Koszul Algebras

Let A be a f.g. standard graded K-algebra (not necessarily commutative).

i.e. $A = \bigoplus_{i \geq 0} A_i$, $A_0 = K$, A gen by A_1 as a K-algebra, $1 \in A_0$, $A_i \cdot A_j \subseteq A_{i+j}$.

Set $V = A_1$. Then all such A are graded quotients of the tensor algebra $T(V) = \bigoplus_{i \geq 0} V^{\otimes i}$.

If $\dim V = n$, can identify $T(V) = K\{x_1, \dots, x_n\}$

$\boxed{\text{Examples}}$ $\quad Sym(V) = \dfrac{T(V)}{\langle [x,y] \mid x,y \in V \rangle} = \dfrac{K\{x_1, \dots, x_n\}}{\langle x_i x_j - x_i x_j \rangle} \cong K[x_1, \dots, x_n]$

$$\Lambda(V) = \frac{T(V)}{\langle v \otimes v \mid v \in V \rangle} = \frac{K\{x_1, \dots, x_n\}}{\langle x_i^2 \rangle + \langle x_i x_j + x_i x_j \rangle}$$

A is <u>quadratic</u> if $Ker(T(V) \twoheadrightarrow A)$ is generated by elements in $V \otimes V$. (Examples: See above)

Given a <u>quadratic</u> algebra $A = T(V)/I$, its <u>quadratic dual algebra</u> $A^! = T(V^*)/I^\perp$, where $I^\perp$ is gen by elements orthogonal to $I_2 \subseteq V \otimes V$ under the natural pairing

$$\langle v_1 \otimes v_2, v_1^* \otimes v_2^* \rangle = \langle v_1, v_1^* \rangle \langle v_2, v_2^* \rangle$$

between $V \otimes V$ & $V^* \otimes V^*$.

Example: $\mathrm{Sym}(V)^! = \Lambda(V^*)$

& $\Lambda(V)^! = \mathrm{Sym}(V^*)$

Example: Given any graded algebra A, the diagonal subalgebra $\bigoplus \mathrm{Ext}^i_A(K,K)_i \cong (qA)^!$

is a quadratic algebra dual to the quadratic part of $A = T(V)/I$, $qA = T(V)/I_2$.

<u>Def</u>: A is called <u>Koszul</u> if one of the following equivalent conditions holds:

- $\mathrm{Ext}^i_A(K,K)_j = 0$ if $i \neq j$
- $\mathrm{Ext}^i_A(A,A)_i$ is gen by $\mathrm{Ext}^1_A(K,K)$
- A is quadratic and $\mathrm{Ext}^i_A(K,K)_i \cong A^!$
- K has a linear free resolution over A.

<u>Example</u>: $A = \mathrm{Sym}(V)$, then the Koszul complex is a (finite) linear free resolution of K over A, i.e.

$$0 \to \Lambda^n V^* \otimes_K A \to \cdots \to \Lambda^2(V^*) \otimes_K A \to \Lambda^1(V^*) \otimes_K A \to A$$

with differential

$$e_1 \wedge e_2 \wedge \cdots \wedge e_j \otimes a \longmapsto \sum_{i=1}^{j} (-1)^i e_1 \wedge \cdots \wedge \hat{e}_i \wedge \cdots \wedge e_j \otimes x_i a$$

remove e_i.

More explicitly: $A = K[x_1, x_2, x_3]$, K has free res:

$$0 \to A \xrightarrow{\begin{pmatrix} x_3 \\ -x_2 \\ x_1 \end{pmatrix}} A^3 \xrightarrow{\begin{pmatrix} x_2 & x_3 & 0 \\ -x_1 & 0 & x_3 \\ 0 & -x_1 & -x_2 \end{pmatrix}} A^3 \xrightarrow{(x_1\ x_2\ x_3)} A$$

. i.e. $Sym(V)$ is Koszul

Example: $A = \Lambda(V)$, then the Cartan Complex is a linear (infinite) free resolution of K over A, i.e.:

$$\cdots \longrightarrow Sym_2(V^*) \otimes A \longrightarrow Sym_1(V^*) \otimes A \to A$$

with differential

$$x_{k_1} \cdots x_{k_j} \otimes a \longmapsto \sum_{i=1}^{j} x_{k_1} \cdots \hat{x}_{k_i} \cdots x_{k_j} \otimes e_{k_i} a$$

More explicitly: $A = \Lambda(Ke_1 \oplus Ke_2 \oplus Ke_3)$, K has res:

$$\to A^{10} \longrightarrow A^6 \xrightarrow{\begin{pmatrix} e_1 & e_2 & e_3 & 0 & 0 & 0 \\ 0 & e_1 & 0 & e_2 & e_3 & 0 \\ 0 & 0 & e_1 & 0 & e_2 & e_3 \end{pmatrix}} A^3 \xrightarrow{e_1\, e_2\, e_3} A$$

i.e. $\Lambda(V)$ is Koszul.

By above, A Koszul $\Rightarrow$ A quadratic.

The converse does not hold.

<u>Example</u>: $R = K[w,x,y,z]/(w^2,x^2,y^2,z^2,xy+xz+xw)$

Res of K looks like:

$$\underbrace{R(-3)^{25} \oplus R(-4)^2}_{} \longrightarrow R(-2)^{11} \longrightarrow R(-1)^4 \longrightarrow R$$

2 nonlinear 3rd syzygies.

$R(-j)_i = R_{i-j}$

On the other hand, there is no "finite check" for Koszul property.

<u>Example (Roos 1993)</u>

Fix an integer $r \geq 2$. Let

$$R = \frac{\mathbb{Q}[x,y,z,u,v,w]}{(x^2,xy,yz,z^2,zu,u^2,vw,w^2,xz+rzw-uw,zw+xu+(r-2)uw)}$$

Then K has an R-linear resolution for exactly r steps. (So not Koszul but difficult to check.)

New Construction: (–, Seceleanu) Same thing for quadratic Artinian Gorenstein rings.

Some interesting examples/problems:

① Conjecture (Backelin? Bøgvad) (1993)

The coordinate ring of a smooth, projective, toric variety is Koszul. (Still open)

e.g. Consider the dth Veronese embedding

$$\mathbb{P}^n_K \hookrightarrow \mathbb{P}_K^{\binom{n+d}{d}-1}$$

all monomials ⟹ toric
can check it is smooth

given by

$$[a_0 : \cdots : a_n] \longmapsto [\underbrace{a_0^d : a_0^{d-1} a_1 : \cdots : a_n^d}_{\text{all monomials in degree } d.}]$$

The image has coordinate ring

$$A = K[x_0^d, x_0^{d-1}x_1, \ldots, x_n^d] = K[y_0, \ldots, y_{\binom{n+d}{d}}]/I$$

& I has a quadratic GB; hence

A is Koszul.

- On the other hand, the 3rd pinched Veronese has coordinate ring

$$K[x^3, x^2y, x^2z, xy^2, xz^2, y^3, y^2z, yz^2, z^3] \quad (\underline{\text{no}}\ xyz)$$

This is a toric variety, but has an isolated singularity.

It is Koszul (Caviglia - 2008) - no known quad GB.

- Other examples:

Segre Embeddings $\mathbb{P}^{n-1} \times \mathbb{P}^{m-1} \hookrightarrow \mathbb{P}^{nm-1}$

$$[a_1,\dots,a_n] \times [b_1,\dots,b_m] \longmapsto [a_1b_1 : \dots : a_nb_n]$$

(also smooth & toric)

- Grassmannians :

The set of r-dimensional subspaces in K^n has the structure of an algebraic variety $Gr(n,r)$.
There is a standard embedding via Plücker coordinates.
Resulting coordinate ring is Koszul (not toric).

② <u>Hyperplane Arrangements</u>

A set $\mathcal{A} = \bigcup_{i=1}^{s} H_i \subseteq \mathbb{C}^r$ of central (contains the origin) hyperplanes defines a connected space

$\mathbb{C}^r \setminus \mathcal{A}$, whose cohomology ring is combinatorially defined by the Orlik-Solomon algebra:

$E =$ exterior algebra on $e_1,\dots,e_s$

$$J = \langle \partial(e_{i_1} \wedge \dots \wedge e_{i_p}) \mid \operatorname{codim}(H_{i_1} \cap \dots \cap H_{i_p}) < p \rangle$$

i.e. $H^*(\mathbb{C}^r \setminus \mathcal{A}; \mathbb{C}) \cong E/J$.

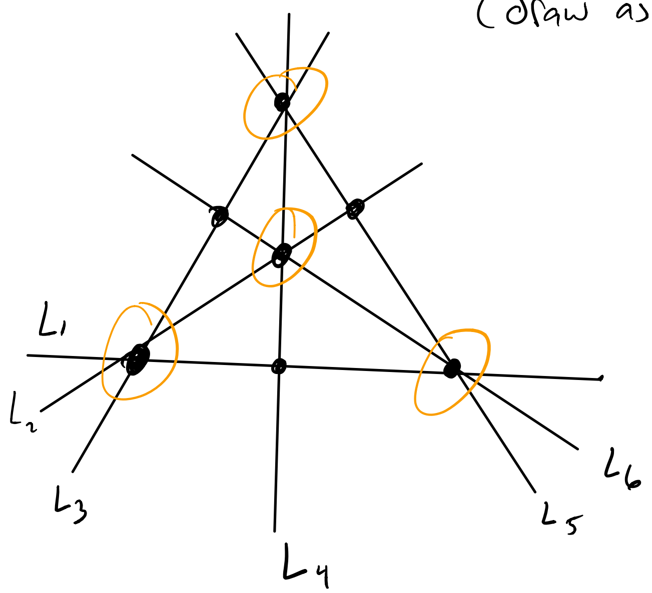
<u>Thm</u> (Peeva) $\mathcal{A}$ is supersolvable (intersection lattice

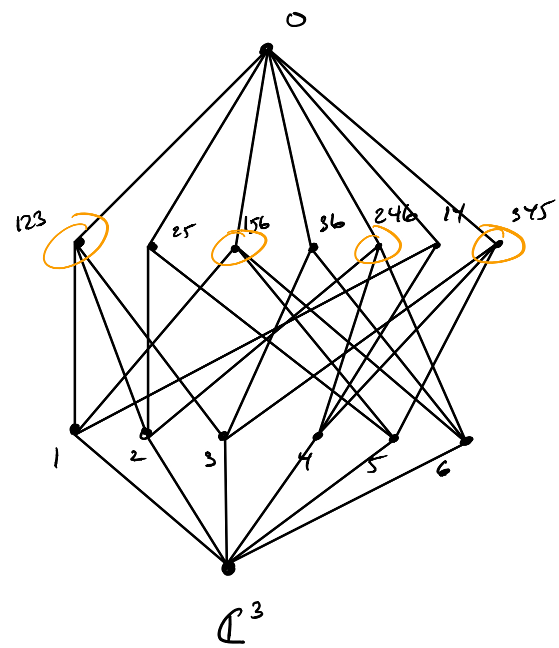
is supersolvable - ($\exists$ a max chain C s.t. $\forall$ max chain C', sublattice gen by $C \cup C'$ is distributive) iff

E/J has a quad. GB.

Example: Braid Arrangement in $\mathbb{C}^3$

(draw as lines in $\mathbb{P}^2_{\mathbb{C}}$)





$$E = \frac{\mathbb{C}\langle e_1, e_2, e_3, e_4, e_5, e_6 \rangle}{\langle \partial(e_1e_2e_3), \partial(e_1e_5e_6), \partial(e_2e_4e_6), \partial(e_3e_4e_5) \rangle}$$

$$= \frac{\mathbb{C}\langle e_1, \dots, e_6 \rangle}{\langle e_1 \wedge e_2 - e_1 \wedge e_3 + e_2 \wedge e_3,\ e_1 \wedge e_5 - e_1 \wedge e_6 + e_1 \wedge e_5, \dots \rangle}$$

clearly quadratic - by thm, this

is Koszul.

(3) Points in $\mathbb{P}^n$:

Given $P_1, \dots, P_s \in \mathbb{P}^n$, $S = K[x_0, \dots, x_n]$

Write $I = \bigcap_{i=1}^{s} I(P_i)$

each is a prime ideal gen by $n-1$ linear forms corresponding to $n-1$ lin. ind. lines through P_i.

Thm (Kempf): If $s \leq 2n$ and $P_1, \dots, P_s$ are in linearly general position, then S/I is Koszul.

(No 3 on a line.)

Next time: Focus on commutative case & how to show something is/isn't Koszul.

Talk 2: Commutative Koszul Algebras

$R = \bigoplus_{i \geq 0} R_i$ standard, graded (commutative) K-algebra

$R_0 = K$, $R_i R_j \subseteq R_{i+j}$, $R = K[R_1] = \dfrac{K[x_1, \dots, x_n]}{I}$ $:= S$

homogeneous ideal

R has minimal S-free resolution:

$$\to F_p \to \quad \cdots \to F_1 \to F_0 = S$$

$F_i = \bigoplus_j S(-j)^{\beta_{ij}}$ where

$S(-j)_i := S_{i-j}$ & $\beta_{ij} = \dim_K \mathrm{Tor}_i^S(R, K)_j$

Total Betti Number $\beta_i^S(R) = \sum_j \beta_{ij} = \dim_K \mathrm{Tor}_i^S(R, K)$

Hilbert Function: $HF_R(i) = \dim_K(R_i)$

2 Generating Functions:

1. Hilbert Series: $HS_R(t) = \sum_{i \geq 0} HF_R(i) t^i$

2. Poincare Series: $P_R(t) = \sum_{i \geq 0} \beta_i^R(K) t^i$

$:= \dim_K \mathrm{Tor}_i^R(K, K)$

$\boxed{\text{Example}}$ $R = \dfrac{K[x,y]}{(x^2, xy, y^2)}$

Hilbert Series : $1+2t$

Free Res of R over $S = K[x,y]$:

$$0 \to S(-3)^2 \longrightarrow S(-2)^3 \longrightarrow S$$

Note: $HS_S(t) = \sum_i \dim_K (K[x,y]) t^i = \binom{2+i-1}{i} t^i$

$$= \sum_i (i+1) t^i$$

So $HS_R(t) = HS_S(t) - 3HS_S(t) \cdot t^2 - 2HS_S(t) \cdot t^3$

$$= (1+2t+3t^2+\cdots) - 3(t^2+2t^3+3t^4+\cdots) - 2(t^3+2t^4+3t^5\cdots)$$

$$= 1+2t$$

Resolution of K over R :

$$\cdots \longrightarrow R(-3)^8 \longrightarrow R(-2)^4 \xrightarrow{\begin{pmatrix} x & y & 0 & 0 \\ 0 & 0 & x & y \end{pmatrix}} R(-1)^2 \xrightarrow{(x,y)} R$$

Poincare Series : $P_R(t) = 1+2t+4t^2+8t^3+\cdots$

$$= \sum_{i \geq 0} 2^i t^i$$

$$= \sum_{i \geq 0} (2t)^i$$

$$= \frac{1}{1-2t}$$

$$= \frac{1}{HS_R(-t)} \quad !!$$

This is not a coincidence

Thm (Fröberg 1999): If R is Koszul, then

$$P_R(t) = \frac{1}{HS_R(-t)}.$$

Can sometimes use this to show a particular algebra is not Koszul.

$\boxed{\text{Example}}$ (Same as talk 1)

$$R = K[w,x,y,z] / (w^2, x^2, y^2, z^2, x(y+z+w))$$

Easy to check $HS_R(t) = 1 + 4t + 5t^2 + t^3$

$$\frac{1}{HS_R(-t)} = 1 + 4t + 11t^2 + \cdots + 71t^7 - 174t^8 + \cdots$$

!

This can't be Koszul. If it were,

$-174 = \dim_K \operatorname{Tor}_i^R(K,K)$, which is impossible.

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2 more invariants: Fix a f.g. graded R-module M.

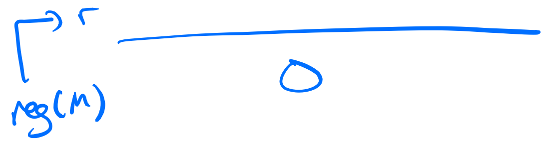
① $$\begin{aligned} \operatorname{pd}_R(M) &= \max\{i \mid \beta_i^R(M) \neq 0\} \\ &= \max\{i \mid \operatorname{Tor}_i^R(M,K) \neq 0\} \\ &= \text{length of resolution of } M \end{aligned}$$

② $$\begin{aligned} \operatorname{reg}_R(M) &= \max\{j \mid \beta_{ij}^R(M) \neq 0\} \\ &= \max\{j \mid \operatorname{Tor}_i^R(M,K)_j \neq 0\} \end{aligned}$$

Betti Table Notation:

$\operatorname{pd}_R(M)$

	0	1	2	---	i	p
0	β_{00}	β_{11}	β_{22}		β_{ii}	
1	β_{01}	β_{12}	β_{23}		$\vdots$	
$\vdots$					$\vdots$	0
j	β_{0j}		$\cdots$		$\beta_{i,i+j}$	



Thm (Auslander-Buchsbaum-Serre)

TFAE:

1. $pd_R(M) < \infty$ $\forall$ f.g. R-module M.
2. $pd_R(K) < \infty$.
3. R is a polynomial ring.

Thm (Avramov - Eisenbud - Peeva)

1. $reg_R(M) < \infty$ $\forall$ f.g. R-module M
2. $reg_R(K) < \infty$.
3. R is Koszul. (equiv. $reg(K) = 0$)

So how do we show a given ring is (or isn't) Koszul)?

1. S/I Koszul $\Longrightarrow$ I gen by quadrics
2. If I is gen by monomials of degree 2,

then S/I is Koszul.

<u>Fact</u>: Fix a monomial order — write $\mathrm{In}_<(I)$ for the initial ideal

$$\mathrm{In}_<(I) = \{\mathrm{In}_<(f) \mid f \in I\}$$

Set $R = S/I$, $A = S/\mathrm{In}_<(I)$.

Then $\beta_{ij}^R(k) \leq \beta_{ij}^A(k)$.

So if I has a quadratic Gröbner basis, $\mathrm{In}_<(I)$ is a quadratic monomial ideal. By (2)

$\beta_{ij}^A(k) = 0$ for $i \neq j$

$\Rightarrow \beta_{ij}^R(k) = 0$ for $i \neq j$

$\Rightarrow R$ is Koszul.

(3) There are Koszul Algebras S/I s.t. I has no quadratic GB w.r.t. any monomial order.

(4) If I is generated by a regular sequence of quadrics $f_1, \ldots, f_t$, then $R = S/I$ is Koszul.

(i.e. f_i is a nzd on $S/(f_1, \ldots, f_{i-1})\ \forall i$)

Tate showed the minimal free res of K over R

is obtained by appending degree-2 terms to the Koszul complex on $x_1, \ldots, x_n$ to kill $H_1(K_\bullet(\underline{x}))$.

5. If I is generated by quadrics and has linear free resolution over S, then S/I is Koszul.

If all else fails, try a filtration argument:

<u>Def</u>: A <u>Koszul filtration</u> of a K-algebra R is a set $\mathcal{F}$ of ideals of R s.t.:

1. Every ideal $I \in \mathcal{F}$ is generated by linear forms.
2. $(0), \mathfrak{m}_R \in \mathcal{F}$
3. $\forall\ 0 \neq I \in \mathcal{F}$, $\exists\ J \in \mathcal{F}$ such that
   - (a) $J \subset I$
   - (b) I/J is cyclic
   - (c) $\mathrm{Ann}(I/J) = J : I \in \mathcal{F}$.

<u>Thm</u> (Conca - Trung - Valla)
If R has a Koszul filtration, R is Koszul.

<u>Example</u>: $R = K[a,b,c,d]/(ac, ad, ab-bd, a^2+bc, b^2)$

$\mathcal{F} = \{(a,b,c,d), (a,c,d), (c,d), (a,c), (c), (a), (0)\}$

is a Koszul filtration since

$$(a,c,d) : (a,b,c,d) = (a,b,c,d)$$
$$(c,d) : (a,c,d) = (a,b,c,d)$$
$$(c) : (c,d) = (a,c)$$
$$(c) : (a,c) = (a,c,d)$$
$$0 : (a) = (c,d)$$
$$0 : (c) = (a)$$

This ideal has 1 fewer generator so I/J is cyclic

Each element in $\mathcal{F}$ appears once

The colon ideal is another ideal in $\mathcal{F}$.

This is a Koszul filtration. Hence R is Koszul.

However, the Hilbert Series of R is

$$HS_R(t) = \frac{1+2t-2t^2-2t^3+2t^4}{(1-t)^2}$$

One checks by exhaustive (but finite!) search that no degree-two monomial ideal has this Hilbert Series. Since

$$\dim_K (S/I)_i = \dim_K (S/\mathrm{In}_<(I))_i$$

I cannot have a quadratic GB w.r.t.
<u>any</u> order.