Interpolation Problems

Tuesday, October 27, 2020 2:00 PM

Notation: k = field of characteristic 0 (eg. $k=\mathbb{C}$)

If V = graded k-vector space, $[V]_d = \langle f \in V \mid \deg(f) = d \rangle$

Hermite I.P. (= interpolation problem):

Fix a set X of pts in $\mathbb{P}^n$, fix $m \in \mathbb{Z}_+$, deduce information about

$\boxed{X^{(m)}} := \{$ all hypersurfaces passing through $X \geq m$ times $\}$

(eg. $H_{X^{(m)}}(d) := \dim_k [X^{(m)}]_d$ (= Hilbert function)

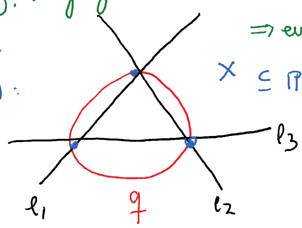
or $\alpha(X^{(m)}) := \min\{ d \mid \exists f \in X^{(m)}$ of $\deg(f) = d \}$)

The case $m=1$ is Lagrange I.P.

[eg. (Cayley–Bacharach thm): If $X = \{9$ pts$\} \subseteq \mathbb{P}^2$ and $H_{X^{(1)}}(3) \geq 2$

$\Rightarrow$ every cubic through 8 pts of X passes through all 9]

Eg:



$X \subseteq \mathbb{P}^2$

$\alpha(X^{(1)}) = 2$ $\quad [q]$

$3 \leq \alpha(X^{(2)}) \leq 4$ $\quad [q^2 \in X^{(2)}]$

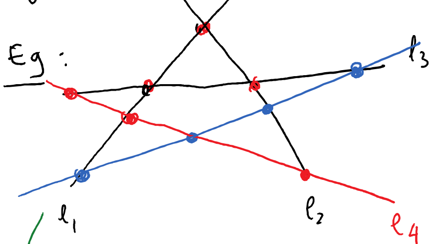
$\boxed{\alpha(X^{(2)}) = 3}$ $\quad \ell_1 \ell_2 \ell_3 \in X^{(2)}$

In fact $\alpha(X^{(2t)}) = 3t$, $\forall t \geq 1$.

Already: $X = 5$ pts in $\mathbb{P}^2$ $\quad \alpha(X^{(m)})$ $\quad \forall m \geq 1$

A Star configuration consists of all the n-wise intersection of $s \geq n$ hyperplanes $\ell_1 = 0, \ldots, \ell_s = 0$ in $\mathbb{P}^n$ meeting properly.

Eg:



= star conf. of $\cancel{3}$ $\binom{3}{2}$ pts in $\mathbb{P}^2$

$\cancel{\binom{4}{2}} = 6$ " "

$\binom{5}{2} = 10$ " "

Thm (..., Bocci–Harbourne '09): X = star conf. $\Rightarrow \alpha(X^{(nt)}) = s \cdot t$, $\forall t \geq 1$

("Pf": $(\ell_1 \cdots \ell_s)^t \in X^{(nt)} \Rightarrow \alpha(X^{(nt)}) \leq st$. Not too hard: "=" holds #)

[eg. $X = 10$ pts $\quad \alpha(X^{(2t)}) \geq 5t$, $\forall t \geq 1$.]

2. Linear Algebra (Lagrange I.P.)

Eg: If $X = r$ pts in $\mathbb{P}^1 \Rightarrow \alpha(X^{(m)}) = mr$, $\forall m \geq 1$.

However, for $\mathbb{P}^n$ $n \geq 2$: Lagrange I.P. is nearly intractable (in general).

Thm (Lagrange I.P. for general pts):

If $X = \{P_1, \ldots, P_r\} \subseteq \mathbb{P}^n$ general pts $\Rightarrow$ $\boxed{\alpha(X^{(1)}) = \min\{ t \mid \binom{n+t}{n} > r \}}$

and $\boxed{H_{X^{(1)}}(d) = \binom{n+d}{n} - r}$, $\forall d \geq \alpha(X^{(1)})$.

...es of degree d

Pf. Let $F \in k[x_0,\dots,x_n]_d$, $F = \sum c_i M_i$ $\quad M_i$ = monomials of degree d, $c_i \in k$

$$F \in X^{(1)} \Leftrightarrow \begin{cases} F(P_1)=0 \\ \vdots \\ F(P_r)=0 \end{cases} \to \text{a homog. lin. syst. of } r \text{ equations and } \binom{n+d}{n} \text{ variables } (=\text{the } c_i\text{'s})$$

$$H_{X^{(m)}}(d) = \dim_k(\text{null-space of } \mathcal{J}) = \binom{n+d}{n} - rk \underset{\substack{\downarrow \\ (X \text{ is general})}}{=} \binom{n+d}{n} - r \underset{\substack{\downarrow \\ (b/c\ d \ge \alpha(X^{(1)}))}}{\ge} 0$$

#

3. Commutative Algebra.

P = a pt in $\mathbb{P}^n \leftrightarrow \mathfrak{p} = (\ell_1,\dots,\ell_n) \subseteq R = k[x_0,\dots,x_n]$

$F=0$ passes through $P \iff F \in \mathfrak{p}$

$F=0$ " " $X = \{P_1,\dots,P_r\} \iff F \in \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \dots \cap \mathfrak{p}_r =: I_X$

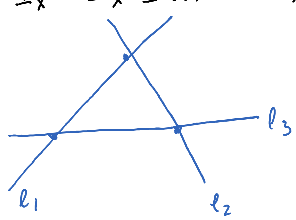
Thm (Zariski-Nagata): $F=0$ lies in $X^{(m)} \iff F \in \mathfrak{p}_1^m \cap \mathfrak{p}_2^m \cap \dots \cap \mathfrak{p}_r^m =: I_X^{(m)}$

= m^{th} symbolic power of I_X.

Rmk: $I_X^m \subseteq I_X^{(m)}$, $\forall m \ge 1$.

$I_X \supseteq I_X^{(2)} \supseteq \dots \supseteq I_X^{(m)} \supseteq \dots$

Eg:



$X = 3$ pts $\to I_X = (\ell_1,\ell_2) \cap (\ell_1,\ell_3) \cap (\ell_2,\ell_3)$
$= (\ell_1\ell_2, \ell_1\ell_3, \ell_2\ell_3)$

$I_X^{(2)} = (\ell_1,\ell_2)^2 \cap (\ell_1,\ell_3)^2 \cap (\ell_2,\ell_3)^2$
$= (\ell_1\ell_2\ell_3) + I_X^2 \supsetneq I_X^2$

Hermite I.P. Fix $X \subseteq \mathbb{P}^n$ pts, deduce info about $I_X^{(m)}$, eg.

(•) $H_{R/I_X^{(m)}}(d) = \dim_k \left[R/I_X^{(m)} \right]_d$ (Hlb. function) $\to$ lots of work

(•) $\alpha(I_X^{(m)}) = \min\{t \mid \exists\, 0 \ne f \in I_X^{(m)}, \deg f = t\}$

(•) Betti table of $I_X^{(m)}$.

Thm: $X = \{P_1,\dots,P_r\} \subseteq \mathbb{P}^n \Rightarrow H_{R/I_X^{(m)}}(d) \le \min\left\{ \binom{n+d}{n}, r\binom{n+m-1}{n} \right\}$, $\forall d, m \in \mathbb{Z}_+$.

~~Defn.~~ Pf. Let $F \in k[x_0,\dots,x_n]_d$, now $0 \ne F \in X^{(m)} \iff F \in I_X^{(m)}$

$\iff$ all $\binom{n+m-1}{n}$ partial derivatives of F of order $m-1$ pass through X

$$\iff \begin{cases} \frac{\partial_\alpha}{\partial x^\alpha} F(P_1) = 0 \\ \vdots \\ \frac{\partial_\alpha}{\partial x^\alpha} F(P_r) = 0 \end{cases} \quad |\alpha| = m-1$$

$\to$ this is a homog. lin. system of $r\binom{n+m-1}{n}$ eq'ns in $\binom{n+d}{n}$ variables.

So $H_{R/I_X^{(m)}}(d)$ = rank of this linear system $\le \min\left\{ \binom{n+d}{n}, r\binom{n+m-1}{n} \right\}$ #

$X^{(m)}$ has exp. dim. in degree d if "=" is achieved.

$H\ (2) = \min\{\binom{2+2}{2}, \binom{2+2-1}{2}\cdot 2\}$

I_X ...

$X^{(m)}$ has exp. dim. in degree d if "$=$" is achieved.

Eg: $X=\{2 \text{ pts}\}\subseteq \mathbb{P}^2$, then $X^{(2)}$ has exp. dim. in degree $2 \iff H_{R/I_X^{(2)}}(2) = \min\left\{\binom{2+2}{2}, \binom{2+2-1}{2}\cdot 2\right\} = 6 = H_R(2)$

$(R=k[x_0,x_1,x_2])$

$\iff \nexists$ quadric in $I_X^{(2)}$

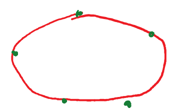
However, $\ell^2 \in I_X^{(2)}$ where ℓ = line through 2 pts

$\Rightarrow X^{(2)}$ does not have expected dim. in degree 2.

Thm (Alexander-Hirschowitz '90s): $X = r$ general pts in $\mathbb{P}^n$, then X does not have exp. dim. in degree $d \iff$

(i) $2 \le r \le n$, $d=2$ [Reason: as above]

or (ii) $r=5$, $n=2$
or $r=9$, $n=3$ $\rightarrow d=4$
or $r=14$, $n=4$

[Reason (r=5, n=2):  q X | $X^{(2)}$ has exp. dim. in degree 4 $\iff \nexists$ quartic in $I_X^{(2)}$. However $q^2 \in I_X^{(2)}$]

or (iii) $r=7$, $n=4$, $d=3$

[Reason: $\exists$ rat'nal normal curve passing through X, its equations are

$$I_2\begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \\ x_2 & x_3 & x_4 \end{pmatrix}$$

Now $X^{(2)}$ has exp. dim. in degree $3 \iff \nexists$ cubic in $X^{(2)}$

However, $\det(\ )$ is a cubic in $X^{(2)}$]

M-Ha': survey paper on this theorem.

Open Problems: How about

(·) general triple pts?

(·) general m-tuple pts in $\mathbb{P}^2$?

(·) even α of $\uparrow$ is wide-open? [Even if $r=10$]

(·) Betti table of $I_X^{(2)}$?

(·) Betti table of I_X?

(·) Hilb. function or Betti table for special sets of pts?

Thm [M '20, BDGMNOS '20]: If X = star config. $\Rightarrow \exists$ explicit formula for Betti table of $I_X^{(m)}$.

Idea of pf. We prove $I_X^{(m)}$ have c.i. quotients, ie. ... $(f_1, \dots, f_i) : f_{i+1}$ = complete ...

Idea of pf. We prove $I_X^{(m)}$ have c.i. quotients, ie.
$I_X^{(m)} = (f_1, \ldots, f_s)$ $\deg f_i \le \deg f_{i+1}$ s.t. $(f_1, \ldots, f_i) : f_{i+1}$ = complete intersection

$\Rightarrow$ Betti table is obtained by mapping cones of Koszul complexes #

Nagata's Conj ('58): $X = r \ge 10$ general pts in $\mathbb{P}^2$
$\Rightarrow \alpha(X^{(m)}) > m \cdot \sqrt{r}$, $\forall m \ge 1$.

known: r = perfect squares (Nagata)
Miranda-Roé '11: study $r = 10$.
Biran '98: Nagata's Conj $\Leftrightarrow$ symplectic packing problem
(very hard) (very hard)

SHGH Conj: conjecturing analogue of Alexander-Hirschowitz in $\mathbb{P}^2$
(for non-uniform I.P.)