \in Irr(\mathcal{A})$ and $Y \in Irr(\mathcal{C})$. Then

$$\dim Hom_{\mathcal{B}}(P_{\mathcal{A}}(X), P_{\mathcal{C}}(Y)) = \begin{cases} 1 & X = \mathbb{1} \text{ and } Y = \mathbb{1}^D \\ 0 & else \end{cases}$$

Proof:

- Let $0 \neq f \in \mathrm{Hom}_{\mathcal{B}}(P_{\mathcal{A}}(X), P_{\mathcal{C}}(Y))$ and let $Z = \mathrm{im}(f)$
- $Z \in \mathcal{A}\cap\mathcal{C}$
- $Z = \mathbb{1}^{\oplus n}$

Exact Factorizations of Finite Tensor Categories

$$\begin{tikzcd} P_{\mathcal{A}}(X) \arrow[rr, "f \neq 0"] \arrow[dr, "i \neq 0"'] & & P_{\mathcal{C}}(Y) \\ & \mathbb{1}^{\oplus n} \arrow[ur, hook, "j \neq 0"'] & \end{tikzcd}$$

- $0 < \dim \operatorname{Hom}_{\mathcal{A}}(P_{\mathcal{A}}(X), \mathbb{1}^{\oplus n}) = n[\mathbb{1} : X]_{\mathcal{A}} = n\delta_{X,\mathbb{1}}$
- $X = \mathbb{1}$
- $0 < \dim \operatorname{Hom}_{\mathcal{C}}(\mathbb{1}, P_{\mathcal{C}}(Y)) = \dim \operatorname{Hom}_{\mathcal{C}}(P_{\mathcal{C}}(Y)^*, \mathbb{1}) = [\mathbb{1}^D : Y]$
- $Y = \mathbb{1}^D$

Exact Factorizations of Finite Tensor Categories

$$\begin{tikzcd}
P_{\mathcal{A}}(\mathbb{1}) \arrow[rr, "{f,f'\neq 0}"] \arrow[dr, "{i,i'\neq 0}"'] & & P_{\mathcal{C}}(\mathbb{1}^D) \\
& \mathbb{1} \arrow[ur, hook, "{j,j'\neq 0}"'] &
\end{tikzcd}$$

- The socle of $P(\mathbb{1}^D)$ is $\mathbb{1}$, so $n = 1$.
- $\dim \mathrm{Hom}_{\mathcal{B}}(P_{\mathcal{A}}(\mathbb{1}), P_{\mathcal{C}}(\mathbb{1}^D)) = 1$

Exact Factorizations of Finite Tensor Categories

Proposition

Let $\mathcal{A}, \mathcal{C}$ be tensor subcategories of a finite tensor category $\mathcal{B}$. Suppose that $\mathcal{B} = \mathcal{A} \bullet \mathcal{C}$. Then

$$FPdim(\mathcal{B}) = FPdim(\mathcal{A})FPdim(\mathcal{C}).$$

Proof:

- $R_{\mathcal{B}} = \sum_{Z \in \mathrm{Irr}(\mathcal{B})} \mathrm{FPdim}(Z) P_{\mathcal{B}}(Z)$
- There exists $\lambda > 0$ such that $R_{\mathcal{A}} R_{\mathcal{C}} = \lambda R_{\mathcal{B}}$
- $\dim \mathrm{Hom}_{\mathcal{B}}(R_{\mathcal{B}}, \mathbb{1}) = 1$

Exact Factorizations of Finite Tensor Categories

$$\begin{aligned}
\lambda &= \dim \mathrm{Hom}_{\mathcal{B}}(\lambda R_{\mathcal{B}}, \mathbb{1}) \\
&= \dim \mathrm{Hom}_{\mathcal{B}}(R_{\mathcal{A}} R_{\mathcal{C}}, \mathbb{1}) \\
&= \dim \mathrm{Hom}_{\mathcal{B}}(R_{\mathcal{A}}, R_{\mathcal{C}}) \\
&= \sum_{X \in \mathrm{Irr}(\mathcal{A}), Y \in \mathrm{Irr}(\mathcal{C})} \mathrm{FPdim}(X)\mathrm{FPdim}(Y) \dim \mathrm{Hom}_{\mathcal{B}}(P_{\mathcal{A}}(X), P_{\mathcal{C}}(Y)) \\
&= 1
\end{aligned}$$

where the last line follows from the lemma.

Exact Factorizations of Finite Tensor Categories

Conjecture

Let $\mathcal{A}, \mathcal{C} \subseteq \mathcal{B}$ be tensor subcategories of a finite tensor category $\mathcal{B}$. Then the following are equivalent:

1. $\mathcal{B} = \mathcal{A} \bullet \mathcal{C}$
2. *Every simple object $Z \in Irr(\mathcal{B})$ can be uniquely expressed in the form $X \otimes Y$ for some $X \in Irr(\mathcal{A})$ and $Y \in Irr(\mathcal{C})$, and also $P_{\mathcal{B}}(Z) = P_{\mathcal{A}}(X) \otimes P_{\mathcal{C}}(Y)$*
3. *$FPdim(\mathcal{B}) = FPdim(\mathcal{A})FPdim(\mathcal{C})$ and $\mathcal{A} \cap \mathcal{C} = Vec$ and $\mathcal{B}$ is exact over $\mathcal{A} \boxtimes \mathcal{C}^{op}$*

Exact Factorizations of Finite Tensor Categories

Questions?

References

- P. Etingof, S. Gelaki. Exact sequences of tensor categories with respect to a module category. *Adv. Math.* **308** (2017), 1187–1208.
- P. Etingof, S. Gelaki, D. Nikshych, V. Ostrik. Tensor Categories. *AMS Mathematical Surveys and Monographs book series* **205** (2015), 362 pp.
- S. Gelaki. Exact Factorizations and Extensions of Fusion Categories. *Journal of Algebra* **480** (2017), 505–518.