Real and complex hyperbolic geometry

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Preface

These notes developed while trying to gather some of the background material while working on certain complex hyperbolic reflection groups. The notes are roughly in two parts. The first part consists of section 1 through 10 and is a quick introduction to some aspects of real and complex hyperbolic geometry. The second part consists of section 11 through 22 and is about hyperbolic reflection groups. One aim to give a quick straight path to the beautiful results of Conway and Borcherds on the reflection group of the even self dual lattice of signature $(25, 1)$ acting on real hyperbolic space of dimension 25 and the result of Allcock on the largest known arithmetic complex hyperbolic reflection group acting on complex hyperbolic space of dimension 13. We wanted to collect all the background material necessary to get to these results and in particular give a detailed exposition to Vinberg's algorithm. We have tried to keep the prerequisites minimal and tried to keep the arguments as self contained as possible. The main exception is that most of the results in section on hyperplanes. facets, chambers of hyperbolic reflection groups are almost verbatim copies of similar results about Euclidean reflection groups from Bourbaki, Lie algebra and Lie groups Chapter 5, section 1. In these cases, we have just referred to Bourbaki for the proofs.

In the first part of the notes we give a mostly uniform introduction to real and hyperbolic spaces, working in the projective model. In particular, the results in section 3 to 8 hold in both real or complex cases. Of course this development of hyperbolic geometry is highly lopsided and completely omits much of the most important topics. We mostly get by only using linear algebra. There is nothing here about negative curvature, or CAT(0) geometry, Fuschian groups and so on. However, we only take about 30 pages to introduce hyperbolic geometry other than bringing out the parallel between real and complex cases, we have found that this approach also quickly introduces the necessary tools for someone interested in computing with real and complex hyperbolic reflection groups.

None of the results here are new. Some of the proofs are our own but we believe they are all well known. The organization and ordering of topics is our own as are all the errors that have inadvertently creeped in. We have freely borrowed from Bridson-Haefliger, Bourbaki, Humphreys and the papers of Borcherds and Allcock and other sources whenever convenient.

1 Geodesic spaces

1.1 Definition. In this section, (X, d) will always denotes a metric space. If A and B are non-empty subsets of X, let $d(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$. If $x \in X$, let $d(A, x) = d(A, \{x\})$. For $A \subseteq X$, define

$$B_r(A) = \{y \in X : d(A, y) < r\}.$$

So $B_r(x) = B_r(\{x\})$ is the open ball of radius r centered at x. Let $\mathrm{cl}(B_r(x))$ denote the closure of $B_r(x)$. A subset of X is *bounded* if it is contained in a ball. A metric space is called *proper* if every closed ball in X is compact and hence, every closed and bounded subset of X compact.

1.2 Lemma. *Let $\emptyset \neq A \subseteq X$.*
(a) If $x, y \in X$, then $|d(A, x) - d(A, y)| \leq d(x, y)$.
(b) The function $d(A, \cdot) : X \to \mathbb{R}$ given by $(x \mapsto d(A, x))$ is continuous.

Proof. Fix $\epsilon > 0$. There exists $a \in A$ such that $d(a, x) < d(A, x) + \epsilon$. Then

$$d(A, y) \leq d(a, y) \leq d(a, x) + d(x, y) \leq d(A, x) + d(x, y) + \epsilon.$$

Since ϵ is arbitrary, we have $(d(A, y) - d(A, x)) \leq d(x, y)$. Part (a) follows. Part (b) follows from part (a). □

1.3 Definition. If $f : [a, b] \to X$ is a continuous function, we say that f is a *parametrized curve in X from $f(a)$ to $f(b)$*. We say that $f(a)$ is the *origin* of f and $f(b)$ is the *endpoint* of f. Let $f : [a, b] \to X$ and $g : [c, d] \to X$ such that $f(b) = g(c)$. Define $f * g : [a, b + d - c] \to X$ by

$$(f * g)(s) = \begin{cases} f(s) & \text{for } s \leq b \\ g(c - b + s) & \text{for } s \geq b. \end{cases}$$

We say that $f * g$ is the concatenation of f and g.

Let $P = (a_1 < \cdots < a_n)$ be a partition of $[a, b]$. Let $\|P\| = \max_i\{|a_i - a_{i-1}|\}$. Define the Riemann sum

$$l(f, P) = \sum_i d(f(a_i), f(a_{i-1})).$$

A parametrized curve f is *rectifiable* if $\sup\{l(f, P) : P \text{ is a partition of } [a, b]\}$ is finite and in that case the supremum is called the *length* of f, denoted $l(f)$. Let f and g be curves in X such that the endpoint of f is the origin of g. Then $f * g$ is rectifiable if and only if f and g are rectifiable and one has

$$l(f * g) = l(f) + l(g).$$

In particular, let $f : [a, b] \to X$ and $c \in [a, b]$, Then f is rectifiable if and only if $f|_{[a,c]}$ and $f|_{[c,b]}$ are rectifiable and $l(f) = l(f|_{[a,c]}) + l(f|_{[c,b]})$.

1.4 Theorem. *Piecewise Lipschitz continuous paths in X are rectifiable. If $f : [a, b] \to X$ is a rectifiable path, then $l(f) = \lim_{n\to\infty} l(f, P_n)$ where P_n is any sequence of partitions such that $P_n \subseteq P_{n+1}$ for all n and $\|P_n\| \to 0$. (Add reference)*

1.5. Arc length parametrization: Let $f : [a, b] \to X$ be a parametrized curve. If ϕ is a non-decreasing function from $[c, d]$ onto $[a, b]$, then $f \circ \phi : [c, d] \to X$ is called a *reparametrization* of f. Reparametrization defines an equivalence relation on the set of parametrized curves in X. An equivalence class will be called an *(oriented) curve* in X. The length of a rectifiable curve is invariant under reparametrization, so length of a curve is well defined.

Let $f : [a, b] \to X$ be a rectifiable parametrized curve of length l. Then $\gamma(t) = l(f|_{[a,t]})$ is a continuous weakly increasing function from $[a, b]$ onto $[0, l]$. If $\gamma(s) = \gamma(t)$ for some $s < t$, then $l(f|_{[s,t]}) = 0$, so $f|_{[s,t]}$ is a constant. It follows that there exists a unique parametrized curve $f_* : [0, l] \to X$ such that $f_*(\gamma(t)) = f(t)$. Then one has $l(f_*|_{[0,u]}) = u$ for all u. We say that f_* is *parametrized by arc length.* So each rectifiable curve has a unique parametrization by arc length.

1.6 Definition (geodesic)**.** Let $x, y \in X$. We say that $f : [0, d_0] \to X$ is a *parametrized geodesic* from x to y if $f(0) = x$, $f(d_0) = y$ and $d(f(s), f(t)) = |s - t|$ for all $s, t \in [0, d_0]$. In particular $d_0 = d(x, y)$. Suppose $f : [0, d_0] \to X$ is a geodesic from x to y. If P is any partition of $[r, s]$, then one verifies that $l(f|_{[r,s]}, P) = (s - r)$. So f is rectifiable and

$$\begin{aligned} l(f|_{[r,s]}) &= (s - r) \\ &= d(f(r), f(s)) \quad \text{for all} \quad 0 \le r \le s \le d_0. \end{aligned}$$

Observe that if f is a geodesic from x to y, then $f^{op} : [0, d_0] \to X$ defined by $f^{op}(r) = f(d_0 - r)$ is a geodesic from y to x. Say that (X, d) is a *geodesic* (resp. *uniquely geodesic*), if there is a geodesic (resp. unique geodesic) joining any two points of X. If there is a unique geodesic joining x and y in X, then this unique geodesic will be denoted by $[x, y]$.

1.7 Lemma. *Let $x, y \in X$ and $d_0 = d(x, y)$. Let $\{0, d_0\} \subseteq I \subseteq [0, d_0]$. Suppose $f : I \to X$ be a function such that $f(0) = x$, $f(d_0) = y$ and*

$$d(f(r), f(s)) \le (s - r) \quad \text{for all} \;\; r < s \quad \text{and} \;\; r, s \in I.$$

Then $d(f(r), f(s)) = (s - r)$ for all $r < s$ and $r, s \in I$.

Proof. Let $r, s \in I$, $r < s$. We have $d(f(0), f(r)) \le r$ and $d(f(r), f(d_0)) \le (d_0 - r)$. So

$$\begin{aligned} d_0 &= d(f(0), f(d_0)) \\ &\le d(f(0), f(r)) + d(f(r), f(d_0)) \\ &\le r + (d_0 - r) = d_0, \end{aligned}$$

so equality must hold everywhere, that is, $d(f(0), f(r)) = r$ for all $r \in I$. Now

$$\begin{aligned} s &= d(f(0), f(s)) \\ &\leq d(f(0), f(r)) + d(f(r), f(s)) \\ &\leq r + (s - r) = s, \end{aligned}$$

so equality must hold everywhere, hence $d(f(r), f(s)) = s - r$. $\square$

1.8 Theorem. *Suppose f is a rectifiable curve in X from x to y such that $l(f) = d(x, y)$. Then f (parametrized by arc length) is a geodesic joining x and y. So X is a geodesic space if and only if any two point in X can be joined by a rectifiable curve whose length attains the distance between the two points.*

Proof. Let $d_0 = d(x, y)$. Parametrize f by arc length. So $f : [0, d_0] \to X$. We have

$$\begin{aligned} d(f(r), f(s)) &\leq l(f|_{[r,s]}) \\ &= (s - r) \text{ for all } 0 < r < s < d_0. \end{aligned}$$

So 1.7 implies $d(f(r), f(s)) = s - r$ for all $0 \leq r \leq s \leq d_0$, that is, f is a geodesic.

Given two points x and y in a geodesic space X, take a rectifiable curve f in X such that $l(f) = d(x, y)$ and parametrize f by arc length. Then f is a geodesic joining x and y. $\square$

The lemma below gives us a condition for point to lie on a geodesic.

1.9 Lemma. *Suppose $x, y, z \in X$ such that $d(x, z) + d(z, y) = d(x, y)$. Let γ_1 be the geodesic joining x and z and γ_2 be the geodesic joining z to y. Parametrize $\gamma_1 * \gamma_2$ by arc length and call it γ. Then γ is a geodesic joining x to y. In particular, if X is geodesic space and $x, y, z \in X$, then z lies on a geodesic joining x and y if and only if $d(x, z) + d(z, y) = d(x, y)$.*

Proof. Note that γ is parametrized by arc length, joins x to y and $l(\gamma) = l(\gamma_1) + l(\gamma_2) = d(x, z) + d(z, y) = d(x, y)$. So 1.8 implies that γ is a geodesic joining x and y. $\square$

2 Midpoints

In this section, (X, d) denotes a metric space.

2.1 Lemma. *Let $x, y \in X$. Let $f : [0, d_0] \to X$ be the unique geodesic from x to y. If σ is an automorphism of (X, d) that interchanges x and y, then $\sigma(f(t)) = f(d_0 - t)$. The automorphism σ fixes a single point of* $\mathsf{Image}(f)$, *namely $f(d_0/2)$.*

Proof. Note that $\sigma^{-1} f^{op}$ and f are both geodesics from x to y, so $f^{op} = \sigma f$, that is

$$\sigma(f(t)) = f(d_0 - t).$$

So σ fixes $f(d_0/2)$. On the other hand, if $\sigma(f(t)) = f(t)$, then

$$\begin{aligned} t &= d(f(0), f(t)) \\ &= d(\sigma(f(0), \sigma(f(t)) \\ &= d(f(d_0), f(t)) \\ &= (d_0 - t). \end{aligned}$$

So $t = d_0/2$. $\square$

2.2 Definition (midpoint)**.** Let $x, y \in X$. We say that $z \in X$ is a *midpoint* of x and y if

$$d(x, z) = d(z, y) = d(x, y)/2.$$

When two points x and y have a unique midpoint, it will be denoted by $m(x, y) = m_d(x, y)$.

2.3 Lemma. *Let (X, d) be a geodesic space. If $x, y \in X$ and m is a midpoint of x and y, then there is a geodesic from x and y that passes through m. In particular, if there is a the unique geodesic f joining x and y, then $f(d(x, y)/2)$ is the unique midpoint of x and y.*

Proof. Follows from lemma 1.9. $\square$

2.4 Theorem. *(a) Suppose (X, d) is a complete metric space. Suppose every pair of points in X has a unique midpoint. Then there is a unique geodesic joining any two points in X.*

(b) Suppose d and d' are two metric on X such that given any $x, y \in X$, there exists a unique midpoint $m_d(x, y) = m_{d'}(x, y)$. Then d and d' determine the same geodesics.

Proof. Let $I = \{l.q \colon q \in [0, 1] \text{ is a diadic rational}\}$. Let $x, y \in X$, $l = d(x, y)$. Suppose $f : [0, l] \to X$ is a geodesic joining $x = f(0)$ and $y = f(l)$. Then $f(\frac{r+s}{2})$ is the midpoint of $f(r)$ and $f(s)$ for all $0 < r < s < l$. In particular $f|_I$ is determined by the endpoints x and y by taking successive midpoints. Since f is continuous, f is determined by $f|_I$. This proves the uniqueness of geodesic and part (b) and also suggests that we may construct the geodesic by taking successive midpoints.

Given $x, y \in X$ with $l = d(x,y)$, let $f(0) = x$, $f(l) = y$. We shall inductively define $f(t)$ for every $t \in I$. Having defined $\{f(0), \cdots, f(\frac{kl}{2^n}), \cdots, f(l)\}$, we let

$$f(\tfrac{(2k+1)l}{2^{n+1}}) = m(f(\tfrac{kl}{2^n}, f(\tfrac{(k+1)l}{2^n})).$$

One verifies that $d(f(\frac{kl}{2^n}), f(\frac{(k+1)l}{2^n})) = \frac{l}{2^n}$ for all n, k.

Let $r, s \in I$ and $r \leq s$. Write $r = \frac{jl}{2^n}$ and $s = \frac{kl}{2^n}$ with $0 \leq j \leq k \leq 2^n$. Then

$$\begin{aligned} d(f(r), f(s)) &= d(f(\tfrac{jl}{2^n}), f(\tfrac{kl}{2^n})) \\ &\leq \sum_{i=j}^{k-1} d(f(\tfrac{il}{2^n}), f(\tfrac{(i+1)l}{2^n})) \\ &= \tfrac{(k-j)l}{2^n} \\ &= (s-r). \end{aligned}$$

Now 1.7 implies that $d(f(r), f(s)) = (s - r)$ for all $r, s \in I$, $r < s$. Since X is a complete metric space $f : I \to X$ extends uniquely to a continuous function $f : [0, l] \to X$ such that $d(f(r), f(s)) = (s - r)$ for all $0 \leq r \leq s \leq l$. So f is a geodesic from x to y. $\square$

The lemma below generalizes 1.9 and gives a condition for a point to be close to a geodesic.

2.5 Lemma ([BH], p 30)**.** *Let X be a proper uniquely geodesic space. Let $x, y \in X$. Given any $\epsilon > 0$, there exists $\eta > 0$ such that if $u \in X$ such that $d(x,u) + d(u,y) < d(x,y) + \eta$, then $d(u, [x,y]) < \epsilon$.*

Proof. Let $S_\epsilon = \{p \in X \colon d(p, [x,y]) = \epsilon\}$. The function $p \mapsto d(p, [x,y])$ is continuous by 1.2, so S_ϵ is closed. Also S_ϵ is bounded (e.g. $S_\epsilon \subseteq B_{d(x,y)+2\epsilon}(x)$). Since X is proper, S_ϵ is compact. Let η be the minimum value of the function

$$f(z) := d(x,z) + d(z,y) - d(x,y)$$

on S_ϵ. Since X is uniquely geodesic, lemma 1.9 implies that $\eta > 0$.

Suppose $u \in X$ such that $d(u, [x,y]) \geq \epsilon$. Since distance from $[x,y]$ measured along a geodesic is a continuous function, any geodesic joining x and u intersects S_ϵ. Let v be an intersection point. Then 1.9 implies that $d(x,u) = d(x,v) + d(v,u)$. So

$$\begin{aligned} d(x,u) + d(u,y) &= d(x,v) + d(v,u) + d(u,y) \\ &\geq d(x,v) + d(v,y) \\ &\geq d(x,y) + \eta \end{aligned}$$

where we have the last inequality since $v \in S_\epsilon$. $\square$

2.6 Lemma. *Let X be a proper uniquely geodesic space. Let $x, y \in X$. Let $d_0 = d(x,y)/2$. Given any $\epsilon > 0$, there exists $\delta > 0$ such that if $m' \in X$ is such that $d(x,m')$ and $d(m',y)$ belong to $[d_0 - \delta, d_0 + \delta]$, then $d(m', m(x,y)) < \epsilon$.*

Sketch of proof. Let $m = m(x, y)$. Suppose $m' \in X$ such that $d(x, m')$ and $d(m', y)$ are both close to d_0. We need to show that m' is close to m. Since $d(x, m') + d(m', y) \approx d(x, y)$, lemma 2.5 implies that there exists $p \in [x, y]$ such that $d(m', p)$ is small. Since $d(x, m') \approx d_0$, we get $d(x, p) \approx d_0$. Since p and m are both on $[x, y]$, we have $d(p, m) = |d(x, p) - d(x, m)|$. It follows that p and m are close, so m and m' are close. □

Lemma 2.6 gives us the following theorem:

2.7 Theorem. *Let X be a proper uniquely geodesic metric space. Then the midpoint function $m : X \times X \to X$ is continuous.*

2.8 Definition. Let (X, d) be a metric space. We say that $A \subseteq X$ is *convex* if any two points in A can be joined by a geodesic and the image of any such geodesic lies in A.

2.9 Lemma. *Let X be a uniquely geodesic metric space. Then a closed subset A of X is convex if and only if A is closed under taking midpoints, that is, $m(x, y) \in A$ for all $x, y \in A$.*

Proof. Suppose A is a closed subset of X and A is closed under taking midpoints. Let $x, y \in A$. Let $f : [0, d_0] \to A$ be the parametrized geodesic joining x and y. Then

$$f((i + j)/2) = m(f(i), f(j)),$$

so $f(i), f(j) \in A$ implies $f((i+j)/2) \in A$. Since $f(0) = x$ and $f(d_0) = y$ belong to A, by induction it follows that $f(kd_0/2^n) \in A$ for all $k, n \in \mathbb{N}$, $k/2^n \leq 1$. Since f is continuous, the image of f is the closure of the set of points of the form $f(kd_0/2^n)$. □

2.10 Theorem. *In a proper uniquely geodesic metric space, closure of a convex set is convex.*

Proof. Let X be proper, uniquely geodesic. Let $A \subseteq X$ be convex. Let $x, y \in A^{cl}$. By 2.9, it suffices to show that $m(x, y) \in \mathrm{cl}(A)$. Fix $\epsilon > 0$. Using theorem 2.7, find a $\delta > 0$ such that $d(x, x') < \delta$ and $d(y, y') < \delta$ implies $d(m(x, y), m(x', y')) < \epsilon$. Since $x, y \in \mathrm{cl}(A)$, we can choose $x', y' \in A$ such that $d(x, x') < \delta$ and $d(y, y') < \delta$. Since A is convex, $m(x', y') \in A$ and $d(m(x, y), m(x', y')) < \epsilon$. So $m(x, y) \in \mathrm{cl}(A)$. □

3 The hyperbolic space

3.1 Definition. Let $F=\mathbb{R}$ or $\mathbb{C}$. If $\alpha\in F$, let $\bar{\alpha}$ denote the complex conjugate of α. If $F=\mathbb{R}$, then $\alpha\mapsto\bar{\alpha}$ is the identity map. Let W be a vetor space over F with a map $\langle\ ,\ \rangle: W\times W\to F$ such that

$$\langle \alpha w'+\beta w'', w\rangle = \alpha\langle w', w\rangle + \beta\langle w'', w\rangle \ \text{ and }\ \langle w, w'\rangle = \overline{\langle w', w\rangle}.$$

for all $w, w'w''\in W$ and $\alpha,\beta\in F$. If $F=\mathbb{C}$, then $\langle\ ,\ \rangle$ is a hermitian form. If $F=\mathbb{R}$, then $\langle\ ,\ \rangle$ is really a symmetric bilinear form. In the next few sections, we shall treat these two cases simultaneously and always refer to $\langle\ ,\ \rangle$ as a hermitian form.

Let $(w_1,\cdots,w_k)\in W$. The matrix $((\langle w_i,w_j\rangle))$ is called the *gram matrix* of $(w_1,\cdots,w_n)$ and is denoted by $\mathsf{gram}(w_1,\cdots,w_k)$. If $A\subseteq W$, then we let

$$A^{\perp}=\{w\in W\colon \langle w,a\rangle=0 \ \text{ for all }\ a\in A\}.$$

We write $W^{\perp}=\mathsf{rad}(W)$ and call it the *radical* of $(W,\langle\ ,\ \rangle)$. Say that $(W,\langle\ ,\ \rangle)$ is *non degenerate* if $\mathsf{rad}(W)=0$. The hermitian form on W naturally induces a non-degenerate hermitian form on $W/\mathsf{rad}(W)$.

Assume that W is a F-vector space with a non-degenerate hermitian form $\langle\ ,\ \rangle$. We say that $(W,\langle\ ,\ \rangle)$ has *signature* (m,n) if W has a basis $(w_1,\cdots,w_{m+n})$ whose gram matrix has m positive eigenvalues and n negative eigenvalues. We write $\mathsf{sgn}(W)=(m,n)$. One has

$$\mathsf{sgn}(W_1\oplus W_2)=\mathsf{sgn}(W_1)+\mathsf{sgn}(W_2).$$

Let $w\in W$. The real number $|w|^2=\langle w,w\rangle$ is called the *norm* of w. A non-zero vector of norm zero is called a *null vector*. We let $W_{<}$ (resp. $W_{\leq}$) denote the set of vectors of W having strictly negative (resp. non-positive) norm.

The vector space F^{m+n+1} has the standard hermitian form of signature $(m,n+1)$ given by

$$\langle(\beta_0,\cdots,\beta_{m+n}),(\alpha_0,\cdots,\alpha_{m+n})\rangle = -\sum_{i=0}^{n}\beta_i\bar{\alpha}_i+\sum_{i=n+1}^{m+n}\beta_i\bar{\alpha}_i.$$

We denote this hermitian space by $F^{m,n+1}$. Any hermitian space of signature $(m,n+1)$ is isometric to $F^{m,n+1}$.

For the next few sections, $(V,\langle\ ,\ \rangle)$ will always denote a hermitian F-vector space of signature $(n,1)$. Let $\mathbb{P}(V)$ be the projective space of V and

$$\mathbb{P}: V\setminus\{0\}\to\mathbb{P}(V)$$

be the projection. Let U be the group of linear automorphisms of the hermitian vector space V. Each $g\in U$, induces an automorphism $\mathbb{P}(g):\mathbb{P}(V)\to\mathbb{P}(V)$ given by $\mathbb{P}(g)\mathbb{P}(v)=\mathbb{P}(gv)$. Let $\mathbb{P}U=\{\mathbb{P}(g)\colon g\in U\}$. Suppose $g\in U$ such that $\mathbb{P}(g)=\mathsf{id}_{\mathbb{P}(V)}$. Then for all $v\in V$, we have $gv=\lambda_v v$ for some scalar λ_v. It follows that $gv = \lambda v$ for some constant $\lambda \in F^*$. So $\mathbb{P}U \simeq U/F^*$. One verifies that $\mathbb{P}U$ acts faithfully and transitively on $\mathbb{P}(V_<)$. Define

$$c : \mathbb{P}(V) \times \mathbb{P}(V) \to \mathbb{R} \text{ by } c(\mathbb{P}(u), \mathbb{P}(v)) = \sqrt{\tfrac{\langle u,v\rangle\langle v,u\rangle}{\langle u,u\rangle\langle v,v\rangle}}.$$

3.2 Theorem. *There is a $\mathbb{P}U$ invariant metric on $\mathbb{P}(V_<)$ given by*

$$d(\mathbb{P}(u), \mathbb{P}(v)) = \cosh^{-1}(c(\mathbb{P}(u), \mathbb{P}(v))).$$

Proof. Clearly $d(\mathbb{P}(u), \mathbb{P}(v)) = \cosh^{-1}(c(\mathbb{P}(u), \mathbb{P}(v)))$ is $\mathbb{P}U$ invariant. It remains to check the triangle inequality. Let $v_1, v_2, v_3 \in V_<$. Let $\alpha = \langle v_1, v_2\rangle$, $\beta = \langle v_3, v_1\rangle$, $\gamma = \langle v_2, v_3\rangle$. Without loss, we may scale v_j's to assume that $|v_j|^2 = -1$. Since $\mathsf{Span}\{v_i, v_j\}$ is either singular or indefinite, we have $|\alpha|^2 \geq 1$, $|\beta|^2 \geq 1$, $|\gamma|^2 \geq 1$. Let $M = \mathsf{gram}(v_1, v_2, v_3)$. If v_1, v_2, v_3 are linearly independent then $\mathsf{Span}\{v_1, v_2, v_3\}$ has signature $(2, 1)$ so $\mathsf{det}(M) < 0$, otherwise $\mathsf{det}(M) = 0$.

$$\begin{aligned}
0 &\leq -\mathsf{det}\begin{pmatrix} -1 & \alpha & \bar\beta \\ \bar\alpha & -1 & \gamma \\ \beta & \bar\gamma & -1 \end{pmatrix} \\
&= 1 - |\alpha|^2 - |\beta|^2 - |\gamma|^2 + 2\,\mathsf{Re}(\alpha\beta\gamma) \\
&\leq 1 - |\alpha|^2 - |\beta|^2 - |\gamma|^2 + 2|\alpha\beta\gamma| \\
&= (|\alpha|^2 - 1)(|\beta|^2 - 1) - (|\gamma| - |\alpha||\beta|)^2.
\end{aligned}$$

Equivalently,

$$|\alpha||\beta| + \sqrt{|\alpha|^2 - 1}\sqrt{|\beta|^2 - 1} \geq |\gamma|.$$

The last inequality is equivalent to $(\cosh^{-1}|\alpha| + \cosh^{-1}|\beta|) \geq \cosh^{-1}|\gamma|$. Equality holds if and only if $\mathsf{det}(M) = 0$, that is, v_1, v_2, v_3 are linearly dependent. ☐

3.3 Remark. Given $u, v \in V_<$, sometimes we denote the distance between $\mathbb{P}(u)$ and $\mathbb{P}(v)$ simply by $d(u, v)$. Similarly, we sometimes write $c(u, v) = c(\mathbb{P}(u), \mathbb{P}(v))$ etcetera.

3.4 Definition. The metric space $(\mathbb{P}(V_<), d)$ is called the *hyperbolic space* of V. We let FH^n be the hyperbolic space of $F^{n,1}$. The metric space FH^n is called the n dimensional hyperbolic space over F. Without loss, for the rest of this section, we assume that $V = F^{n,1}$. Let $B_1(F^n)$ denote the unit ball in F^n (with the usual positive definite Euclidean metric). The ball $B_1(F^n)$ has the Euclidean metric topology. We have a bijection $j : \mathbb{P}(V_<) \to B_1(F^n)$ given by

$$j(\mathbb{P}(\alpha_0; \alpha_1, \cdots, \alpha_n)) = (\tfrac{\alpha_1}{\alpha_0}, \cdots, \tfrac{\alpha_n}{\alpha_0}).$$

with inverse given by $j^{-1}(\beta_1, \cdots, \beta_n) = \mathbb{P}(1; \beta_1, \cdots, \beta_n)$. If $g \in \mathbb{P}U$, then g acts on $B_1(F^n)$ by $g|_{B_1(F^n)} = jgj^{-1}$.

3.5 Lemma. *(a) Let $y = \mathbb{P}(1; 0, 0, \cdots, 0)$, so $j(y) = 0 \in B_1(F^n)$. Then $j(B_r(y))$ is equal to the Euclidean ball of radius* $\tanh(r)$ *around* 0 *in* $B_1(F^n)$.

(b) If $g \in \mathbb{P}U$, then $g|_{B_1(F^n)} : B_1(F^n) \to B_1(F^n)$ is a homeomorphism.

Proof. (a) Let $v = (v_0; v_1 \cdots, v_n) \in V_<$. One has $d(\mathbb{P}(v), y) < r$ if and only if

$$\begin{aligned}\cosh^2(r) &> \frac{|v_0|^2}{|v_0|^2 - |v_1|^2 - \cdots |v_n|^2} \\ &= \frac{1}{1 - |j(\mathbb{P}(v))|^2}.\end{aligned}$$

Equivalently, $|j(\mathbb{P}(v))| < (1 - \cosh^{-2}(r))^{1/2} = \tanh(r)$.

(b) Let $\tilde{g} \in U$ such that $\mathbb{P}(\tilde{g}) = g$, i.e., $\mathbb{P}(\tilde{g}v) = g\mathbb{P}(v)$ for all $v \in V$. Let $\tilde{j} : V_< \to B_1(F^n)$ be the map $\tilde{j}(v_0; v_1, \cdots, v_n) = (\frac{v_1}{v_0}, \cdots, \frac{v_n}{v_0})$. Let $\tilde{i} : B_1(F^n) \to V_<$ be the map $\tilde{i}(\beta_1, \cdots, \beta_n) = (1, \beta_1, \cdots, \beta_n)$. Then jgj^{-1} is equal to the composition of the following maps each of which is continuous

$$B_1(F^n) \xrightarrow{\tilde{i}} V_< \xrightarrow{\tilde{g}} V_< \xrightarrow{\tilde{j}} B_1(F^n).$$

So jgj^{-1} is continuous and it has a continuous inverse given by $jg^{-1}j^{-1}$. $\square$

3.6 Remark. Let $\mathcal{T}$ and $\mathcal{T}'$ be two topologies on a set X. Let $a \in X$. Let $N_a(\mathcal{T}) = \{U \in \mathcal{T} : a \in U\}$ be the set of neighborhoods of a in the topology $\mathcal{T}$. If $N_x(\mathcal{T}) = N_x(\mathcal{T}')$ for each $x \in X$, then $\mathcal{T} = \mathcal{T}'$. Suppose G is a transitive group of homeomorpshisms of a topological space $(X, \mathcal{T})$ and let $x_0 \in X$. Since $g \in G$ induces a bijection between $N_{x_0}(\mathcal{T})$ and $N_{gx_0}(\mathcal{T})$, the topology on X is determined by the set of neighborhoods of x_0.

3.7 Theorem. *Identify $\mathbb{P}(V_<)$ and $B_1(F^n)$ via j. Then,*

(a) the hyperbolic metric d and the Euclidean metric E induce the same topology on $B_1(F^n)$.

(b) A sequence in $B_1(F^n)$ is convergent (resp. bounded) in the metric d if and only if it is convergent (resp. bounded) in the Euclidean metric E.

(c) $(\mathbb{P}(V_<), d)$ is a complete and proper metric space.

Proof. (a) Part (a) of 3.5 shows that the 0 has the same set of neighborhoods in both topology. The group $\mathbb{P}U$ is a transitive group of homeomorphisms of $B_1(F^n)$ with hyperbolic topology, since the hyperbolic metric is $\mathbb{P}U$ invariant. Now 3.5(b) shows that $\mathbb{P}U$ is also a transitive group of homeomorphisms of $B_1(F^n)$ with the Euclidean topology. Part (a) follows by remark 3.6.

(b) Part (b) is immediate from part (a).

(c) Suppose $\{x_n : n \in \mathbb{N}\}$ is Cauchy with respect to d. Then $\{x_n\}$ is bounded with respect to d and hence E. By Bolzano-Weierstrass theorem, there exists a subsequence $\{x_{n_k}\}$ that converges with respect to E in $B_1(F^n)$, hence it converges with respect to d. Since $\{x_n\}$ is Cauchy with respect to d and has a convergent subsequence, it converges in d. $\square$

3.8 Definition (points at infinity)**.** We want to extend the topology on $\mathbb{P}(V_<)$ to a topology on $\mathbb{P}(V_\leq)$. If $x \in V \setminus \{0\}$ such that $x^2 = 0$, then the the sets of the form $\{\mathbb{P}(x)\} \cup \mathbb{P}\{y \in V_< : |\langle x, y\rangle| / \sqrt{-|y|^2} < r\}$ are declared to be a local basis at $\mathbb{P}(x)$. This defines a topology on $\mathbb{P}(V_\leq)$ containing $\mathbb{P}(V_<)$ as an open dense subset and its boundary $\partial\mathbb{P}(V) = \mathbb{P}(V_\leq) \setminus \mathbb{P}(V_<)$ as a totally disconnected subset. The elements of $\partial\mathbb{P}(V_<)$ are called the points at infinity or cusps of the hyperbolic space $\mathbb{P}(V_<)$ We shall discuss cusps more in the section on horoballs.

The homeomorphism $j : \mathbb{P}(V_<) \to B_1(F^n)$ defined in 3.4 extends to a give a bijection j_1 from $\mathbb{P}(V_\leq)$ to the closed unit ball in F^n. However, j_1 is not a homeomorphism since the points at infinity have different set of neighborhoods.

3.9 Remark. The space $\mathbb{C}H^1$ (called the complex hyperbolic line) and $\mathbb{R}H^2$ (called the real hyperbolic plane) are isometric upto scaling. Recall that $\mathbb{C}H^1 = \mathbb{P}(\mathbb{C}^{1,1}_<)$, where $\mathbb{C}^{1,1}$ be a two dimensional complex hermitian space with the form $\langle (x_0, x_1), (y_0, y_1)\rangle = x_1\bar{y}_1 - x_0\bar{y}_0$. Let $B_1(\mathbb{C})$ be the unit ball in $\mathbb{C}$. We have a bijection $j : \mathbb{C}H^1 \to B_1(\mathbb{C})$ given by $j(\mathbb{P}x) = x_1/x_0$. Pick $\mathbb{P}(x), \mathbb{P}(y) \in \mathbb{C}H^1$. Let $d = d(\mathbb{P}(x), \mathbb{P}(y))$, $u = x_1/x_0$ and $v = y_1/y_0$. Now

$$\begin{aligned}
\tfrac{1}{2}(\cosh(2d) - 1) &= (\cosh^2 d - 1) \\
&= \tfrac{|x_1\bar{y}_1 - x_0\bar{y}_0|^2}{(|x_0|^2 - |x_1|^2)(|y_0|^2 - |y_1|^2)} - 1 \\
&= \tfrac{|u\bar{v} - 1|^2}{(1-|u|^2)(1-|v|^2)} - 1 \\
&= \tfrac{|u-v|^2}{(1-|u|^2)(1-|v|^2)}
\end{aligned}$$

So $2d = \cosh^{-1}\bigl(1 + \frac{2|u-v|^2}{(1-|u|^2)(1-|v|^2)}\bigr)$. This is the standard metric on the hyperbolic plane in the Poincare disc model (see Wikipedia).

One can see the isometry without computation. Note that $PU(1,1)$ acts transitively on both $\mathbb{C}H^1$ (linearly) and the Poincare disc $B_1(\mathbb{C})$ (by linear fractional transformations). One can show that the metric on $\mathbb{C}H^1$ and $B_1(\mathbb{C})$ are the unique (upto scale) metic on these two spaces invariant under the $PU(1,1)$ action (How?). Since j is $PU(1,1)$ equivariant, it must be an isomorphism of metric spaces (upto scale).

4 Geodesics and midpoints in hyperbolic space

Recall that V denote a hermitian F-vector space of signature $(n, 1)$ (see 3.1).

4.1 Definition (reflections)**.** Let $s \in V$ such that $s^2 \neq 0$ and $\alpha \in F$ such that $|\alpha| = 1$. define $R_s^\alpha : V \to V$ by

$$R_s^\alpha(x) = x - (1 - \alpha)|s|^{-2}\langle x, s\rangle s.$$

Then R_s^α is an automorphism of V which fixes the hyperplane $s^\perp$ pointwise. The automorphism of $\mathbb{P}(V_<)$ induced by R_s^α will also be denoted by the same symbol. If $s^2 > 0$ and α is a root of unity, then we say that R_s^α, is the α-*reflection* in the vector s. The hyperplane $s^\perp$ (or $\mathbb{P}(s^\perp_<)$) is called the *mirror* of this reflection. We write $R_s = R_s^{(-1)}$; these are called real reflections. The automorphism group of V is generated by these reflections (give reference or include proof).

4.2 Theorem. *(a) Let $x, y, z \in V_<$ such that $c(x, y) = c(x, z)$. Then there exists an automorphism of V that fixes x and takes $\mathbb{P}(y)$ to $\mathbb{P}(z)$.*

(b) For each $d_0 > 0$, the group of automorphisms $\mathbb{P}U$ acts transitively on the set of ordered pairs of points in $\mathbb{P}(V_<)$ that are at distance d_0.

Proof. (a) Without loss, we may assume that $|y|^2 = |z|^2 = -1$. Next, by changing y by a scalar of absolute value 1 if necessary, we may assume that $\langle z, x\rangle = \langle y, x\rangle$. Let $s = y - z$ and $\alpha = -(1 + \langle y, z\rangle)(1 + \langle z, y\rangle)^{-1}$. Note that $s \in x^\perp$, which is positive definite since $|x|^2 < 0$. Now,

$$\begin{aligned}\frac{|s|^2}{\langle y, s\rangle} &= \frac{|y - z|^2}{\langle y, y - z\rangle} \\ &= \frac{-2 - \langle y, z\rangle - \langle z, y\rangle}{-1 - \langle y, z\rangle} \\ &= 1 - \alpha.\end{aligned}$$

So

$$\begin{aligned}R_s^\alpha(y) &= y - (1 - \alpha)|s|^{-2}\langle y, s\rangle s \\ &= y - s \\ &= z.\end{aligned}$$

Also R_s^α fixes x since $\langle s, x\rangle = 0$.

(b) Let (x_0, y_0) and (x, y) such that $d(x, y) = d(x_0, y_0)$. Since $\mathbb{P}U$ is transitive on $\mathbb{P}(V_<)$, there exists $g \in U$ such that $g(x) = x_0$. Since $g(y)$ and y_0 are equidistant from x_0, part (a) implies that there is an automorphism that fixes x_0 and takes $g(y)$ to y_0. □

4.3 Lemma. *Let $y, z \in V_<$, $\mathbb{P}(y) \neq \mathbb{P}(z)$. Let $y_1 = y/\sqrt{-|y|^2}$ and $z_1 = cz/\sqrt{-|z|^2}$ where c is chosen so that $\langle y_1, z_1\rangle \in \mathbb{R}_<$. Then there is a real reflection that interchanges $\mathbb{P}(y)$ and $\mathbb{P}(z)$ and fixes only one point of $\mathbb{P}((Fy + Fz)_>)$, namely $\mathbb{P}(y_1 + z_1)$.*

Proof. By scaling y and z, we may assume, without loss, that $|y|^2 = |z|^2 = -1$ and $\langle y, z\rangle \in \mathbb{R}_{\leq}$. Observe that

$$0 > \mathsf{det}(\mathsf{gram}(y, z)) = 1 - |\langle y, z\rangle|^2.$$

So $\langle y, z\rangle < -1$. Note that $|y \pm z|^2 = 2(-1 \pm \langle y, z\rangle)$, so $|y-z|^2 > 0$ and $|y+z|^2 < 0$.

One verifies that R_{y-z} interchanges y and z. Suppose $c = \mathbb{P}(ry + sz)$ is fixed by R_{y-z}. Then

$$\mathbb{P}(\alpha y + \beta z) = R_{y-z}\mathbb{P}(\alpha y + \beta z) = \mathbb{P}(\alpha z + \beta y).$$

Since $\mathbb{P}(y) \neq \mathbb{P}(z)$, it follows that, $\alpha^2 = \beta^2$. So $c = \mathbb{P}(y \pm z)$, but $(y - z)$ has negative norm. □

4.4 Definition. Let $y \in \mathbb{P}(V_<)$ and $\rho \in \partial\mathbb{P}(V_<)$. A *ray* γ from y to ρ is a path $\gamma : [0, \infty) \to \mathbb{P}(V_<)$ such that $\lim_{t\to\infty} \gamma(t) = \rho$. Such a γ is called a *parametrized geodesic ray* form y to ρ if restriction to γ to any closed interval is a parametrized geodesic.

4.5. Exercise: Define $\gamma : [0, \infty) \to \mathbb{P}(V_<)$ by

$$\gamma_t = \mathbb{P}(\cosh(t); \sinh(t), 0, \cdots, 0).$$

Note that $\lim_{t\to\infty} \gamma(t) = \mathbb{P}(1; 1, 0, 0, \cdots, 0)$. Using $\cosh(s - t) = \cosh s \cosh t - \sinh s \sinh t$, verify that $d(\gamma_s, \gamma_t) = |s - t|$ for all t, s, so γ is a parametrized geodesic ray from $\mathbb{P}(1; 0, 0, \cdots, 0)$ to $\mathbb{P}(1; 1, 0, \cdots, 0)$. (Compare with the parametrization of great circle, which is geodesic in the the sphere).

4.6 Theorem. *Two points in $\mathbb{P}(V_<)$ have a unique midpoint. So $\mathbb{P}(V_<)$ is uniquely geodesic.*

Proof. Let γ_t be the geodesic in 4.5. By 4.2(b), it suffices to show that γ_0 and γ_{2t} has a unique midpoint for any $t > 0$. Note that γ_t is a midpoint of γ_0 and γ_{2t}. Let $v = (v_0; v_1, v_2, \cdots)$ such that $|v|^2 = -1$ and $\mathbb{P}(v)$ is a midpoint of γ_0 and γ_{2t}. The equation $d(\gamma_0, v) = t$ translates into $\cosh^{-1}|v_0| = t$, so after changing v by a root of unity, we may assume that $v_0 = \cosh t$. Since $|v|^2 = -1$, we have

$$|v_1|^2 \leq \sum_{j=1}^{n} |v_j|^2 = |v_0|^2 - 1 = \sinh^2(t).$$

So $|v_1| \leq \sinh t$. The equation $d(\gamma_{2t}, v) = t$ translates into

$$|-\cosh(t)\cosh(2t) + v_1 \sinh(2t)| = \cosh(t),$$

or equivalently $(\cosh(2t) - 2v_1 \sinh(t)) = e^{i\theta}$ for some θ. Rearranging, this equation becomes

$$\cosh^2 t - v_1 \sinh t = (e^{i\theta} + 1)/2. \tag{1}$$

Note that $|\cosh^2 t - v_1 \sinh t| \geq (\cosh^2 t - |v_1| \sinh t) \geq (\cosh^2 t - \sinh^2 t) = 1$. So the absolute value of left hand side of (1) is alteast 1 while that of the right hand side is atmost 1. Equality can hold if and only if $e^{i\theta} = 1$. Now (1) implies $v_1 = \sinh t$. Since $|v|^2 = -1$, we must have $v_2 = v_3 = \cdots = 0$. So $v = \gamma_t$. So γ_t is the unique midpoint of γ_0 and γ_{2t}. □

4.7 Theorem. *Upto scaling, there exists a unique $\mathbb{P}U$-invariant uniquely geodesic, complete metric on $\mathbb{P}(V_<)$.*

Proof. Let d be a $\mathbb{P}U$-invariant uniquely geodesic, complete metric on $\mathbb{P}(V_<)$. Let $\mathbb{P}(x), \mathbb{P}(y) \in \mathbb{P}(V_<)$, $l = d(\mathbb{P}(x), \mathbb{P}(y))$. Without loss assume that $|x|^2 = |y|^2 = -1$ and $\langle x, y\rangle \in \mathbb{R}_\leq$. Let $f : [0, l] \to X$ be the geodesic joining $\mathbb{P}(x)$ and $\mathbb{P}(y)$. Any automorphism that fixes $\mathbb{P}(x)$ and $\mathbb{P}(y)$, must fix f. Given any hyperplane H containing x and y, there is a reflection whose fixed point set is equal to H. So $\mathsf{Image}(f)$ must be contained in any such hyperplane. So $\mathsf{Image}(f) \subseteq \mathbb{P}((Fx + Fy)_>)$.

By 4.3, there is an automoprhism g that interchanges $\mathbb{P}(x)$ and $\mathbb{P}(y)$ and fixes just one point of $\mathbb{P}((Fx + Fy)_>)$, namely $\mathbb{P}(x + y)$. On the other hand, by uniqueness of geodesic, g fixes only one point of $\mathsf{Image}(f)$; the midpoint $f(l/2)$. It follows that $f(l/2) = \mathbb{P}(x + y)$ must be the unique mid-point of $\mathbb{P}(x)$ and $\mathbb{P}(y)$. So 2.4 implies that $\{\mathbb{P}((l - t)x + ty) \colon t \in [0, l]\}$ must be the image of the geodesic segment joining $\mathbb{P}(x)$ and $\mathbb{P}(y)$.

Now suppose d_1 and d_2 be two $\mathbb{P}U$ invariant uniquely geodesic metric on $\mathbb{P}(V_<)$. Let $l_j = d_j(x, y)$ and $\gamma_j : [0, l_j] \to X$ be the geodesic from x to y with respect to d_j. Since d_1 and d_2 determine the same midpoints, we find $\gamma_1(ql_1) = \gamma_2(ql_2)$, for all diadic rational $q \in [0, 1]$, By continuity, it follows that $\gamma_1(tl_1) = \gamma_2(tl_2)$ for all $t \in [0, 1]$. Let $c = l_2/l_1$. Since γ_1 and γ_2 are geodesics, we have

$$\begin{aligned} d_2(\gamma_1(r), \gamma_1(s)) &= d_2(\gamma_2(cr), \gamma_1(cs)) \\ &= c(s - r) \\ &= cd_1(\gamma_1(r), \gamma_1(s)), \end{aligned}$$

that is, d_1 and d_2 agree upto scale along any geodesic segment.

Fix $x \in V_<$ and $z \in V$ with $|z|^2 = 0$, so that $\langle x, z\rangle \in \mathbb{R}_\geq$. Let $\Gamma = \{\mathbb{P}(x + tz) \colon t \in \mathbb{R}_\geq\}$ be the infinite geodesic ray from $\mathbb{P}(x)$ to the cusp $\mathbb{P}(z)$. Since d_1 and d_2 agree on any geodesic segment upto scale, it follows that d_1 and d_2 agree on Γ upto scale. Without loss, assume that d_1 and d_2 agree on Γ. Since d_1 and d_2 are $\mathbb{P}U$ invariant, theorem 4.2(b) implies that d_1 and d_2 agrees everywhere. ◻

4.8 Theorem. *Let $x, y \in \mathbb{P}(V_<)$, $|x|^2 = |y|^2$ and $\langle x, y\rangle \in \mathbb{R}_<$. Then one has the following:*

(a) $\mathbb{P}(x + y)$ is the midpoint of $\mathbb{P}(x)$ and $\mathbb{P}(y)$.

(b) $\{\mathbb{P}(tx + (1 - t)y) \colon t \in [0, 1]\}$ is the image of the geodesic joining $\mathbb{P}(x)$ and $\mathbb{P}(y)$.

(c) For each cusp $\mathbb{P}(\rho)$, there is a unique geodesic ray joining $\mathbb{P}(\rho)$ and $\mathbb{P}(x)$. If $\langle x, \rho\rangle \in \mathbb{R}_<$, then the image of this geodesic ray is $\{\mathbb{P}(x + t\rho) \colon t \in [0, \infty)\}$.

Proof. Proof of part (a) is contained in the proof of 4.7. Part (b) follows from part (a) and part (c) follows from part (b). ◻

5 Projections onto subspaces and some distance formulae

Recall that V denote a hermitian F-vector space of signature $(n,1)$ (see 3.1).

5.1 Definition. Given $x \in V$, define $h_x : \mathbb{P}(V_<) \to \mathbb{R}_\geq$ by

$$h_x(\mathbb{P}(a)) = |\langle x, a\rangle|^2/(-|a|^2).$$

If $a \in V \setminus \{0\}$, then $h_x(a) = h_x(\mathbb{P}(a))$. Let $y \in \mathbb{P}(V_<)$ and $A \subseteq \mathbb{P}(V_<)$. We say that $p \in \mathrm{cl}(A)$ is the *projection* of y onto A if the function $(a \mapsto d(y,a))$ has a unique minima on A^{cl} attained attained at p. The projection of y onto A, if it exists, will be denoted by $p = \mathsf{pr}_A(y)$.

5.2 Lemma. *Let $a, b, \tau \in V$ with $|a|^2, |b|^2 < 0$ and $\tau \neq 0$. Assume that $\langle \tau, a\rangle \neq 0$ or $\langle \tau, b\rangle \neq 0$* [1]*. Let $\gamma(s)$, $s \in [0, d(a,b)]$ be the parametrized geodesic ray joining $\mathbb{P}(a)$ and $\mathbb{P}(b)$. Then $h_\tau(\gamma(s))$ is a strictly convex C^∞ function of s.*

Proof. Assume $\mathbb{P}(a) \neq \mathbb{P}(b)$. Without loss we may assume $|a|^2 = |b^2| = -1$ and $t = \langle a, b\rangle$ is a non-positive real number. Since the $\mathsf{span}\{a,b\}$ has signature $(1,1)$, we have $|a|^2|b^2| - |\langle a,b\rangle|^2 < 0$, so $t < -1$. Since $a+b$ represents the midpoint of $\mathbb{P}(a)$ and $\mathbb{P}(b)$, it suffices to check that $(h_\tau(a) + h_\tau(b))/2 > h_\tau(a+b)$. Now

$$\begin{aligned}
h_\tau(a) + h_\tau(b) - 2h_\tau(a+b) &= |\langle \tau, a\rangle|^2 + |\langle \tau, b\rangle|^2 + 2\frac{|\langle \tau, a\rangle + \langle \tau, b\rangle|^2}{|a+b|^2} \\
&= |\langle \tau, a\rangle|^2 + |\langle \tau, b\rangle|^2 + \frac{|\langle \tau, a\rangle + \langle \tau, b\rangle|^2}{t-1} \\
&= \frac{t|\langle \tau, a\rangle|^2 + t|\langle \tau, b\rangle|^2 + 2\,\mathsf{Re}(\langle a, \tau\rangle\langle \tau, b\rangle)}{t-1} \\
&= \frac{(t+1)(|\langle \tau, a\rangle|^2 + |\langle \tau, b\rangle|^2) - |\langle \tau, a\rangle - \langle \tau, b\rangle|^2}{t-1}
\end{aligned}$$

In the final expression, the numerator and denominator are both negative. ☐

5.3 Lemma. *Let W be a Lorentzian subspace of V. Let $a \in W_<$. Then each element of $\mathbb{P}(W_<)$ can be represented in the form $\mathbb{P}(a+w)$ for some $w \in W_> \cup \{0\}$ and $\langle a, w\rangle = 0$.*

Proof. Let $(a, a_2, \cdots, a_m)$ be a orthogonal basis of W. Let $\beta \in \mathbb{P}(W_<)$. Then we can write $\beta = \mathbb{P}(ca + c_2a_2 + \cdots c_m a_m)$. Since $|ca + c_2a_2 + \cdots c_m a_m|^2 < 0$ and $\mathsf{span}\{a_2, \cdots, a_m\}$ is positive definite, c must be nonzero. So $\beta = \mathbb{P}(a+w)$ where $w = (c_2a_2 + \cdots + c_m a_m)/c$. ☐

The linear algebra lemma below will be used freely in the subsequent computations without explicit reference.

[1] Note that this assumption is only required if $|\tau|^2 > 0$.

5.4 Lemma. *(a) Let W be a subspace of V. Then one has $W = W^{\perp\perp}$ and $\mathsf{rad}(W) = W \cap W^{\perp} = \mathsf{rad}(W^{\perp})$. If either W or $W^{\perp}$ is a definite, then $V = W \oplus W^{\perp}$.*

(b) Let $0 \neq W$ be a proper subspace V of dimension k. Then one of the following three exclusive possibilities hold:

(i) $\mathbb{P}(W)$ meets the hyperbolic space $\mathbb{P}(V_<)$. In this case $\mathsf{sgn}(W) = (k-1, 1)$ and $\mathsf{sgn}(W^{\perp}) = (n-k+1, 0)$.

(ii) $\mathbb{P}(W)$ does not meet $\mathbb{P}(V_{\leq})$. In this case $sgn(W) = (k, 0)$ and $\mathsf{sgn}(W^{\perp}) = (n-k, 1)$.

(iii) $\mathbb{P}(W)$ meets $\mathbb{P}(V_{\leq})$ but does not meet $\mathbb{P}(V_<)$. In this case both $W \cap W^{\perp}$ is the one dimensional radical of W and $W^{\perp}$.

In case (i) and (ii), the restriction of of the hermitian form to W and $W^{\perp}$ are non-degenerate, $V = W \oplus W^{\perp}$ and $W^{\perp\perp} = W$.

Proof. (a) One has $\mathsf{dim}(W) + \mathsf{dim}(W^{\perp}) = \mathsf{dim}(V)$ for any subspace W since the hermitiam form on V is non-degenerate. So $\mathsf{dim}(W) = \mathsf{dim}(W^{\perp\perp})$. But clearly $W \subseteq W^{\perp\perp}$, so we have $W = W^{\perp\perp}$. It follows that $\mathsf{rad}(W) = \mathsf{rad}(W^{\perp})$. Suppose Either W or $W^{\perp}$ is definite. Then we must have $W \cap W^{\perp} = 0$, so $V = W \oplus W^{\perp}$.

(b) (i) Suppose $\mathbb{P}(W)$ meets $\mathbb{P}(V_<)$. Pick $v \in W$ such that $|v|^2 < 0$. Part (a) implies that $v^{\perp}$ is positive definite. So $W^{\perp} \subseteq v^{\perp}$ is also positive definite. So $\mathsf{sgn}(W^{\perp}) = (n+1-k, 0)$. Part (a) now implies that $V = W \oplus W^{\perp}$. So

$$\mathsf{sgn}(W) = \mathsf{sgn}(V) - \mathsf{sgn}(W^{\perp}) = (k-1, 1).$$

(ii) Suppose $\mathbb{P}(W)$ does not meet $\mathbb{P}(V_<)$. Then W is positive definite, so $\mathsf{sgn}(W) = (k, 0)$ and part (a) implies that $V = W \oplus W^{\perp}$. So

$$\mathsf{sgn}(W^{\perp}) = \mathsf{sgn}(V) - \mathsf{sgn}(W) = (n-k, 1).$$

(iii) Let ρ be a null vector in V. Since the form on V is non-degenerate, we can pick a vector $x \in V$ such that $\langle \rho, x \rangle \neq 0$. Then $\rho' = x - |x|^2 \langle x, \rho \rangle^{-1} \rho$ has norm 0. and $\langle \rho, \rho' \rangle \neq 0$. After changing ρ' by a scalar, we may assume $\langle \rho, \rho' \rangle = 1$. Let $H = F\rho + R\rho'$. Then H has signature $(1,1)$, so $V = H \oplus H^{\perp}$ (by part (b)(i)). Now it follows that $\rho^{\perp} = H^{\perp} + F\rho$. Suppose $\mathbb{P}(W)$ meets $\mathbb{P}(V_{\leq})$ but does not meet $\mathbb{P}(W)$. Then W contains a null vector ρ. Then

$$\rho F \subseteq W^{\perp} \subseteq \rho^{\perp} = H^{\perp} + \rho F.$$

Since $\rho F \subseteq \mathsf{rad}(\rho^{\perp})$, the quotient of $\rho^{\perp}$ by ρF is a hermitian vector space and we have $W^{\perp}/\rho F \subseteq H^{\perp}$. Since $H^{\perp}$ is positive definite, so is $W^{\perp}/\rho F$. So $W^{\perp}$ has a one dimensional radical spanned by ρ. ∎

5.5 Theorem. *Let W be a Lorentzian subspace of V. Let $x \in V$. Assume that either (i) $|x|^2 \leq 0$ and $x \notin W$, or (ii) $|x|^2 > 0$ and $\mathbb{P}(x^{\perp}_{\leq})$ and $\mathbb{P}(W_{\leq})$ does not intersect. Then*

(a) $(W^{\perp} + Fx) \cap W$ is a negative definite vector space of dimension one.

(b) The restriction of h_x to $\mathbb{P}(W_<)$ has a unique minima at the point $\mathbb{P}((W^{\perp} + Fx) \cap W)$.

Proof. (a) Note that either $x^\perp$ is does not have a negative norm vector or $x^\perp$ does not meet $W_<$. In either case $x^\perp$ does not contain W. So $x \notin W^\perp$ and hence $\mathsf{dim}(W^\perp + Fx) = \mathsf{dim}(W^\perp) + 1$. Since $W \oplus W^\perp = V$, it follows that $(W^\perp + Fx) \cap W$ is one dimensional. This one dimensional subspace contains a non-zero vector of the form $a = (x + v)$ where $v \in W^\perp$. Observe that $\langle a, v\rangle = 0$, so $\langle x, v\rangle = -|v|^2$. Also

$$\begin{aligned} |a|^2 &= \langle x + v, a\rangle \\ &= \langle x, a\rangle \\ &= \langle x, x + v\rangle \\ &= |x|^2 + \langle x, v\rangle \\ &= |x|^2 - |v|^2. \end{aligned}$$

Assume (i). Since $x \notin W$, we must have $v \neq 0$, so $|v|^2 > 0$ and $|a|^2 < 0$.

Now assume (ii). Then $(Fx + W^\perp)^\perp = x^\perp \cap W$ is positive definite. So $(Fx + W^\perp)$ is indefinite. So there exists $r_1 \in W^\perp$ such that $|r_1|^2 = 1$ and $\mathsf{Span}\{x, r_1\}$ has signature $(1, 1)$. Extend r_1 to a orthogonal basis $\{r_1, \cdots, r_m\}$ for $W^\perp$ such that $|r_i|^2 = 1$ for all i. Then remembering $\langle a, r_i\rangle = 0$ for all i, we get $a = x - \sum\langle x, r_i\rangle r_i$ and

$$\begin{aligned} -|a|^2 &= -\langle x, a\rangle \\ &= -|x|^2 + \sum_{i=1}^{m} |\langle r_i, x\rangle|^2 \\ &= (|\langle r_1, x\rangle|^2 - |r_1|^2|x|^2) + \sum_{i=2}^{m} |\langle r_i, x\rangle|^2. \end{aligned}$$

The final expression is strictly positive, since $\mathsf{Span}\{r_1, x\}$ has signature $(1, 1)$.

(b) Lemma 5.3 implies that an arbitrary element of $\mathbb{P}(W_<)$ can be represented in the form $\mathbb{P}(a + w)$ for some $w \in W_> \cup \{0\}$ and $\langle a, w\rangle = 0$. Now $\langle x, w\rangle = \langle x + v, w\rangle = \langle a, w\rangle = 0$. So

$$\begin{aligned} h_x(\mathbb{P}(a + w)) &= \tfrac{|\langle x, a+w\rangle|^2}{-|a+w|^2} \\ &= \tfrac{|\langle x, a\rangle|^2}{(-|a|^2) - |w|^2} \\ &= \tfrac{|a|^4}{(-|a|^2) - |w|^2}. \end{aligned}$$

Since $w \in W_> \cup \{0\}$, it follows that h_x has a unique minima at $w = 0$. □

5.6 Corollary. *Let $x \in V_\leq$. Let $\{r_1, \cdots, r_m\}$ be a orthogonal basis for a positive definite subspace of V. Let $W = r_1^\perp \cap \cdots \cap r_m^\perp$. Then the projection of $\mathbb{P}(x)$ onto $\mathbb{P}(W_<)$ is given by $\mathbb{P}(\mathsf{pr}_W(x))$ where*

$$\mathsf{pr}_W(x) = x - \sum |r_i|^{-2} \langle x, r_i\rangle r_i.$$

is simply the linear projection of the vector x onto the subspace W.

Proof. Verify that $(x - \sum |r_i|^{-2}\langle x, r_i\rangle r_i)$ belongs to $W \cap (Fx + W^\perp)$. Now the result follows from 5.5. $\square$

5.7 Lemma. *(a) Let $v, y \in V$ with $|v|^2 \neq 0$ and $|y|^2 = 1$. Let $a = v - \langle v, y\rangle y$. Then*

$$\tfrac{|\langle v,a\rangle|^2}{|v|^2|a|^2} = \tfrac{|a|^2}{|v|^2} = 1 - \tfrac{|\langle v,y\rangle|^2}{|v|^2}.$$

(b) If $v \in V_<$ and $y \in V_>$, then $c(v, \mathsf{pr}_{y^\perp}(v))^2 = 1 - c(v, y)^2$.

Proof. Note that $\langle y, a\rangle = 0$. So $|a|^2 = \langle v, a\rangle = |v|^2 - |\langle v, y\rangle|^2$. Part (a) follows. Part (b) follows from part (a) since we can assume without loss that $|y|^2 = 1$ and then $\mathsf{pr}_{y^\perp}(v) = v - \langle v, y\rangle y$. $\square$

5.8 Theorem. *(a) Let $r \in V_>$ and $x \in V_< \setminus r^\perp$. Then*

$$d(\mathbb{P}(x), \mathbb{P}(r^\perp_<)) = \sinh^{-1}\sqrt{-c(x,r)^2}.$$

(b) Let $r, x \in V_>$ such that $\mathsf{span}\{r, x\}$ is Lorentzian. Then

$$d(\mathbb{P}(r^\perp_<), \mathbb{P}(x^\perp_<)) = \cosh^{-1} c(r, x).$$

Proof. Theorem 5.5 implies that $d(x, r^\perp) = d(x, \mathsf{pr}_{r^\perp}(x))$. One has

$$\begin{aligned}\sinh^2 d(x, r^\perp) &= \cosh^2 d(x, r^\perp) - 1\\ &= \cosh^2 d(x, \mathsf{pr}_{r^\perp}(x)) - 1\\ &= c(x, \mathsf{pr}_{r^\perp}(x))^2 - 1\\ &= -c(x, r)^2.\end{aligned}$$

(b) Without loss assume $|x|^2 = |r|^2 = 1$. Let $v \in r^\perp_<$. The distance between $\mathbb{P}(v)$ and $\mathbb{P}(x^\perp_<)$ is minimized uniquely at $\mathsf{pr}_{x^\perp}(v)$. So

$$\begin{aligned}\cosh^2 d(v, x^\perp) &= c(v, \mathsf{pr}_{x^\perp}(v))^2\\ &= 1 - c(v, x)^2\\ &= 1 + h_x(v).\end{aligned}$$

Theorem 5.5 (under assumption (ii)) implies that h_x has a unique minima on $\mathbb{P}(r^\perp_<)$ at $\mathbb{P}((Fr + Fx) \cap r^\perp) = \mathbb{P}(x - \langle x, r\rangle r)$. So $\inf\{d(\alpha, \mathbb{P}(x^\perp_<)) \colon \alpha \in \mathbb{P}(r^\perp_<)\}$ is attained at the unique point $\mathbb{P}(a)$ where $a = x - \langle x, r\rangle r$. So

$$\begin{aligned}\cosh^2 d(x^\perp, r^\perp) &= 1 - c(a, x)^2\\ &= 1 - \tfrac{|\langle x,a\rangle|^2}{|x|^2|a|^2}\\ &= 1 - (1 - \tfrac{|\langle x,r\rangle|^2}{|x|^2})\\ &= c(x, r)^2.\end{aligned}$$

where the third equality uses part (a) of lemma 5.7. Part (b) follows. $\square$

6 Horoballs

6.1 Definition. Let $\rho \in V$ be a null vector. Define $d_\rho : \mathbb{P}(V_<) \to \mathbb{R}$ by

$$d_\rho(\mathbb{P}(x)) = \log(h_\rho(x))^{1/2} = \log(|\langle \rho, x\rangle|/(-|x|^2)^{1/2}).$$

If $x \in V_<$, we let $d_\rho(x) = d_{\bar\rho}(\mathbb{P}(x))$. The set

$$B_k(\rho) = \{\mathbb{P}(x) \in \mathbb{P}(V_<) \colon h_\rho(x) < k^2\}$$

is called an *open horoball centered at* ρ.

6.2 Remark. Let $\bar\rho$ be a cusp of $\mathbb{P}(V_<)$. Note that if ρ and ρ_1 are two null vectors such that $\mathbb{P}(\rho) = \mathbb{P}(\rho_1) = \bar\rho$, then d_ρ and d_{ρ_1} differ by a constant, so $((x, y) \mapsto (d_\rho(x) - d_\rho(y)))$ depends only on the cusp $\bar\rho$ and not on the choice of the lift ρ. So we can write $d_\rho(x) - d_\rho(y) = d_{\bar\rho}(x) - d_{\bar\rho}(y)$. If $d_\rho(x) = k$, then it is convenient to think that the distance of $\mathbb{P}(\rho)$ and x is $(\infty + k)$, so that the difference of the distances $(d_\rho(x) - d_\rho(y))$ is well defined. So $B_k(\rho)$ is a ball or radius $(\infty + k)$ around ρ.

6.3 Theorem. *Let $\bar\rho$ be a cusp of V. Let $x, y \in V_<$. One has*

(a) The triangle inequality for ideal triangles: $|d_{\bar\rho}(x) - d_{\bar\rho}(y)| \le d(x, y)$.

(b) The equality $d_{\bar\rho}(x) - d_{\bar\rho}(y) = d(x, y)$ holds if and only if $\mathbb{P}(y)$ lies on the geodesic ray joining $\bar\rho$ and $\mathbb{P}(x)$.

Proof. (a) Choose ρ such that $\mathbb{P}(\rho) = \bar\rho$. Let $\alpha = \langle \rho, x\rangle$, $\beta = \langle y, \rho\rangle$ and $\gamma = \langle x, y\rangle$. By changing ρ, x, y by units if necessary, we may assume, without loss, that, $|x|^2 = |y|^2 = -1$. If ρ, x, y are linearly independent then their span has signature $(2, 1)$, so $\mathsf{det}(\mathsf{gram}(\rho, x, y)) < 0$, otherwise $\mathsf{det}(\mathsf{gram}(\rho, x, y)) = 0$. So we have

$$\begin{aligned} 0 \ge \mathsf{det}(\mathsf{gram}(\rho, x, y)) = \mathsf{det}\begin{pmatrix} 0 & \alpha & \bar\beta \\ \bar\alpha & -1 & \gamma \\ \beta & \bar\gamma & -1 \end{pmatrix} &= |\alpha|^2 + |\beta|^2 + 2\,\mathsf{Re}(\alpha\beta\gamma) \\ &\ge |\alpha|^2 + |\beta|^2 - 2|\alpha\beta\gamma|. \end{aligned}$$

It follows that

$$\begin{aligned} \cosh d(x, y) = |\gamma| \ge \tfrac{1}{2}\left(\tfrac{|\alpha|}{|\beta|} + \tfrac{|\beta|}{|\alpha|}\right) &= \tfrac{1}{2}(e^{d_\rho(x) - d_\rho(y)} + e^{d_\rho(y) - d_\rho(x)}) \\ &= \cosh|d_\rho(x) - d_\rho(y)|. \end{aligned}$$

Since $(t \mapsto \cosh t)$ is strictly increasing for $t \in [0, \infty)$, the triangle inequality follows.

(b) Suppose $\mathbb{P}(y)$ lies on the geodesic ray joining $\mathbb{P}(\rho)$ and $\mathbb{P}(x)$. Then ρ, x, y are linearly dependent, so the calculation in part (a) show that $d(x, y) = |d_\rho(x) - d_\rho(y)|$. Now, without loss, assume $\langle \rho, x\rangle \in \mathbb{R}_<$ and $|x|^2 = -1$. If $\mathbb{P}(y)$ is on the geodesic ray joining $\mathbb{P}(x)$ and $\mathbb{P}(\rho)$, then 4.8(c) implies that $\mathbb{P}(y) = \mathbb{P}(x + t\rho)$ for some $t \ge 0$. So

$$e^{2d_\rho(y)} = \tfrac{|\langle x, \rho\rangle|^2}{-|x + t\rho|^2} = \tfrac{|\langle x, \rho\rangle|^2}{1 - 2t\langle x, \rho\rangle} < |\langle x, \rho\rangle|^2 = e^{2d_\rho(x)}.$$

So $d_\rho(x) > d_\rho(y)$ and hence $d(x, y) = d_\rho(x) - d_\rho(y)$.

Conversely, suppose y is such that $d_\rho(x) - d_\rho(y) = d(x, y)$. One computes $\cosh^2 d(\mathbb{P}(x + t\rho), \mathbb{P}(x)) = (1 - t\langle x, \rho\rangle)^2/(1 - 2t\langle x, \rho\rangle)$. Note that both the numerator and the denominator are positive and the numerator is quadratic in t while the denominator is linear, so $\cosh^2 d(\mathbb{P}(x + t\rho), \mathbb{P}(x)) \to \infty$ as $t \to \infty$. So we can choose y' such that $\mathbb{P}(y')$ is on the geodesic joining $\mathbb{P}(x)$ and $\mathbb{P}(\rho)$ and $d(x, y') = d(x, y)$. Then

$$d_\rho(y') = d_\rho(x) - d(x, y') = d_\rho(x) - d(x, y) = d_\rho(y).$$

If possible, suppose $\mathbb{P}(y) \neq \mathbb{P}(y')$. Let m be the midpoint of $\mathbb{P}(y)$ and $\mathbb{P}(y')$. Then 5.2 implies that $d_\rho(m) < d_\rho(y)$ and $d(x, m) < d(x, y)$. So using part (a), we have

$$d_\rho(x) \leq d_\rho(m) + d(m, x) < d_\rho(y) + d(y, x) = d_\rho(x).$$

which is a contradiction. So we must have $\mathbb{P}(y) = \mathbb{P}(y')$. □

The lemma 6.4 below shows that a horoball around ρ is the limsup of a sequence of balls as the center of the balls converge to ρ and the radius tends to infinity in an appropriate rate.

6.4 Lemma. *Let ρ be a null vector and $x \in V_<$ such that $\langle \rho, x\rangle = -1$. Let B_ϵ be the ball of radius $\cosh^{-1}(\frac{k}{\sqrt{2\epsilon}})$ centered at $\mathbb{P}(\rho + \epsilon x)$. Then $B_k(\rho) = \lim_{c\to 0} \cup_{\epsilon < c} B_\epsilon$.*

Proof. One has $\mathbb{P}(y) \in B_\epsilon$ if and only if

$$\begin{aligned} k^2 &> 2\epsilon \frac{|\langle y, \rho\rangle + \epsilon\langle y, x\rangle|^2}{|y|^2|\rho + \epsilon x|^2} \\ &= \frac{|\langle y, \rho\rangle + \epsilon\langle y, x\rangle|^2}{|y|^2(\frac{1}{2}\epsilon|x|^2 - 1)} \\ &=: f(\epsilon) \ \ \text{(say)} \end{aligned}$$

Since $0 \leq -\mathsf{det}(\mathsf{gram}(\rho, x, y)) = 2\,\mathsf{Re}(\langle x, y\rangle\langle y, \rho\rangle) + |x|^2|\langle y, \rho\rangle|^2 + |y|^2$, we have

$$\frac{d}{d\epsilon} f(\epsilon)\Big|_{\epsilon=0} = \frac{2\,\mathsf{Re}(\langle x, y\rangle\langle y, \rho\rangle) + \frac{1}{2}|x|^2|\langle y, \rho\rangle|^2}{-|y|^2} \geq \frac{-\frac{1}{2}|x|^2|\langle y, \rho\rangle|^2 - |y|^2}{-|y|^2} \geq 1.$$

Note that $\mathbb{P}(y) \in \lim_{c\to 0} \bigcup_{\epsilon<c} B_\epsilon$ if and only if there exists a sequence $\epsilon_n \to 0$ such that $\mathbb{P}(y) \in B_{\epsilon_n}$ for all n, that is, $k^2 > f(\epsilon_n)$ for all n. Since f is an increasing function at 0, we have $k^2 > f(0) = |\langle y, \rho\rangle|^2/(-|y|^2)$, that is, $\mathbb{P}(y) \in B_k(\rho)$. Conversely, if $\mathbb{P}(y) \in B_k(\rho)$, then $k^2 > f(0)$, so $k^2 > f(\epsilon)$ for ϵ small enough, that is $\mathbb{P}(y) \in \lim_{c\to 0} \bigcup_{\epsilon<c} B_\epsilon$. □

7 Convexity and projections

7.1 Definition (convex hull)**.** Let $a, b \in \mathbb{P}(V_<)$. Recall that $[a, b]$ denotes the real geodesic segment joining a and b. Say that $K \subseteq \mathbb{P}(V_<)$ is *(geodesically) convex*, if $a, b \in K$ implies $[a, b] \subseteq K$.

Let K be a subset of $\mathbb{P}(V_<)$. Since an intersection of convex sets is convex, the intersection of all the convex sets containing K is the smallest convex set containing K; it is called the *convex hull* of K and denoted by $\mathsf{hull}(K)$. Let $K_0 = K$ and define K_n inductively by $K_{n+1} = \cup_{x,y \in K_n}[x, y]$. One verifies that $K_n \subseteq \mathsf{hull}(K)$ and $\cup_n K_n$ is convex, so $\mathsf{hull}(K) = \cup_n K_n$.

7.2 Lemma. *If K is a subspace of V, then $\mathbb{P}(K_<)$ is a convex. The open and closed balls and horoballs are convex.*

Proof. The geodesic joining $\mathbb{P}(a)$ and $\mathbb{P}(b)$ is contained in $\mathbb{P}(\mathsf{span}(a, b))$, So for every subspace K, $\mathbb{P}(K_<)$ is convex. Open balls and horoballs are sets of the form $\{\mathbb{P}(v) \colon h_\tau(v) < r\}$ for some $\tau \in V_\leq$ and $r \in \mathbb{R}_>$. So 5.2 implies that balls and horoballs are convex. $\square$

7.3 Lemma. *Let $a, b \in V_<$, and $\tau \in V \setminus \{0\}$. Assume that $h_\tau(b) > h_\tau(a) > 0$. Let $\gamma : [0, d(a, b)] \to \mathbb{P}(V_<)$ be the geodesic from $\mathbb{P}(a)$ to $\mathbb{P}(b)$.*

(a) The following are equivalent:

(i) $t \mapsto h_\tau(\gamma(t))$ is a strictly increasing function.

(ii) h_τ increases if we move a little bit from $\mathbb{P}(a)$ towards $\mathbb{P}(b)$ along γ, or in other words, $\frac{d}{dt} h_\tau(\gamma(t))|_{t=0} > 0$.

(iii) $\mathsf{Re}\Big(\frac{\langle \tau, b\rangle\langle a, a\rangle}{\langle a, b\rangle\langle \tau, a\rangle}\Big) > 1$.

(b) The function h_τ restricted to γ has a unique maxima at b.

(c) If $c \in V_<$ such that $\mathbb{P}(b)$ lies on the geodesic joining $\mathbb{P}(a)$ and $\mathbb{P}(c)$, then h_τ is strictly increasing along the geodesic going from $\mathbb{P}(b)$ to $\mathbb{P}(c)$. In particular $h_\tau(c) > h_\tau(b)$.

Proof. (a) The equivalence of (i) and (ii) follows from 5.2. Let $b_1 = -\langle b, a\rangle^{-1} b$ so that $\langle a, b_1\rangle = -1$. Now $\mathbb{P}(a + tb_1)$, $t \geq 0$, parametrizes the geodesic ray starting at a and moving towards b. So (ii) is equivalent to $\frac{d}{dt}\big(h_\tau(a + tb_1)\big)\Big|_{t=0} > 0$. One has

$$\frac{d}{dt}\big(h_\tau(a + tb_1)\big)\Big|_{t=0} = \frac{2\,\mathsf{Re}(\langle a, \tau\rangle\langle \tau, b_1\rangle)}{-|a|^2} - \frac{2|\langle \tau, a\rangle|^2}{(-|a|^2)^2}.$$

The quantity on the right is positive if and only if $\mathsf{Re}\Big(\frac{\langle \tau, b_1\rangle\langle a, a\rangle}{\langle \tau, a\rangle(-1)}\Big) > 1$.

(b) Part (b) holds since $h_\tau|_{[a,b]}$ is strictly convex and $h_\tau(a) < h_\tau(b)$.

(c) Since $h_\tau|_{[a,b]}$ attains its maximum at b, the function $h_\tau|_{[a,c]}$ is strictly increasing at b (when we move towards c). Since $h_\tau|_{[a,c]}$ is convex, it remains strictly increasing on $[b, c]$. $\square$

7.4 Remark. Let $a, b \in \mathbb{P}(V_<)$ and γ be the infinite geodesic ray that contains them. Lemma 5.2 implies that the distance function f from a point in the hyperbolic space or from a cusp or from a mirror measured along γ looks like a parabola, attaining its minima at a unique point $m \in \gamma$. Lemma 7.3 lets us decide whether a and b are on the same side of m or not. If they are on the same side of m, then the distance function f is monotone along the geodesic segment $[a, b]$, otherwise it decreases first and then increases.

7.5 Lemma. *Let B be an open ball or a horoball inside $\mathbb{P}(V_<)$. Then the geodesic segment joining a point inside B and a point outside B intersects ∂B at a unique point.*

Proof. Let B be an open ball or horoball centered at a. Let $u \in B$ and $x \in \mathbb{P}(V_<) \setminus B$. Let $\gamma : [0,1] \to \mathbb{P}(V_<)$ be the parametrized geodesic joining u and x. Let $t_0 = \inf\{t \colon \gamma(t) \notin B\}$. Then verify that $v = \gamma(t_0) \in \partial B$. Since $d(a, v) > d(a, u)$ and v lies on the geodesic joining u and x, 7.3(c) implies that the function $d(a, \cdot)$ is strictly increasing along $[v, x]$, so $\gamma(t) \notin \mathrm{cl}(B)$ for any $t \geq t_0$. On the other hand $\gamma(t) \in B$ for all $t \leq t_0$ by definition of t_0. $\square$

7.6 Lemma. *Let C be a convex subset of $\mathbb{P}(V_<)$ and $x \in \mathbb{P}(V_<)$. Then there exists a unique $y \in \mathrm{cl}(C)$ such that $d(x, y) = d(x, C)$. So $y = \mathsf{pr}_C(x)$.*

Proof. Since $d(x, C) = d(x, \mathrm{cl}(C))$ and closure of a convex set is convex (see 2.10), assume without loss that C is closed and convex. Suppose y and y' were two distinct points in C such that $d(x, y) = d(x, y') = d(x, C)$. Since C is convex, the midpoint $m(y, y') \in C$ and 5.2 implies $d(x, m(y, y')) < d(x, C)$. This contradiction proves that y must be unique.

If $x \in C$, then $d(x, C) = 0$, so $x = \mathsf{pr}_C(x)$. Assume, $x \notin C$. Let $d_0 = d(x, C)$. Since the hyperbolic space is a proper metric space, $C_n = \mathrm{cl}(B_{d_0 + 1/n}(x)) \cap C$ are a decreasing sequence of non-empty compact sets for $n \geq 1$. So $\cap_n C_n \neq \emptyset$. If $y \in \cap_n C_n$, then y satisfies $d_0 = d(x, C)$. $\square$

7.7 Remark. If A is a bounded subset of $\mathbb{P}(V_<)$ and C is a convex set, then there exists $y \in \mathrm{cl}(C)$ such that $d(A, y) = d(A, C)$ but such an y need not be unique. For example, let $f : [0, 3] \to \mathbb{P}(V_<)$ be a parametrized geodesic. Let $A = \{f(0), f(3)\}$ and $C = \mathsf{Im}(f|_{[1,2]})$. Then both $y = f(1)$ and $y = f(2)$ satisfy $d(A, y) = d(A, C)$.

The following lemma may sometimes be used to calculate the projection onto a convex set.

7.8 Lemma. *Let C be a convex subset of $\mathbb{P}(V_<)$ and $x \in \mathbb{P}(V_<) \setminus C$. Then any automorphism of $\mathbb{P}(V_<)$ that fixes both C and x also fixes $\mathsf{pr}_C(x)$.*

Proof. Follows from the uniqueness of the projection. $\square$

7.9 Theorem. *Let $a \in V_\leq$, let B be a ball (or horoball) centered at a and $x \in \mathbb{P}(V_<) \setminus B$. Then $\mathsf{pr}_B(x) = [a, x] \cap \partial B$.*

Proof. Let $B = B_r(a)$ be a ball. Let $d_0 = d(a, x)$. Lemma 7.2 and 7.6 implies that $p = \mathsf{pr}_B(x)$ exists. Let $y = [a, x] \cap \partial B_r(a)$ (see 7.5). Then $d(y, x) =$

$d(a,x) - d(a,y) = d_0 - r$. So $d(p,x) \le (d_0 - r)$. Since $p \in \mathrm{cl}(B)$, we have $d(a,p) \le r$. But

$$\begin{aligned} d_0 &= d(a,x) \\ &\le d(a,p) + d(p,x) \\ &\le (d_0 - r) + r. \end{aligned}$$

So both the previous inequalities must be equalities. Hence $d(a,p) = r$ and $d(p,x) = d_0 - r$, and $d(a,x) = d(a,p) + d(p,x)$. So p is on the geodesic $[a,x]$ by 1.9. This proves the lemma for a ball. The argument for a horoball is similar, using 6.3. ☐

7.10 Remark. From 7.9 and 6.3 we find that if ρ is a null vector and $x \notin B_1(\rho)$, then $d_\rho(x) = d(B_1(\rho), x)$.

7.11 Definition. Let A and K be non-empty subsets of $\mathbb{P}(V_<)$. If K is bounded, define

$$\mathsf{md}_A(K) = \inf\{t \ge 0 \colon K \subseteq B_t(A)\} = \sup\{d(A,z) \colon z \in K\}.$$

In other words, $\mathsf{md}_A(K)$ is the radius of the smallest closed ball around A that contains K.

7.12 Lemma. *Let $A \subseteq \mathbb{P}(V_<)$.*

(a) If $K_1 \subseteq K_2$ are bounded subsets of $\mathbb{P}(V_<)$, then $\mathsf{md}_A(K_1) \le \mathsf{md}_A(K_2)$.

(b) Let $\{K_i \colon i \in I\}$ be a collection of subsets of $\mathbb{P}(V_<)$ such that $\cup_{i \in I} K_i$ is bounded. Then $\mathsf{md}_A(\cup_i K_i) = \sup\{\mathsf{md}_A(K_i) \colon i \in I\}$.

Proof. Routine exercise. ☐

7.13 Lemma. *Let A be a point or a hyperplane in $\mathbb{P}(V_<)$. Then one has*

(a) If $x, y \in \mathbb{P}(V_<)$, then $\mathsf{md}_A([x,y]) = \max\{d(A,x), d(A,y)\}$.

(b) If K is any bounded subset of $\mathbb{P}(V_<)$, then $\mathsf{md}_A(\mathsf{hull}(K)) = \mathsf{md}_A(K)$.

Proof. (a) If $A \in \mathbb{P}(V_<)$, let $\tau \in V_<$ such that $A = \mathbb{P}(\tau)$. If A is a hyperplane, let $\tau \in V_>$ such that $A = \mathbb{P}^+(\tau^\perp_<)$. In either case, $d(A,y)$ is a increasing function of $h_\tau(y)$. So part (a) follows from lemma 5.2.

(b) Recall that $\mathsf{hull}(K) = \cup_n K_n$, where $K_0 = K$ and $K_{n+1} = \cup_{x,y \in K_n}[x,y]$. Suppose $D \ge 0$ such that $\mathsf{md}_A(K_n) < D$. Let $x, y \in K_n$. Then $d(A,x) \le D$ and $d(A,y) \le D$. So part (a) implies $\mathsf{md}_A([x,y]) \le D$. So, using 7.12, we have

$$\begin{aligned} \mathsf{md}_A(K_{n+1}) &= \mathsf{md}_A(\cup_{x,y \in K_n}[x,y]) \\ &= \sup\{\mathsf{md}_A([x,y]) \colon x, y \in K_n\} \\ &\le D. \end{aligned}$$

By induction on n, it follows that $\mathsf{md}_A(K_n) \le \mathsf{md}_A(K)$ for all n. So

$$\begin{aligned} \mathsf{md}_A(\mathsf{hull}(K)) &\le \sup\{\mathsf{md}_A(K_n) \colon n \ge 0\} \\ &\le \mathsf{md}_A(K). \end{aligned}$$

☐

8 The exponential map

8.1 Definition. Let G be an abelian group. A G-torsor is a set X with a simple transitive action of G. Each $x \in G$ gives us a bijection $r_x : G \to X$ given by $r_x(g) = g + x$. Let B be a set. Then G acts on $\mathsf{Fun}(B, X)$, the action of g takes a function $(b \mapsto f(b))$ to the function $(b \mapsto g + f(b))$. So f and f' are in the same orbit of G if and only if there is a $g \in G$ such that $f(b) = g + f'(b)$ for all $b \in B$. For each $x \in X$, we have an isomorphism

$$\mathsf{Fun}(B, G) \to \mathsf{Fun}(B, X) \text{ given by } \phi \mapsto r_x \circ \phi.$$

Observe that if $x, y \in X$, then $r_x \circ \phi \equiv r_y \circ f \bmod G$. So The above map induces a surjection $\mathsf{Fun}(B, G) \to \mathsf{Fun}(B, X)/G$ that does not depend on the choice of x. One verifies that in the above correspondence $\phi \equiv \phi' \bmod G$ if and only if $r_x \circ \phi \equiv r_y \circ \phi' \bmod G$ for all $x, y \in X$, $\phi, \phi' \in \mathsf{Fun}(B, G)$. Thus there is a natural isomorphism

$$\mathsf{Fun}(B, X)/G \simeq \mathsf{Fun}(B, G)/G.$$

Let U is a vector space. An U torsor is called an affine space for U. We let $\mathsf{Affine}(U)$ denote an affine space of U.

The discussion above obviously does not require G to be abelian. We made that assumption only to maintain consistency of notation, when we later use the notion of a torsor for an affine space.

8.2 Definition. Let $F = \mathbb{R}$ or $\mathbb{C}$. Let V be a hermitian F-vector space of signature $(n, 1)$. Let $\mathbb{P}(V_<)$ be the hyperbolic space of V. Let v be a nonzero vector of V such that $|v|^2 \leq 0$. If $|v|^2 \neq 0$, then choose $v' = -|v|^{-2} v$. If $|v|^2 = 0$, choose any $v' \in V$ such that $|v'|^2 = 0$ and $\langle v, v' \rangle = -1$. Define

$$j_v : V_< \to v^\perp \text{ by } j_v(x) = -\langle x, v \rangle^{-1} x - v'.$$

j_v induces a map $\mathbf{j}_v : \mathbb{P}(V_<) \to v^\perp$ such that $\mathbf{j}_v \circ \mathbb{P} = j_v$.

8.3 Remark.

- Suppose $|v|^2 < 0$. Then $\mathbf{j}_v(x) = -\mathsf{proj}_{v^\perp}(x)/\langle x, v \rangle$. Note that $\mathbb{P}(v)$ determines the map $\mathbf{j}_v$ as an element of $\mathsf{Fun}(\mathbb{P}(V_<), \mathbb{P}(v^\perp))$. If $c \in F$, then $j_{cv}(x) = \bar{c}^{-1} j_v(x)$. So we often scale v and assume that $|v|^2 = -1$.

- Suppose $|v|^2 = 0$. In this case, the definition of the map j depends on the choice of v'. There is a $v^\perp$ worth of choice of v'. So v determines j_v upto translation by an element of $v^\perp$; in other words v determines j_v as an element of $\mathsf{Fun}(\mathbb{P}(V_<), \mathsf{Affine}(v^\perp))/v^\perp$.

8.4 Lemma. *Suppose $|v|^2 = -1$. Define $s_v : B_1(v^\perp) \to V_<$ by $s_v(u) = v + u$. Then*

(a) Then $\mathbf{j}_v$ and $\mathbb{P} \circ s_v$ are mutually inverse homeomorphisms between $\mathbb{P}(V_<)$ and $B_1(v^\perp)$. The homeomorphism $\mathbf{j}_v$ takes 0 to $\mathbb{P}(v)$ and a ball or radius r around $\mathbb{P}(v)$ in $\mathbb{P}(V_<)$ to a ball or radius $\tanh(r)$ around 0 in $v^\perp$.

(c) $\mathbf{j}_v$ maps lines and hyperplanes in $\mathbb{P}(V_<)$ passing through $\mathbb{P}(v)$ to lines and hyperplanes in $v^\perp$ passing through 0.

(d) The map $\mathbf{j}_v : \mathbb{P}(V_<) \to v^\perp$ is equivariant under $\mathsf{Stab}_v(\mathsf{Aut}(V))$. The map $\mathbb{P} \circ \mathbf{j}_v : \mathbb{P}(V_<) \to \mathbb{P}(v^\perp)$ is equivariant under $\mathsf{Stab}_{\mathbb{P}(v)}(\mathsf{Aut}(\mathbb{P}(V_<)))$.

(e) Let H be a hyperplane through $\mathbb{P}(v)$. Under the identification of $\mathbb{P}(V_<)$ and $v^\perp$ by $\mathbf{j}_v$, the hyperbolic reflection in H correspond to ordinary Euclidean reflection in the hyperplane $\mathbf{j}_v(H)$.

Proof. (b) Pick $x \in V_<$. One has $|\langle x, v\rangle|^2/(-|x|^2) = \cosh^2 d(x, v)$. Let $p = \mathsf{proj}_{v^\perp}(x) = x + \langle x, v\rangle v$. Then $\langle v, p\rangle = 0$, so $\langle p, p\rangle = \langle x, p\rangle = |x|^2 + |\langle v, x\rangle|^2$. Note that $j_v(x) = -p/\langle x, v\rangle$. So

$$|j_v(x)|^2 = 1 + \tfrac{|x|^2}{|\langle v,x\rangle|^2} = 1 - \cosh^{-2} d(x, v) = \tanh^2 d(x, v).$$

If u is in the unit ball in $v^\perp$, then $(u + v) \in V_<$ and one verifies directly that $u \mapsto \mathbb{P}(v + u)$ is the inverse to the map $\mathbf{j}_v$.

(c) Let H be a hyperplane in $\mathbb{P}(V_<)$ through $\mathbb{P}(v)$. Then $H = \mathbb{P}(s^\perp_<)$ for some non-zero vector $s \in v^\perp$. Let $x \in V_<$. Note that

$$\langle s, j_v(x)\rangle = \langle s, x\rangle/\langle v, x\rangle.$$

So $\mathbb{P}(x) \in H^\perp$ if and only if $\langle x, s\rangle = 0$ if and only if $\langle s, j_v(x)\rangle = 0$ if and only if $j_v(x) \in s^\perp \cap v^\perp$. □

8.5 Remark. Assume $|v|^2 = -1$. One may scale $\mathbf{j}_v$ to make sure that the euclidean distance between $\mathbf{j}_v(\mathbb{P}(v)) = 0$ and $\mathbf{j}_v(\mathbb{P}(x))$ is the same as the hyperbolic distance between $\mathbb{P}(v)$ and $\mathbb{P}(x)$. This defines the maps

$$\log_v(\mathbb{P}(x)) = \frac{d(x, v)}{\tanh(d(x, v))} j_v(x) = \frac{\tanh^{-1}|j_v(x)|}{|j_v(x)|} j_v(x)$$

and its inverse

$$\exp_v(u) = \mathbb{P}(|u|^{-1}\tanh(|u|)u + v).$$

The map $\exp_v : \{\text{ unit ball in } v^\perp\} \to \mathbb{P}(V_<)$ has the property that it maps 0 to v, takes straight lines through 0 to geodesic rays and preserves lengths along these rays. So this is the exponential map in the sense of Riemannian geometry.

8.6 Lemma (Need to be fixed, since the definition of j_v has been changed by a sign)**.** *Suppose $|v|^2 = 0$. Fix v' such that $|v'|^2 = 0$ and $\langle v, v'\rangle = -1$. Let $M = v^\perp \cap v'^\perp$.*

(a) $j(\mathbb{P}(v')) = 0$ and $j(\mathbb{P}(v)) = \infty$.

(b) The map j maps the horoball $\mathbb{P}\{x \in V_< : \frac{|\langle v,x\rangle|}{\sqrt{-|x|^2}} < r\}$ isomorphically onto

$$\mathcal{C}_{v,r} = \{w - \tfrac{c}{2}v : w \in M, c \in F, |w|^2 < \mathsf{Re}(c) - r^{-2}\}.$$

Let $\mathcal{C}_v = \mathcal{C}_{v,\infty}$. Then $j_v^{-1} : \mathcal{C}_v \to \mathbb{P}(V_<)$ is given by $j_v(u) = \mathbb{P}(u - v')$.

(c) j maps geodesic rays and hyperplanes through $\mathbb{P}(v)$ to affine lines and affine hyperplanes in $v^\perp$ and in $v^\perp/v$.

(d) Let $g \in \mathsf{Aut}(V)$ such that $gv = v$. Then there exists $w \in v^\perp$ such that

$$j_v(g\xi) = g j_v(\xi) + w \quad \text{for all} \quad \xi \in \mathbb{P}(V_<).$$

So j_v is $\mathsf{Stab}_v(\mathsf{Aut}(V))$-equivariant, upto translation by an element of $v^\perp$.

(e) Let $F = \mathbb{R}$. Consider the map $J : \mathbb{P}(V_<) \to v^\perp/v$ determined by j. Let H be a hyperplane in $\mathbb{P}(V_<)$ through $\mathbb{P}(v)$. Then $J(H)$ is an affine hyperplane in $v^\perp/v$ and $R_{J(H)} \circ J = J \circ R_H$.

Proof. (b) Each $u \in v^\perp$ can be uniquely written in the form $u = w - \frac{c}{2}v$ where $w \in M$, $c \in F$. We calculate

$$|u - v'|^2 = |w|^2 - \mathsf{Re}(c).$$

If $u = j(x)$ for some $x \in \mathbb{P}(V_<)$, then $|u - v'|^2 < 0$, so $|w|^2 < \mathsf{Re}(c)$, that is, $j : \mathbb{P}(V_<) \to \mathcal{C}_v$. Conversely, if $u \in \mathcal{C}_v$, then $(u - v')$ has negative norm, so $k(u) = \mathbb{P}(u - v')$ is a well defined map from $k : \mathcal{C}_v \to \mathbb{P}(V_<)$. One verifies that k and j are inverse to each other.

Let $x \in V_<$. Write $j(x) = w - \frac{c}{2}v$ with $w \in M$. Then $|w|^2 - \mathsf{Re}(c) = |j(x) - v'|^2 = |x|^2/|\langle v, x\rangle|^2$. So x belongs to the given horoball if and only if $|x|^2/|\langle v, x\rangle|^2 < -1/r^2$, if and only if $|w|^2 - \mathsf{Re}(c) < -1/r^2$, that is $j(x) \in \mathcal{C}_{v,r}$.

(e) Choose $s \in V_<$ such that $s^2 = 2$ and $s^\perp = H$. Since H passes through $\mathbb{P}(v)$, one has $\langle s, v\rangle = 0$. So

$$\langle s, j(x)\rangle = \langle s, x\rangle\langle v, x\rangle^{-1} + \langle s, v'\rangle.$$

It follows that $\mathbb{P}(x) \in s^\perp$ if and only if $j(x) \in \{u \in v^\perp \colon \langle s, u\rangle = \langle s, v'\rangle\}$. So $j(H)$ is an affine hyperplane in $v^\perp$. Note that $u \in j(H)$ implies $u + v\mathbb{R} \subseteq j(H)$, that is, the hyperplane $j(H)$ is parallel to the line $v\mathbb{R}$. So $j(H)$ projects onto a proper hyperplane in $v^\perp/v$.

Let $x \in V_<$. Let $j(x) = u$. One has $x = j^{-1}(u) = \mathbb{P}(u - v')$. Now

$$J \circ R_H(x) = J \circ R_H(\mathbb{P}(u - v')) = J(u - v' - \langle s, u - v'\rangle s) = u - \langle s, u - v'\rangle s \bmod v,$$

where the last equality uses $\langle v, u - v' - \langle s, u - v'\rangle s\rangle = -\langle u, v'\rangle = 1$. It follows that $J \circ R_H(x) = R_s(u) + \langle s, v'\rangle s \bmod v$. Observe that $(R_s(u) \bmod v)$ only depends on $(u \bmod v)$. To finish, note that $(u \bmod v) \mapsto (R_s(u) + \langle s, v'\rangle s) \bmod v$ is the affine reflection in the hyperplane $J(H)$. ∎

9 The tangent space

Recall that the topology on $\mathbb{P}(V)$ is induced from V by the quotient map $\mathbb{P} : V \to \mathbb{P}(V)$. In other words, a subset U of $\mathbb{P}(V)$ is open if and only $\mathbb{P}^{-1}(U)$ is open in V in its Euclidean topology.

9.1 Lemma. *Let $\alpha \in \mathbb{P}(V)$ be a line containing a non-null vector. Then*

$$U_\alpha = \{\beta \in \mathbb{P}(V) \colon \beta \nsubseteq \alpha^\perp\}$$

is an open neighborhood of α. One has an isomorphism $\phi_\alpha : U_\alpha \to \mathsf{Hom}(\alpha, \alpha^\perp) \simeq \mathrm{Hom}(\alpha, V/\alpha)$.

Proof. The set U_α is open since $\mathbb{P}^{-1}(U_\alpha) = V - \alpha^\perp$ is open in V. (Note that U_α is open even when α is a line spanned by a null vector but in this case $\alpha \notin U_\alpha$).

Given $\beta \in U_\alpha$, define $\phi_\alpha(\beta) : \alpha \to \alpha^\perp$ to be the linear map that takes $\pi_\alpha(b)$ to $\pi_{\alpha^\perp}(b)$ where $b \in \beta \setminus \{0\}$. Conversely, for $\phi : \mathrm{Hom}(\alpha, \alpha^\perp)$, let $\mathsf{graph}(\phi) = \{v + \phi(v) : v \in \alpha\} \in U_\alpha$ be the graph of this linear map. One verifies that ϕ_α and graph are well defined and are mutually inverse isomorphisms. Finally the isomorphism $\mathsf{Hom}(\alpha, \alpha^\perp) \simeq \mathrm{Hom}(\alpha, V/\alpha)$ is induced by the isomorphism $\pi_\alpha : V/\alpha \to \alpha^\perp$. □

9.2. Next we want to describe the tangent space $T_\alpha(\mathbb{P}(V_<))$ to $\mathbb{P}(V_<)$ at a point α in a concrete manner. By this we mean the tangent space to $\mathbb{P}(V_<)$ as a differentiable manifold in the real case and the holomorphic tangent space in the complex case. A tangent vector is specified by (the germ of) a curve in $\mathbb{P}(V_<)$. By a curve we mean a differentiable (resp. holomorphic) map from a small neighborhood of 0 to $\mathbb{P}(V_<)$ in the real (resp. complex) case.

Let $\alpha \in \mathbb{P}(V_<)$. Then $\mathbb{P}(V_<) \subseteq U_\alpha$ since $\alpha^\perp$ is positive definite. So $\phi_\alpha : \mathbb{P}(V_<) \hookrightarrow \mathrm{Hom}(\alpha, \alpha^\perp)$ lets us identify the projective model of the complex hyperbolic space inside the vector space $\mathrm{Hom}(\alpha, \alpha^\perp)$.

Choose $v \in \alpha$ with $v^2 = -1$. We get an identification $\mathbf{j}_v : \mathbb{P}(V_<) \to B_1(v^\perp)$. This gives an identification $\sigma : B_1(v^\perp) \hookrightarrow \mathrm{Hom}(\alpha, \alpha^\perp)$. Unfolding the definitions, we find that $(\sigma(y) : v \mapsto y)$. Note that σ is a linear map (or rather a restriction of a linear map to $B_1(v^\perp)$).

Thus we have identified $\mathbb{P}(V_<)$ with $B_1(\alpha^\perp)$ and also with a subset of $\mathrm{Hom}(\alpha, \alpha^\perp)$ so that one can go between these identifications by a linear map σ. A curve γ in $\mathbb{P}(V_<)$ with $\gamma(0) = \alpha$ determines a curve in $B_1(\alpha^\perp)$ (resp. in $\mathrm{Hom}(\alpha, \alpha^\perp)$) and by differentiating at 0 we get elements in the tangent space of $T_0 B_1(\alpha^\perp)$ (resp. in $T_0 \mathrm{Hom}(\alpha, \alpha^\perp)$). As is usual for Euclidean spaces, we have canonical identifications: $\alpha^\perp = T_0 B_1(\alpha^\perp)$ and $\mathrm{Hom}(\alpha, \alpha^\perp) = T_0 \mathrm{Hom}(\alpha, \alpha^\perp)$. Thus we get two concrete realizations of a tangent vector in $\mathbb{P}(V_<)$: as a vector in $\mathrm{Hom}(\alpha, \alpha^\perp)$ and as a vector in $\alpha^\perp$. The first description has the advantage that the map $\phi_\alpha : \mathbb{P}(V_<) \hookrightarrow \mathrm{Hom}(\alpha, \alpha^\perp)$ and hence the identification $(\phi_\alpha)_* : T_\alpha(\mathbb{P}(V_<)) \simeq \mathrm{Hom}(\alpha, \alpha^\perp)$ is canonical. In contrast, the identification of $\mathbb{P}(V_<) \simeq B_1(\alpha^\perp)$ depends on the choice of a norm -1 vector in α. However the second description has the advantage that $B_1(\alpha^\perp)$, comes with ready made coordinates. We can easily move between these descriptions via σ: note that the derivative of σ is equal to σ since σ is linear.

9.3 Lemma. *Let $u \in V_<$ and $x \in T_u(V_<)$. Choose $v \in V$ such that $|v|^2 = -1$ and $\mathbb{P}(u) = \mathbb{P}(v)$. Then $dj_v : T_v(V_<) \to T_0(B_1(v^\perp))$ is given by $dj_v(x) = \left(\frac{v}{u}\right)\pi_{v^\perp}(x)$ when we make the identifications $T_u(V_<) = V$ and $T_0(B_1(v^\perp)) = v^\perp$.*

Proof. Recall that $j_v(x) = -\pi_{v^\perp}(x)/\langle x, v\rangle$. Let $\Gamma(t) = u + tx$. Then $x = \Gamma'(0)$. One has

$$(j_v \circ \Gamma)'(t) = \frac{d}{dt}\Big(\frac{-\pi_{v^\perp}(u+tx)}{\langle u+tx, v\rangle}\Big) = \frac{-\pi_{v^\perp}(x)}{\langle u+tx, v\rangle} + \frac{\pi_{v^\perp}(u+tx)}{\langle u+tx, v\rangle^2}\langle x, v\rangle$$

For the second equality, note that differentiation commutes with $\pi_{v^\perp}$ since projection to a subspace is a linear map. Substituting $t = 0$, and remembering $u \in v^\perp$, we get

$$dj_v(x) = (j_v \circ \Gamma)'(0) = -\frac{\pi_{v^\perp}(x)}{\langle u, v\rangle} = \frac{u}{v}\pi_{v^\perp}(x).$$

□

9.4 Lemma. *Let $u_1, u_2 \in V_<$ such that $\mathbb{P}(u_1) = \mathbb{P}(u_2)$. Let $x_j \in T_{u_j}(V_<) = V$ for $j = 1, 2$. Choose v such that $|v|^2 = -1$ and $\mathbb{P}(v) = \mathbb{P}(u_1)$. Then the following are equivalent:*

(i) $d\mathbb{P}(x_1) = d\mathbb{P}(x_2)$.

(ii) $dj_v(x_1) = dj_v(x_2)$.

(iii) $\pi_{v^\perp}(x_2) = (u_2/u_1)\pi_{v^\perp}(x_1)$.

Proof. Lemma 9.3 implies that (ii) holds if and only if $(v/u_2)\pi_{v^\perp}(x_2) = (v/u_1)\pi_{v^\perp}(x_1)$ or equivalently $\pi_{v^\perp}(x_2) = (u_2/u_1)\pi_{v^\perp}(x_1)$. This proves the equivalence of (ii) and (iii). The equivalence of (i) and (ii) follows from 9.2. □

9.5 Remark. It is instructive to note the following alternative way to show (iii) implies (i) in 9.4. Let $\Gamma_j(t) = (u_j + tx_j)$. Then $x_j = \Gamma_j'(0)$. Assume (iii). Then $\pi_{u_1^\perp}((u_1/u_2)x_2) = \pi_{u_1^\perp}(x_1)$, so $(u_1/u_2)x_2 = x_1 + \lambda u_1$ for some scalar λ. Now $\mathbb{P}(u_2 + tx_2) = \mathbb{P}(u_1 + t(u_1/u_2)x_2) = \mathbb{P}(u_1 + t(x_1 + \lambda u_1)) = \mathbb{P}(u_1 + \frac{t}{1+t\lambda}x_1)$, that is $\mathbb{P} \circ \Gamma_2(t) = \mathbb{P} \circ \Gamma_1(t/(1+\lambda t))$, that is, Γ_1 and Γ_2 determine the same curve in $\mathbb{P}(V_<)$ upto reparametrization. This implies (i).

9.6 Theorem. *(a) Let $v \in V$ such that $|v|^2 = -1$. Then $y \in T_v(V_<)$ corresponds to the tangent vector $(a \mapsto (\frac{a}{v})\pi_{v^\perp}(y)) \in \mathrm{Hom}(\mathbb{P}(v), v^\perp)$.*

(b) Let $\gamma(t)$ be a curve in $\mathbb{P}(V_<)$ defined for t in a neighborhood of 0. The tangent vector $\gamma'(0)$ corresponds to the vector $(a \mapsto \frac{a}{\Gamma(0)}\Gamma'(0))$ in $\mathrm{Hom}(\gamma(0), \gamma(0)^\perp)$ where $a \in \gamma(0)$ and $\Gamma(t)$ is any lift of $\gamma(t)$ to $V_<$.

Choose $v \in \gamma(0)$ having norm -1. Then the tangent vector $\gamma'(0)$ corresponds to the vector $(\mathbf{j}_v \circ \gamma)'(0) \in v^\perp = T_0(B_1(v^\perp))$. One can go between the two descriptions of the tangent vector via the map σ.

Proof. (a) Recall from 9.2 that the tangent vector $d\mathbb{P}(y) \in \mathrm{Hom}(\mathbb{P}(v), v^\perp)$ corresponds to the tangent vector $dj_v(y) \in T_0(B_1(v^\perp)$ under $d\sigma = \sigma$. Applying lemma 9.3 with $u = v$, we obtain $dj_v(y) = \pi_{v^\perp}(y)$. So $d\mathbb{P}(y)$ is given by the map $(v \mapsto \pi_{v^\perp}(y)) \in \mathrm{Hom}(\mathbb{P}(v), v^\perp)$. This proves part (a). Part (b) follows from part (a) and 9.3. $\square$

9.7. We shall write $\mathcal{T}_\alpha = \mathrm{Hom}(\alpha, \alpha^\perp)$. There is a natural positive definite hermitian form g_α on $\mathcal{T}_\alpha$ coming from the hermitian form on $\alpha^\perp$. Given $\sigma_1, \sigma_2 \in \mathcal{T}_\alpha$, define

$$g_\alpha(\sigma_1, \sigma_2) = \langle \sigma_1(a), \sigma_2(a) \rangle / (-\langle a, a \rangle)$$

where a is any non-zero vector in α. If $g_\alpha(\sigma_1, \cdot) = 0$, then $\langle \sigma_1(a), \sigma_2(a) \rangle = 0$ for all σ_2. So $\langle \sigma_1(a), y \rangle = 0$ for all $y \in \alpha^\perp$ and this implies $\sigma_2(a) = 0$ since $\alpha^\perp$ is positive definite. It follows that $\sigma_2 = 0$. So g_α is a positive definite hermitian form on $\mathcal{T}_\alpha$. This Hermitian form is clearly invariant under the $U(n,1)$ action. In the complex case, this gives $\mathbb{P}(V_<)$ the stucture of a hermitian manifold. The Riemannian metric on $\mathbb{P}(V_<)$ is given by the real part of g and $Im(g)$ makes $\mathbb{P}(V_<)$ into a symplectic manifold.

9.8 Lemma. *Let v be a vector in $V_<$ of norm -1 and let $\alpha = \mathbb{P}(v)$. Let $v_1, v_2 \in V$ be two vectors representing tangent vectors at v. Then they determine tangent vectors $\mathbb{P}_*(v_1)$, $\mathbb{P}_*(v_2)$ in $T_\alpha \mathbb{P}(V_<)$. One has*

$$g_\alpha(\mathbb{P}_*(v_1)), \mathbb{P}_*(v_2)) = \langle \pi_{v^\perp}(v_1), \pi_{v^\perp}(v_2) \rangle = \tfrac{\langle v_1, v_2 \rangle \langle u, u \rangle - \langle v_1, u \rangle \langle u, v_2 \rangle}{\langle u, u \rangle}$$

where u is any nonzero vector in α.

9.9. Computing distances between points : Let d_g be the distance function on $\mathbb{P}(V_<)$ determined by the Riemannian metric $\mathsf{Re}(g)$. To compute d_g, first note that d_g is clearly $U(n,1)$ invariant, since g is. Since the $U(n,1)$ action is transitive on the set of equidistant points, it is enough to calculate d_g on one geodesic ray. Consider the curve in $V_<$ given by $\gamma(t) = (\cosh t; \sin t, 0, \cdots, 0)$ and let $\Gamma(t) = \mathbb{P}(\gamma(t))$. Let us calculate $d_g(\Gamma(0), \Gamma(T))$. One has $\gamma'(t) = (\sinh t; \cosh t, 0, \cdots, 0)$. So $\langle \gamma(t), \gamma'(t) \rangle = 0$ for all t and $|\gamma'(t)|^2 = 1 = -|\gamma(t)|^2$. Lemma 9.8 gives

$$|\Gamma'(t)|_g^2 := g_{\Gamma(t)}(\Gamma'(t), \Gamma'(t)) = (|\gamma'(t)|^2 |\gamma(t)|^2 - |\langle \gamma(t), \gamma'(t) \rangle|^2) / \langle \gamma(t), \gamma(t) \rangle = 1.$$

So $d_g(\Gamma(0), \Gamma(t)) = \int_0^t |\Gamma'(s)|_g^2 ds = \int_0^t 1 \cdot ds = t = d(\Gamma(0), \Gamma(t))$. Given any two points $a, b \in \mathbb{P}(V_<)$, there exists $g \in PU(n,1)$ such that $ga = \Gamma(0)$ and $gb = \Gamma(t)$ for some t. Since d_g and d are both $U(n,1)$ invariant and they agree on the ray $\Gamma(t)$, it follows that they agree everywhere.

10 Real hyperbolic space

10.1 Definition (the hyperboloid model and the projective model)**.** Let $\mathbb{R}^{n,1}$ denote the vector space $\mathbb{R}^{n+1}$ with the signature $(n, 1)$ bilinear form

$$\langle x, y\rangle = -x_0y_0 + x_1y_1 + \cdots + x_ny_n$$

where $x = (x_0, x_1, \cdots, x_n)$ and $y = (y_0, y_1, \cdots, y_n)$. We write $x^2 = \langle x, x\rangle$.

The hyperboloid model of the n dimensional real hyperbolic space is:

$$\mathcal{H}^n = \{x \in \mathbb{R}^{n,1} \colon x^2 = -1, x_0 > 0\}.$$

Note that if $x \in \mathcal{H}^n$, then $x_0^2 = 1 + x_1^2 + \cdots + x_n^2$ and $x_0 > 0$, so $x_0 > 1$.

The projective model of the n dimensional real hyperbolic space is:

$$\mathbb{P}_-(\mathbb{R}^{n,1}) = \mathbb{P}\{x \in \mathbb{R}^{n,1} \colon x^2 < 0\}.$$

The map $(x \mapsto \mathbb{P}(x))$ sets up the isomorphism between the two models.

10.2 Remark. Let $S = \{x = (x_0, \cdots, x_n) \in \mathbb{R}^{n,1} \colon x^2 = -1\}$. If $x \in S$, then $|x_0| \geq 1$, so the hyperplane $\{x \in \mathbb{R}^{n,1} \colon x_0 = 0\}$ divides S into two connected components, $\mathcal{H}^n$ and $-\mathcal{H}^n$. Fix $y \in \mathcal{H}^n$. Since $\mathcal{H}^n$ is connected and $y^\perp$ does not meet $\mathcal{H}^n$, the continuous function $v \mapsto \langle v, y\rangle$ retains the same sign on all of $\mathcal{H}^n$. Since $\langle y, y\rangle < 0$, it follows that $\langle v, y\rangle < 0$ for all $v \in \mathcal{H}^n$, and hence $\langle u, y\rangle > 0$ for all $u \in -\mathcal{H}^n$. So, if we fix $y \in \mathcal{H}^n$, then we can write

$$\mathcal{H}^n = \{x \in \mathbb{R}^{n,1} \colon x^2 = -1, \langle x, y\rangle < 0\}.$$

This is an useful observation. Given $x \in S$, if we want to check $x \in \mathcal{H}^n$, we can just verify $\langle x, y\rangle < 0$ for a suitable chosen $y \in \mathcal{H}^n$. In the definition of $\mathcal{H}^n$ we had chosen $y = (1, 0, \cdots, 0)$.

In general, let V be a real inner product space of signature $(n, 1)$. To define a hyperboloid model in V, we need to choose and fix a vector $y \in V$ such that $y^2 = -1$ and then define the hyperbolic space

$$\mathcal{H}(V, y) = \{x \in V \colon x^2 = -1, \langle x, y\rangle < 0\}.$$

10.3 Lemma. *Let $x, y \in \mathcal{H}^n$. Then $\langle x, y\rangle \leq -1$, or equivalently, $(x - y)^2 \geq 0$. Equality holds if and only if $x = y$.*

Proof. Let $x \in \mathcal{H}^n$. If x and y are linearly dependent, then $x = cy$, taking norm $c = \pm 1$, so $x = \pm y$, but $-y \notin \mathcal{H}^n$, so $x = y$. So if $x \neq y$, then x and y are linearly independent, so $\mathsf{span}\{x, y\}$ has signature $(1, 1)$, hence $x^2y^2 - \langle x, y\rangle^2 = 1 - \langle x, y\rangle^2 < 0$. Since we have seen already that $\langle x, y\rangle < 0$ for all $x \in \mathcal{H}^n$, it follows that $\langle x, y\rangle < -1$.

Here an alternative proof just using algebra: Note that

$$(x_0^2 - 1)(y_0^2 - 1) = 1 - x_0^2 - y_0^2 + x_0^2y_0^2 \leq 1 - 2x_0y_0 + x_0^2y_0^2 = (x_0y_0 - 1)^2.$$

Taking square root and remembering that $x_0, y_0 \geq 1$, we get

$$\sqrt{(x_0^2 - 1)(y_0^2 - 1)} \leq x_0 y_0 - 1 \tag{2}$$

and equality holds if and only if $x_0 = y_0$. The Cauchy-Schwarz inequality implies $\left(\sum_{i=1}^n x_i y_i\right) \leq \left(\left(\sum_{i=1}^n x_i^2\right)\left(\sum_{i=1}^n y_i^2\right)\right)^{1/2} = ((x_0^2-1)(y_0^2-1))^{1/2}$. Using (2), we get, $\langle x, y\rangle = x_1 y_1 + \cdots + x_n y_n - x_0 y_0 \leq (x^2-1)^{1/2}(y^2-1)^{1/2} - x_0 y_0 \leq -1$. Now suppose $\langle x, y\rangle = -1$. Then all the inequalities above are equalities. If $(x_1, \cdots, x_n) = (y_1, \cdots, y_n) = (0, \cdots, 0)$, then $x_0 = y_0 = 1$, so $x = y$. So without loss we may assume that $(x_1, \cdots, x_n) \neq (0, \cdots, 0)$. Equality holds in Cauchy-Schwarz if and only if $(y_1, \cdots, y_n) = c(x_1, \cdots, x_n)$ for some $c \in \mathbb{R}$. Also since equality holds in (2), we get $x_0 = y_0$. So $\sum_{i=1}^n x_i^2 = x_0^2 - 1 = y_0^2 - 1 = \sum_{i=1}^n y_i^2$. It follows that $c = \pm 1$. So either $y = x$ or $y = (x_0, -x_1, \cdots, -x_n)$. In the later case, verify that $\langle x, y\rangle < -1$. □

10.4 Definition (distance formula, projection)**.** We know that in the projective model, the hyperbolic distance between $\alpha, \beta \in \mathbb{P}_-(\mathbb{R}^{n,1})$ is given by

$$d(\alpha, \beta) = \cosh^{-1}\sqrt{\frac{|\langle a, b\rangle|^2}{a^2 b^2}}$$

where $a, b \in \mathbb{R}^{n,1}$ are vectors such that $\mathbb{P}(a) = \alpha$ and $\mathbb{P}(b) = \beta$. Using lemma 10.3, we find that the distance between $x, y \in H^n$ is given by

$$d(x, y) = \cosh^{-1}(-\langle x, y\rangle) = \cosh^{-1}(1 + \tfrac{1}{2}(x-y)^2).$$

Let $r \in \mathbb{R}^{n,1}$ such that $r^2 = 1$ and $x \in \mathcal{H}^n$. Let $y = x - \langle x, r\rangle r$. We know that $\mathbb{P}(y)$ represents the projection of $\mathbb{P}(y)$ onto $\mathbb{P}(r^\perp)$. Note that $\langle x, y\rangle = x^2 - \langle x, r\rangle^2 = x^2 r^2 - \langle x, r\rangle^2 < 0$, so $y/\sqrt{-y^2} \in \mathcal{H}^n$. So $y/\sqrt{-y^2}$ is the projection of x onto the hyperplane $r^\perp \cap \mathcal{H}^n$ in the hyperboloid model.

10.5 Definition (reflections)**.** Let H be a hyperplane in $\mathbb{R}^{n,1}$. The linear functional $\mathbb{R}^{n,1} \to \mathbb{R}^{n,1}/H \simeq \mathbb{R}$ with kernel H is representable in the form $\langle s, \cdot\rangle$ since the Lorentzian form on $\mathbb{R}^{n,1}$ is non-degenerate. So $H = s^\perp$ for some $s \in \mathbb{R}^{n,1}$. One of the three following possibilities hold:

- $s^2 < 0$, $s^\perp$ is positive definite and does not meet H^n.
- $s^2 = 0$, $s^\perp$ is singular positive semidefinite and is asymptotic to H^n.
- $s^2 > 0$, $s^\perp$ has signature $(n-1, 1)$ and $s^\perp \cap H^n$ is isomorphic to the hyperbolic space H^{n-1}.

In the third case we say $s^\perp \cap H^n$ is a hyperplane in H^n. Sometimes we just write $s^\perp$ instead of $s^\perp \cap H$ to denote a hyperplane in H^n.

Let s be a nonzero norm vector in $\mathbb{R}^{n,1}$. Let

$$R_s(x) = x - 2\langle s, x\rangle s/s^2.$$

be the orthogonal reflection in s. Since R_s is an isometry of $\mathbb{R}^{n,1}$, it defines an isometry of the metric space $\mathbb{P}_-(\mathbb{R}^{n,1})$.

Note that $\{x \in \mathbb{R}^{n,1} : x^2 = -1\} = H^n \cup (-H^n)$ where H^n and $-H^n$ are the two sheets of the hyperboloid. Since R_s is a continous isometry $\mathbb{R}^{n,1}$ of order 2 (with Euclidean topology), we must have $R_s(H^n) = \pm H^n$. If $s^2 < 0$, then without loss we can take $s \in H^n$ and $R_s(s) = -s \in -H^n$. So in this case R_s interchanges the two sheets of the hyperboloid.

Now assume $s^2 > 0$. Then the reflection R_s fixes the hyperplane $s^\perp \cap H^n$ pointwise byR_s, so it must fix each hyperboloid. Thus R_s is an isometry of (H^n, d). In fact one verifies that R_s is the unique isometry of order 2 of (H^n, d) which fixes the hyperplane $s^\perp \cap H^n$ pointwise. If H denotes the hyperplane orthogonal to s, then sometimes we write $R_s = R_H$.

The hyperplane $s^\perp$ or its image in the hyperbolic space is called the mirror of the reflection R_s. The reflection R_s interchanges $\{x \in H^n : \langle x, s\rangle > 0\}$ and $\{x \in H^n : \langle x, s\rangle < 0\}$. These two sets are called the two sides of the mirror $s^\perp$ or the two *open half spaces* in H^n bounded by the mirror $s^\perp$.

10.6 Definition (Boundary of the hyperboloid model and cusps)**.** Consider the n-sphere $C_1^+ = \{x = (x_0, \cdots, x_n) \in \mathbb{R}^{n,1} : x^2 = 0, x_0 = 1\}$ Let $C^+ = \mathbb{R}_> C_1^+$ be the cone on C_1^+. Then $C = C^+ \coprod (-C^+)$ is the set of non-zero null vectors in $\mathbb{R}^{n,1}$. If $x \in C_1^+, y \in \mathcal{H}^n$, then

$$\langle x, y\rangle = \sum_{i=1}^n x_i y_i - y_0 \leq \Big(\sum_{i=1}^n x_i^2\Big)^{1/2}\Big(\sum_{i=1}^n y_i^2\Big)^{1/2} - y_0 = (y_0^2 - 1)^{1/2} - y_0 < 0$$

If y is any vector in $\mathcal{H}^n$, then C^+ and $-C^+$ are on opposite sides of $y^\perp$. So C^+ and $-C^+$ are the two connected components of C; these are called the positive and negative light cones respectively. The positive light cone C^+ is the the component of C that is asymptotic to the hyperboloid $\mathcal{H}^n$. One has $\partial\mathcal{H}^n = C/\mathbb{R}^\times = C^+/\mathbb{R}_>$, so the boundary of $\mathcal{H}^n$ is in natural bijection with the rays lying on the cone C^+. When working in the hyperboloid model of $\mathcal{H}^n$, we shall always choose the representatives for ∂H^n from C^+. If $\rho \in C^+$, then a horoball around ρ has the form

$$B_r(\rho) = \{y \in \mathcal{H}^n : \langle \rho, y\rangle \in [-r, 0)\}.$$

Let $v \in C^+$. Then $v^\perp$ does not intersect $C - \mathbb{R}^\times v$. Since $(C^+ - \mathbb{R}_> v)$ and $-(C^+ - \mathbb{R}_> v)$ are both connected, they are the connected components of $C - \mathbb{R}^\times v$ and they are on either sides of $v^\perp$. So $(C^+ - \mathbb{R}_> v)$ is on the same side of $v^\perp$ as $\mathcal{H}^n$. Since $\langle x, v\rangle < 0$ for all $v \in \mathcal{H}^n$, we have $\langle x, v\rangle < 0$ for all $x \in (C^+ - \mathbb{R}_> v)$. It follows that $\langle u, v\rangle \leq 0$ for all $u, v \in C^+$ and $\langle u, v\rangle = 0$ if and only if $v \in \mathbb{R}_> u$.

11 Hyperplanes, facets, chambers

We present the basic definitions and results about hyperplane arrangement in the real hyperbolic space. Most of it follows directly in [B], Chapter 5, section 1 once we replace "affine space E" by "hyperbolic space $\mathcal{H}^n$", "straight lines" by "geodesics" and so on. In many such cases we have refered to the result in [B] and omitted the proof.

11.1. Let V be a real vector space. A codimension one subspace in V is called a *hyperplane* in V. A coset of a hyperplane in V is called an *affine hyperplane* in V. Let H be an affine hyperplane in V. Then $(V \setminus H)$ has two connected components, these are called *open half spaces* and their closures are called *closed half spaces* and H is called the bounding hyperplane of these half spaces. If A is a connected subset of V that does not meet H, then let $D_H(A)$ be the open half space of V bounded by H that contains A. If $\mathfrak{N}$ is a collection of affine hyperplanes in V and A is a connected subset of V that is disjoint from each hyperplane in $\mathfrak{N}$, then we write $D_{\mathfrak{N}}(A) = \cap_{H \in \mathfrak{N}} D_H(A)$.

11.2 Definition. The notation and assumptions introduced in this subsection will remain in force for the rest of the section. Let $V = \mathbb{R}^{n,1}$ and let $\mathcal{H}^n$ be the n-dimensional real hyperbolic space:

$$\mathcal{H}^n = \{x = (x_0, \cdots, x_n) \in V : x_0 > 0 > x^2\} \simeq \mathbb{P}(V_<).$$

If $a, b \in \mathcal{H}^n$, we let (a, b) (resp. $[a, b]$) denote the open (resp. closed) geodesic segment in $\mathcal{H}^n$ joining a and b etc.

Let A be a half space (resp. a hyperplane or a subspace) of V. If A meets $\mathcal{H}^n$ but does not contain it, then $A \cap \mathcal{H}^n$ is called a half space (resp. a hyperplane or a subspace) of $\mathcal{H}^n$; by abuse of notation sometimes we simply write A instead of $A \cap \mathcal{H}^n$. Let A be a half space (resp. a hyperplane or a subspace) in $\mathcal{H}^n$. Then A determines a half space (resp. a hyperplane or a subspace) in V which will be denoted by A_V, so $A = A_V \cap \mathcal{H}^n$.

More precisely, verify that a subspace A of $\mathcal{H}^n$ has the form $A = A_V \cap \mathcal{H}^n \simeq \mathbb{P}_<(A_V)$ whre A_V is the smallest linear subspace of V that contains A. The subspace A_V has signature $(k, 1)$ for some $k \geq 0$. One has $A \simeq \mathcal{H}^k$, with the induced metric (and hence topology) from $\mathcal{H}^n$. We say A is a subspace of $\mathcal{H}^n$ of dimension k.

Let H be hyperplane in $\mathcal{H}^n$. Let φ be any functional on V whose kernel is H_V. Then the two half spaces of $\mathcal{H}^n$ bounded by H are $\{x \in \mathcal{H}^n : \varphi(x) > 0\}$ and $\{x \in \mathcal{H}^n : \varphi(x) < 0\}$. These are called the two sides of H.

Let $\mathfrak{H}$ be a locally finite collection of hyperplanes in $\mathcal{H}^n$. Let $\mathfrak{H}_V = \{H_V : H \in \mathfrak{H}\}$ be the corresponding set of hyperplanes in V. Notice that $\mathfrak{H}_V$ need not be a locally finite collection since infinitely many hyperplanes may meet at a point near the boundary of $\mathcal{H}^n$. For $A \subseteq \mathcal{H}^n$, we let

$$\mathfrak{H}(A) = \{H \in \mathcal{H} : A \subseteq H\}.$$

Let $a \in \mathcal{H}^n$. Then only finitely many hyperplanes of $\mathfrak{H}$ meet $B_r(a)$ for any fixed r. This shows that $\{d(a, H) : H \in \mathfrak{H}\}$ is a discrete subset of $\mathbb{R}_{\geq}$. Given any a, there exists a open convex neighborhood U of a such that U only meets the hyperplanes of $\mathfrak{H}$ that pass through a. Such a neighborhood U of a will be called a *small neighborhood* (or an $\mathfrak{H}$-small neighborhood) of a. If $B_r(a)$ is a small neighborhood of a, we call it a *small ball* around a.

Given $x, y \in \mathfrak{H}$, say x and y are equivalent with respect to $\mathfrak{H}$ if, for each $H \in \mathfrak{H}$, both x and y are either on H or strictly on the same side of H. This defines an equivalece relation on $\mathcal{H}^n$. An equivalence class is called a *facet*. Let F be a facet. Let $\mathsf{supp}(F) = \cap_{H \in \mathfrak{H}(F)} H$ and $\mathsf{supp}_V(F)$ be the corresponding subspace of V, so $\mathsf{supp}_V(F) \cap \mathcal{H}^n = \mathsf{supp}(F)$. The set $\mathsf{supp}(F)$ (resp. $\mathsf{supp}_V(F)$) is a subspace of $\mathcal{H}^n$ (resp. V) and is called the *support* of F in $\mathcal{H}^n$ (resp V). If F is a facet, then define $\mathsf{dim}(F) = \mathsf{dim}(\mathsf{supp}(F)) = \mathsf{dim}(\mathsf{supp}_V(F)) - 1$.

A facet that is not contained in any hyperplane of $\mathfrak{H}$ is called a *chamber*. Let C be a chamber. A *face* of C is a facet contained in the closure of C whose support is a hyperplane. A *wall* of C is a hyperplane in $\mathfrak{H}$ that is the support of a face of C. Let $\mathsf{Wall}(C)$ be the set of walls of C.

11.3 Lemma. *Let H be a hyperplane in $\mathcal{H}^n$. Then the two half spaces H^+ and H^- bounded by H are the two connected components of $\mathcal{H}^n \setminus H$. The subspaces and open and closed half spaces in $\mathcal{H}^n$ are convex. Let L be a geodesic ray in $\mathcal{H}^n$ such that $L \nsubseteq H$ and $L \nsubseteq H^{\pm}$. Then L meets H at a unique point x, and each neighborhood of x meets both H^+ and H^-. If $a \in L - H$, then $[a, x) \subseteq D_H(a)$.*

Proof. Exercise. □

11.4 Lemma (restricting to a subspace)**.** *Let L be a subspace of $\mathcal{H}^n$. Let H be a hyperplane in $\mathcal{H}^n$ with two sides H^+ and H^-. Suppose $\emptyset \neq L \cap H \neq L$. Then $L \cap H$ is a hyperplane in the hyperbolic space L and its two sides are $H^+ \cap L$ and $H^- \cap L$.*

Proof. Let L be a k-dimensional subspace of $\mathcal{H}^n$. Then the subspace L_V of V has signature $(k, 1)$. Since $L_V \nsubseteq H_V$, we have $L_V + H_V = V$, so $\mathsf{dim}(L_V \cap H_V) = \mathsf{dim}(L_V) + \mathsf{dim}(H_V) - \mathsf{dim}(V) = \mathsf{dim}(L_V) - 1 = k$. Since $L \cap H \neq \emptyset$, the vector space $L_V \cap H_V$ has a negative norm vector, so it has signature $(k-1, 1)$. So $L_V \cap H_V$ determines a $(k-1)$ dimensional subspace $L \cap H$ in L. In other words, $L \cap H$ is a hyperplane in L.

Choose $r \in V$ such that $r^2 = 1$ and $H_V = r^{\perp}$. Then $l \mapsto \langle r, l \rangle$ is a functional on L whose kernel is $L \cap H$. So the two sides of $L \cap H$ in L are the sets $\{l \in L \colon \langle r, l \rangle > 0\}$ and $\{l \in L \colon \langle r, l \rangle < 0\}$. These are precisely $L \cap H^+$ and $L \cap H^-$. □

11.5 Lemma. *(a) Let F be a facet and $a \in F$. If B is a hyperplane in $\mathfrak{H}$ or a half space bounded by a hyperplane in $\mathfrak{H}$, then $a \in B$ if and only if $F \subseteq B$.*

(b) Let F be a facet and $a \in F$. Then $F = \mathsf{supp}(F) \cap \big(\cap_{H \in \mathfrak{H} \setminus \mathfrak{H}(F)} D_H(a)\big)$ and $\bar{F} = \mathsf{supp}(F) \cap \big(\cap_{H \in \mathfrak{H} \setminus \mathfrak{H}(F)} \overline{D_H(a)}\big)$. In particular, the facets and their closures are convex subsets of $\mathcal{H}^n$.

(c) Let C be a chamber and $A \subseteq C$. Then $C = \cap_{H \in \mathfrak{H} \setminus \mathfrak{H}(F)} D_H(A)$ and $\bar{C} = \cap_{H \in \mathfrak{H} \setminus \mathfrak{H}(F)} \overline{D_H(A)}$.

Proof. Part (a) follows from definition of a facet. The expression for F in part (b) follows from part (a). Since $F' = \mathsf{supp}(F) \cap \left(\cap_{H \in \mathfrak{H} \setminus \mathfrak{H}(F)} \overline{D_H(a)}\right)$ is closed, we have $\bar{F} \subseteq F'$. Conversely, suppose $x \in F'$. Then $[a, x) \subseteq \mathsf{supp}(F)$ and $[a, x) \subseteq D_H(a)$ for all $H \in \mathfrak{H} \setminus \mathfrak{H}(F)$, so $[a, x) \in F$, hence $x \in \bar{F}$. This proves part (b). Part (c) follows from from part (b) since $\mathsf{supp}(C) = \emptyset$ and $D_H(a) = D_H(A)$ for any $a \in A$. $\square$

11.6 Lemma ([B] p 62-63, Prop. 3)**.** *Let F be a facet and $L = \mathsf{supp}(F)$. Then*

(i) The set F is a convex open subset of the hyperbolic space L.

(ii) The closure of F is the union of F and facets of dimension strictly smaller than that of F.

(iii) In the topological space L, the set F is the interior of its closure.

11.7 Corollary ([B] p 63)**.** *Let F and F' be two facets. If $\bar{F} = \bar{F}'$, then $F = F'$.*

11.8 Lemma ([B] p 63-64, Prop. 4)**.** *Let F be a facet. Let L be a subspace of $\mathcal{H}^n$ which is an intersection of hyperplanes belonging to $\mathfrak{H}$. Let $\mathfrak{N}$ be the set of hyperplanes in $\mathfrak{H}$ that do not contain L. TFAE:*

(i) There exists a facet with support L that meets $\bar{F}$.

(ii) There exists a facet with support L that is contained in $\bar{F}$.

(iii) There exists $x \in \bar{F} \cap L$ such that x does not belong to any hyperplane of $\mathfrak{N}$.

If these conditions are satisfied then $L \cap D_{\mathfrak{N}}(F)$ is the unique facet with support L, that is contained in $\bar{F}$.

11.9 Lemma. *The chambers are exactly the connected components of $\mathcal{H}^n \setminus (\cup \mathfrak{H})$.*

Proof. Let A be a connected component of $U = \mathcal{H}^n \setminus (\cup \mathfrak{H})$. Pick $x \in A$ and let C be the facet containing x. Then C does not meet $\cup \mathfrak{H}$, so C is a chamber. For each $H \in \mathfrak{H}$, one has $A \subseteq D_H(a) = D_H(C)$. So $A \subseteq \cap_{H \in \mathfrak{H}} D_H(C) = C$. On the other hand, C is a connected subset of U and A is maximal connected subset of U, so $C = A$, i.e. each connected component of U is a chamber. Conversely each chamber C, being a connected subset of U, is contained in some connected component A of U. But we just saw each such connected component A is a chamber. So $A = C$. $\square$

11.10 Lemma ([B] p 64-65, prop. 5)**.** *Let C be a non-empty subset of $\mathcal{H}^n$. Assume that there exists a subset $\mathfrak{H}'$ of $\mathfrak{H}$ with the following properties:*

(a) For any $H \in \mathfrak{H}'$, there exists an open half space D_H bounded by H such that $C = \cap_{H \in \mathfrak{H}'} D_H$.

(b) The set C does not meet any hyperplane belonging to $\mathfrak{H} \setminus \mathfrak{H}'$.

Under these conditions, C is a chamber defined by $\mathfrak{H}$ and $D_H = D_H(C)$ for all $H \in \mathfrak{H}'$.

11.11 Lemma ([B] p 65, prop. 6)**.** *Every point of $\mathcal{H}^n$ is in the closure of atleast one chamber.*

11.12 Lemma ([B] p 65, prop. 7)**.** *Let L be a subspace of $\mathfrak{H}^n$ and Ω be a non-empty open subset of L.*

(i) There exists a point $a \in \Omega$ that does not belong to any of the hyperplanes of $\mathfrak{H}$ that do not contain L.

(ii) If L is a hyperplane and $L \notin \mathfrak{H}$, there exists a chamber that meets Ω.

(iii) If L is a hyperplane and $L \in \mathfrak{H}$, then there exists a point a in Ω that does not belong to any hyperplane in $\mathfrak{H} \setminus \{L\}$.

Proof. The proof in [B] goes through. Lemma 11.4 implies that the restriction of the hyperplane arrangement to L gives a hyperplane arrangement in L. □

11.13 Lemma (recognizing a wall)**.** *Let C be a chamber and let $L \in \mathfrak{H}$. TFAE:*

(i) The hyperplane L is a wall of C.

(ii) One has $C \neq D_{\mathfrak{H}\setminus\{L\}}(C)$.

(iii) There exists x in the closure of C such that $\mathfrak{H}(x) = \{L\}$.

Suppose the equivalent conditions above hold. Then the wall L is the support of a unique face of C, namely $F = L \cap D_{\mathfrak{H}\setminus\{L\}}(C)$. Any $x \in F$ satisfies condition (iii) above.

Proof. Assume (i). Let F be a facet in $\mathrm{cl}(C)$ with $\mathsf{supp}(F) = L$. Let $H \in \mathfrak{H}\setminus\{L\}$. Then $F \subseteq \mathrm{cl}(C) \subseteq \mathsf{cl}(D_H(C))$, and F does not meet H, so $F \subseteq D_H(C)$. So $F \subseteq D_{\mathfrak{H}\setminus\{L\}}(C)$, but $F \nsubseteq C$. Thus (i) implies (ii).

Assume (ii). Write $D = D_{\mathfrak{H}\setminus\{L\}}(C)$. One has $C = D \cap D_L(C)$. So D meets $D_L(C)$. If L does not meet D, then D will be on one side of C, which would imply $D \subseteq D_L(C)$ and hence $D = D \cap D_L(C) = C$, contradicting our assumption (ii). So L meets D. Choose $x \in L \cap D$. Then $x \in \mathrm{cl}(D_L(C)) \cap D \subseteq \mathrm{cl}(C)$ and $\mathfrak{H}(x) = \{L\}$. Thus (ii) implies (iii).

Assume (iii). Let $\mathfrak{N} = \mathfrak{H} \setminus \{L\}$; these are the hyperplanes of $\mathfrak{H}$ that do not contain L. Note that x does not belong to any hyperplane of $\mathfrak{N}$. So the implication ((iii) $\implies$ (ii)) in 11.8 tells us there exists a facet $F_1 \in \bar{C}$ such that $\mathsf{supp}(F_1) = L$, so L is a wall of C. Thus (iii) implies (i) and also any $x \in L \cap D$ satisfies (iii).

Assume the equivalent conditions holds. Then the last statement of 11.8 implies that the wall L is the support of a unique face of C, namely $L \cap D_{\mathfrak{H}\setminus\{L\}}(C)$. □

The next proposition shows that the walls of a chamber are enough to cut out the chamber from the hyperbolic space and they form the samllest such collection.

11.14 Lemma ([B] prop 9, p 66)**.** *Let C be a chamber of $\mathfrak{H}$ and let $\mathfrak{M}$ be the set of walls of C. Then $C = D_{\mathfrak{M}}(C)$. If $\mathfrak{L}$ is a subset of $\mathfrak{H}$ such that $D_{\mathfrak{L}}(C) = C$, then $\mathfrak{L}$ contains $\mathfrak{M}$. A subset F of $\mathrm{cl}(C)$ is a facet with respect to the arrangement $\mathfrak{H}$ if and only if it is a facet with respect to the arrangement $\mathfrak{M}$. In particular, C is a chamber with respect to the arrangement $\mathfrak{M}$. So $\mathrm{cl}(C) = \cap_{H \in \mathfrak{M}} \mathrm{cl}(D_H(C))$.*

11.15 Lemma. *Let H_1 and H_2 be two distinct walls of a chamber C. For $j = 1, 2$, choose $x_j \in \bar{C}$ such that $\mathfrak{H}(x_j) = H_j$. Then the open geodesic segment joining x_1 and x_2 is contained in C.*

Proof. Let $H' \in \mathfrak{H} \setminus \{H_1, H_2\}$. Lemma 11.13 implies $x_j \in D_{H'}(C)$ for $j = 1, 2$, so $[x_1, x_2] \subseteq D_{H'}(C)$. The geodesic ray x_1 and x_2 is not contained in H_1 since $x_2 \notin H_1$, so this ray meets H_1 only at x_1. Lemma 11.13 implies $x_2 \in D_{H_1}(C)$, so $(x_1, x_2] \subseteq D_{H_1}(C)$. Similarly $[x_1, x_2) \subseteq D_{H_2}(C)$. Thus $(x_1, x_2) \subseteq D_H(C)$ for all $H \in \mathfrak{H}$. $\square$

11.16 Lemma. *Let C be a chamber and H_1 and H_2 be two walls of C. Suppose H_1 and H_2 intersect in $\mathcal{H}^n$. Let L be a hyperplane in $\mathcal{H}^n$ such that $H_1 \cap H_2 \subseteq L$ and L meets $D_{H_1}(C) \cap D_{H_2}(C)$.f Then L meets C.*

Proof. (see [B] p. 68, prop. 10). Choose unit normals r_j to $D_{H_j}(C)$, so $H_j = r_j^\perp$ and $C \subseteq D_{r_j^\perp}(r_j) = D_{H_j}(C)$. Choose r such that $r^2 = 1$ and $L = r^\perp$. Since $H_1 \cap H_2 \subseteq L$, we get $(\mathsf{span}\{r_1, r_2\})^\perp \subseteq r^\perp$, so $r = \lambda_1 r_1 + \lambda_2 r_2$ for some $\lambda_1, \lambda_2 \in \mathbb{R}$. Pick $y \in L \cap D_{H_1}(C) \cap D_{H_2}(C)$. Then $0 = \langle r, y\rangle = \lambda_1 \langle r_1, y\rangle + \lambda_2 \langle r_2, y\rangle$. Since both $\langle r_1, y\rangle$ and $\langle r_2, y\rangle$ are positive, it follows that $\lambda_1 \lambda_2 < 0$.

For $j = 1, 2$, let x_j be a generic point of $\bar{C} \cap H_j$. Then $x_2 \in D_{r_1^\perp}(r_1)$, so $\langle r_1, x_2\rangle > 0$. Similarly, $\langle r_2, x_1\rangle > 0$. Now $\langle r, x_1\rangle = \lambda_2 \langle r_2, x_1\rangle$ and $\langle r, x_2\rangle = \lambda_1 \langle r_1, x_2\rangle$. Since $\lambda_1 \lambda_2 < 0$, we find that x_1 and x_2 are strictly on opposite sides of L. So L meets (x_1, x_2). Lemma 11.15 implies that $(x_1, x_2) \subseteq C$. $\square$

12 Opposite half spaces

12.1 Definition. Let $V = \mathbb{R}^n$ or $V = \mathbb{R}^{n,1}$. Let r be a positive norm vector in V. Define $D(r) = D_{r^\perp}(r)$. Note that $D(r) = \{x \in V : \langle x, r\rangle > 0\}$ is the open half space of V bounded by the hyperplane $r^\perp$ that contains r.

Let $r_1, r_2 \in V$ be vectors of norm 1 such that $\mathsf{span}\{r_1, r_2\}$ is positive definite of dimension 2. Let B_1 and B_2 be affine half spaces in V that are translates of $D(r_1)$ and $D(r_2)$ respectively. Define the dihedral angle between the half spaces B_1 and B_2 to be $\cos^{-1}(-\langle r_1, r_2\rangle) \in [0, \pi)$. Note that, if $V = \mathbb{R}^n$, then this is the angle that the cone $D(r_1) \cap D(r_2)$ makes at $r_1^\perp \cap r_2^\perp$.

12.2 Definition. Let B be an open half space in $\mathcal{H}^n$ (resp. an open half affine half space in $\mathbb{R}^n$). Then there exists a unique vector r in $\mathbb{R}^{n,1}$ (resp. $\mathbb{R}^n$) such that $r^2 = 1$ and $B = D(r)$ (resp. B is a translate of $D(r)$). We say that r is the *unit normal vector* to to the half space B (in more precise terms, r is the unit normal to ∂B pointing towards B). Let B_1 and B_2 be two half spaces in $\mathcal{H}^n$ or two affine half spaces in $\mathbb{R}^n$. Let r_i be the unit normal to B_i. We say that B_1 and B_2 are *opposite* half spaces if $\langle r_1, r_2\rangle \leq 0$.

12.3 Lemma. *Let B_1, B_2 be two half spaces in $\mathcal{H}^n$. Assume $\partial B_1 \cap \partial B_2 = \emptyset$.*

(a) The following are equivalent:
(i) $\partial B_1 \subseteq B_2$ and $\partial B_2 \subseteq B_1$.
(ii) $B_2^{op} \subseteq B_1$.
(iii) $B_1^{op} \cap B_2^{op} = \emptyset$.
(iv) $B_1^{op} \subseteq B_2$.
(v) B_1 and B_2 meet but do not contain each other.

(b) Of the four pairs (B_1, B_2), (B_1, B_2^{op}), (B_1^{op}, B_2), (B_1^{op}, B_2^{op}), there is exactly one pair that do not intersect. This pair consists of the two half spaces that do not meet $\partial B_1 \cup \partial B_2$.

Proof. (a) Assume (i). Since $\partial B_1 \subseteq B_2$, the half space B_2^{op} does not meet ∂B_1, so is contained in one side of it, that is, either $B_2^{op} \subseteq B_1$ or $B_2^{op} \subseteq B_1^{op}$. But the second possibility would imply $\partial B_2 \subseteq \mathsf{cl}(B_1^{op})$ which contradicts $\partial B_2 \subseteq B_1$. So (i) implies (ii).

Assume (ii). Then $B_1^{op} \cap B_2^{op} \subseteq B_1^{op} \cap B_1 = \emptyset$. Thus (ii) implies (iii).

Assume (iii). Then $B_1^{op} \subseteq \mathrm{cl}(B_2)$, so $\partial B_1 \subseteq \mathrm{cl}(B_2) = B_2 \cup \partial B_2$. But ∂B_1 is disjoint from ∂B_2, so $\partial B_1 \subseteq B_2$. Similarly $\partial B_2 \subseteq B_1$. Thus (iii) implies (i). Interchanging the role of B_1 and B_2, we find (i), (ii), (iii), (iv) are equivalent.

Assume (i) through (iv). If $B_1 \subseteq B_2$, then (iv) would imply $B_1 \cup B_1^{op} \subseteq B_2$, so $\mathcal{H}^n \subseteq \mathrm{cl}(B_2)$ which is absurd. So $B_2 \not\supseteq B_1$; similarly $B_1 \not\supseteq B_2$. Choose $x \in \partial B_1 \subseteq B_2$. There is a neighborbood of x that is contained in B_2, and any such neighborhood would intersect B_1, so $B_1 \cap B_2 \neq \emptyset$. Thus (i) implies (v).

Finally assume (v). If B_2 did not meet ∂B_1, then B_2 would be contained in one side of ∂B_1. However (v) implies $B_2 \not\subseteq B_1$ and $B_2 \not\subseteq B_1^{op}$. So ∂B_1 must meet B_2. But since ∂B_1 does not meet ∂B_2, the hyperplane ∂B_1 is contained in one side of ∂B_2. Hence $\partial B_1 \subseteq B_2$. For similar reason $\partial B_2 \subseteq B_1$. This proves part (a). For part (b), pick the side C_i of ∂B_i that contains the other hyperplane. Then (a) implies that (C_1^{op}, C_2^{op}) is the only pair that do not intersect. $\square$

12.4 Lemma (pair of non-intersecting hyperplanes)**.** *Let* $r, s \in \mathbb{R}^{n,1}$*,* $r^2 = s^2 = 1$*. Let* $H_r = r^\perp \cap \mathcal{H}^n$ *and* $H_s = s^\perp \cap \mathcal{H}^n$*.*

(a) One has $H_r \cap H_s = \emptyset$ *if and only if* $\mathsf{span}\{r, s\}$ *has signature* $(1, 1)$*, if and only if* $\langle r, s\rangle^2 > 1$*.*

(b) Assume $H_r \cap H_s = \emptyset$*. Then one of the four following mutually exclusive possiblities hold:*

(i) $\langle r, s\rangle > 1$*,* $H_r \subseteq D(s)$*,* $H_s \subseteq D(-r)$ *and* $D(-s) \cap D(r) = \emptyset$*.*

(ii) $\langle r, s\rangle > 1$*,* $H_r \subseteq D(-s)$*,* $H_s \subseteq D(r)$ *and* $D(s) \cap D(-r) = \emptyset$*.*

(iii) $\langle r, s\rangle < -1$*,* $H_r \subseteq D(s)$*,* $H_s \subseteq D(r)$ *and* $D(-s) \cap D(-r) = \emptyset$*.*

(iv) $\langle r, s\rangle < -1$*,* $H_r \subseteq D(-s)$*,* $H_s \subseteq D(-r)$ *and* $D(s) \cap D(r) = \emptyset$*.*

Proof. (b) Clearly four possibilities listed are mutually exclusive. Let $t = \langle r, s\rangle$, $c = 1/\sqrt{t^2 - 1}$, $s' = r - ts$ and $r' = s - tr$. Then

$$\langle s', s\rangle = \langle r', r\rangle = 0 \quad \text{and} \quad s'^2 = \langle s', r\rangle = -1/c^2 = \langle r', s\rangle = r'^2.$$

So $(cs')^2 = (cr')^2 = -1$ and $cr' \in D(-s)$, $cs' \in D(-r)$. We compute

$$\langle cr', cs'\rangle = c^2\langle r - ts, s - tr\rangle = c^2(t - t - t + t^3) = c^2 t(t^2 - 1) = t.$$

Since $r^\perp$ and $s^\perp$ do not meet in $\mathcal{H}^n$, either $\langle r, s\rangle > 1$ or $\langle r, s\rangle < -1$.

First consider the case $\langle r, s\rangle > 1$. Then $\langle cr', cs'\rangle > 0$. So either $cr', -cs' \in \mathcal{H}^n$ or $-cr', cs' \in \mathcal{H}^n$. If $-cr', cs' \in \mathcal{H}^n$, then $cs' \in D(-r) \cap H_s$, so $H_s \subseteq D(-r)$ and $-cr' \in H_r \cap D(s)$, so $H_r \subseteq D(s)$. Now lemma 12.3 implies that $D(-s) \cap D(r) = \emptyset$, so *(i)* holds. Similarly, if $-cr', cs' \in \mathcal{H}^n$, then *(ii)* holds.

Now consider the case $\langle r, s\rangle < -1$. Then $\langle cr', cs'\rangle < 0$. So either $-cr', -cs' \in \mathcal{H}^n$ or $cr', cs' \in \mathcal{H}^n$. We find that *(iii)* and *(iv)* holds respectively in these two cases. ☐

12.5 Lemma. *Let* B_1 *and* B_2 *be two half spaces in* $\mathcal{H}^n$ *or two affine half spaces in* $\mathbb{R}^n$*. Show that the following are equivalent:*

(a) B_1 *and* B_2 *are opposite half spaces.*

(b) One of the following three conditions hold (i) ∂B_1 *and* ∂B_2 *intersect in* $\mathcal{H}^n$ *(resp. in* $\mathbb{R}^n$*) and the dihedral angle between* B_1 *and* B_2 *is in* $[0, \pi/2]$*, (ii)* $\partial B_1 \subseteq B_2$ *and* $\partial B_2 \subseteq B_1$*, (iii)* $B_1 \cap B_2 = \emptyset$*.*

Proof. Let r_1 and r_2 be unit normals to B_1 and B_2. If ∂B_1 and ∂B_2 intersect, then the definition of dihedral angle shows that $\langle r_1, r_2\rangle \leq 0$ if and only if condition (i) of (b) holds. Suppose $\partial B_1 \cap \partial B_2 = \emptyset$. Then conditions (ii) and (iii) in (b) hold if and only if we are in the cases (iii) and (iv) respectively in lemma 12.4. Thus the equivalence of (a) and (b) in this case follows from lemma 12.4. ☐

12.6 Lemma. *Let* B_1 *and* B_2 *be two half spaces in* $\mathcal{H}^n$*. Assume that* ∂B_1 *and* ∂B_2 *do not meet and* B_1 *and* B_2 *are not opposite. Then one of the half spaces contain the closure of the other.*

Proof. Since B_1 and B_2 are not opposite, without loss, we may assume $\partial B_1 \nsubseteq B_2$. Since $\partial B_1 \cap \partial B_2 = \emptyset$, the hyperplane ∂B_1 is contained in one side of ∂B_2. So we must have $\partial B_1 \subseteq B_2^{op}$. This implies B_2 does not meet ∂B_1, so either $B_2 \subseteq B_1$ or $B_2 \subseteq B_1^{op}$. The second case is impossible since it would mean B_1 and B_2 do not meet, hence are opposite. So $B_2 \subseteq B_1$. Hence $\partial B_2 \subseteq \bar{B}_1 = B_1 \cup \partial B_1$. Since ∂B_2 and ∂B_1 do not meet, it follows that $\partial B_2 \subseteq B_1$. So $\bar{B}_2 \subseteq B_1$. $\square$

13 Real hyperbolic reflection groups

13.1. Setup: For this section, fix $\mathfrak{H}$ to be a collection of hyperplanes in $\mathcal{H}^n$. Let W be the group generated by reflections in the hyperplanes in $\mathfrak{H}$. We assume that

$$W \textit{ acts properly discontinuously on } \mathcal{H}^n \textit{ and } \mathfrak{H} \textit{ is stable under } W.$$

We say that W is a (hyperbolic) reflection group. The hyperplanes in $\mathfrak{H}$ are called *mirrors* (of W).

Let $z \in \partial\mathcal{H}^n$. Say that z is a *cusp* of $\mathfrak{H}$ if there exists a horoball around z that only meets the hyperplanes that pass through z. Let $\mathrm{cl}_{\mathfrak{H}}(\mathcal{H}^n)$ be the union of $\mathcal{H}^n$ and the cusps of $\mathfrak{H}$. We define a topology on $\mathrm{cl}_{\mathfrak{H}}(\mathcal{H}^n)$ by the following prescription:

- $\mathrm{cl}_{\mathfrak{H}}(\mathcal{H}^n)$ contains $\mathcal{H}^n$ as an open dense set.
- If $z \in \mathrm{cl}_{\mathfrak{H}}(\mathcal{H}^n) \setminus \mathcal{H}^n$, a basis for open sets around z is given by sets of the form $B \cup \{z\}$ where B is an open horoball around z.

The hyperplanes, half spaces, chambers etc will from now on be considered as subsets of $\mathrm{cl}_{\mathfrak{H}}(\mathcal{H}^n)$. The action of the reflection group W on $\mathcal{H}^n$ extends to an action on $\mathrm{cl}_{\mathfrak{H}}(\mathcal{H}^n)$.

13.2 Lemma ([B], p 77, lemma 1)**.** *The hyperplanes $\mathfrak{H}$ form a locally finite collection.*

13.3 Lemma. *Let H be a hyperplane H^n. Let $a \in H^n \setminus H$, $y \in \mathcal{H}^n$. Then $y \in D_H(a)$ if and only if $d(a, R_H(y)) > d(a, y)$. One has $y \in H$ if and only if $d(a, R_H(y)) = d(a, y)$.*

Proof. Choose s such that $H = s^\perp$ and $s^2 = 1$. One has

$$\cosh(d(a, R_s(y)) - \cosh(d(a, y)) = -\langle a, R_s(y)\rangle + \langle a, y\rangle = 2\langle a, s\rangle\langle y, s\rangle.$$

So $d(a, R_s(y)) > d(a, y)$ if and only if $\langle a, s\rangle\langle y, s\rangle > 0$, that is, a and y are strictly on the same side of $s^\perp$. One has $d(a, R_s(y)) = d(a, y)$ if and only if $\langle a, s\rangle\langle y, s\rangle = 0$, if and only if $\langle y, s\rangle = 0$ (since $\langle a, s\rangle \neq 0$). □

13.4 Lemma ([B], p 77, lemma 2)**.** *Let C be a chamber with respect to $\mathfrak{H}$. Let S be the set of reflections with respect to the walls of C.*

(a) Let $y \in \mathcal{H}^n$. Then there exists $w \in \langle S\rangle$ such that $wy \in \mathrm{cl}(C)$.

(b) The group W acts transitively on the set of chambers.

(c) One has $\langle S\rangle = W$, in other words, the reflections in the walls of a chamber generate the reflection group.

Proof. Let $P = \langle S\rangle$ be the subgroup of W generated by S.

(a) Fix $a \in C$. We shall show that the orbit Py meets $\mathrm{cl}(C)$. Choose $z \in Py$ such that $d(z, a) \leq d(x, a)$ for all $x \in Py$. Such a point z exists since only finitely many points of Py meets a fixed ball around a. Let H be a wall of C.

Then $R_H \in P$, so by our choice of z, we have $d(z,a) \leq d(R_H(z),a)$. Lemma 13.3 implies that $z \in \mathrm{cl}(D_H(a)) = \mathrm{cl}(D_H(C))$. So by lemma 11.14, $z \in \mathrm{cl}(C)$.

(b) Choose a chamber C' and $y \in C'$. By (a), there is $w \in P$ such that $wy \in \mathrm{cl}(C)$. But wy does not belong to any hyperplane of $\mathfrak{H}$, since y does not. So $wy \in C$. So $wC' \cap C \neq \emptyset$. So $wC' = C$. So P acts transitively on the chambers.

(c) Let H be a mirror of W. It suffices to show $R_H \in P$. Choose a chamber C' such that H is a wall of C'. By (b), there exists $w \in P$ such that $wC' = C$, so wH is a wall of C. So $wR_Hw^{-1} = R_{wH} \in P$. It follows that $R_H \in P$. ☐

For background on Coxeter systems see [B] Ch. IV, §1.

13.5 Theorem ([B] p 78, theorem 1)**.** *Let C be a chamber with respect to $\mathfrak{H}$. Let S be the set of reflections with respect to the walls of C.*

(a) (W,S) is a Coxeter system.

(b) Let $H \in \mathfrak{H}$. If $w \in W$ such that $l(R_Hw) > l(w)$, then C and wC are on the same side of H.

(c) The group W acts simply transitively on the set of chambers.

Proof. See [B]. ☐

13.6 Remark. The reflection groups of Lorentzian lattices are examples of hyperbolic reflection groups. Keep in mind that in the setup of 13.4, the chamber C may have infinitely many walls so (W,S) may be Coxeter system with infinitely many generators. Indeed, there is a very interesting hyperbolic reflection group acting on $\mathcal{H}^{25}$ where S is in bijection with the vectors of the Leech lattice.

Next we study the stabilizers of points in $\mathcal{H}^n$ and the stabilizers of cusps.

13.7 Lemma. *Let F be a facet or a cusp of $\mathfrak{H}$. Let C_1 and C_2 be two chambers such whose closures contain F. Let H be a hyperplane that seperates C_1 and C_2. Then H contains F.*

Proof. If F is a facet, pick $v \in F$. If F is a cusp, let $v = F$. Suppose H is a hyperplane that does not contain F. Then there is an open ball (or horoball) B around v that does not meet H. For $j = 1,2$, since $v \in \mathrm{cl}(C_j)$, the neighborhood B meets C_j. Pick $x_j \in C_j \cap B$. Since B is convex, there is a path inside B joining x_1 and x_2 and this path does not meet H since $H \cap B = \emptyset$. So H does not seperate C_1 and C_2. ☐

13.8 Theorem. *Let F be a facet or a cusp of $\mathfrak{H}$. Let $\mathcal{C}_F$ be the set of chambers whose closures contain F. The pointwise stabilizer of F in W is generated by the reflections in the mirrors containing F and this stabilizer acts simply transitively on $\mathcal{C}_F$.*

Proof. Let P be the group generated by the reflections in the mirrors containing F. Fix a chamber $C_1 \in \mathcal{C}_F$ and pick $a \in C_1$. Let $C' \in \mathcal{C}_F$. Choose $y \in C'$. First we are going to show that the orbit Py meets C_1. As in proof of 13.4, choose $z \in Py$ such that $d(z,a) \leq d(x,a)$ for all $x \in Py$. Note that if R is a reflection in a mirror containing F, then R permutes the chambers in $\mathcal{C}_F$. Since $y \in C' \in \mathcal{C}_F$ and $z \in Py$, it follows that $z \in C_2$ for some $C_2 \in \mathcal{C}_F$. If $C_1 \neq C_2$, let H be any mirror that seperates C_1 and C_2. Lemma 13.7 implies that H contains F. So $R_H \in P$. So by our choice of z, we have $d(z, a) \leq d(R_H(z), a)$. Lemma 13.3 implies that $z \in \mathrm{cl}(D_H(a))$. This contradiction proves $C_1 = C_2$. So P acts transitively on $\mathcal{C}_F$.

Let $w \in W$ such that w fixes F pointwise. Then $wC_1 \in \mathcal{C}_F$. By part (a), there exists $p \in P$ such that $wC_1 = pC_1$. Since W is simply transitive on the chambers, we get $w = p \in P$. □

13.9 Lemma. *Let C be a chamber with respect to $\mathfrak{H}$ and H_1, H_2 be two distinct walls of C. Then $D_{H_1}(C)$ and $D_{H_2}(C)$ are opposite half spaces.*

Proof. Suppose $H_1 \cap H_2 = \emptyset$. Note that $C \subseteq D_{H_1}(C)$, so $\mathrm{cl}(C) \subseteq \mathrm{cl}(D_{H_1}(C))$. Since H_2 meets $\mathrm{cl}(C)$, this hyperplane must meet $\mathrm{cl}(D_{H_1}(C)) = D_{H_1}(C) \cup H_1$. But $H_2 \cap H_1 = \emptyset$, we find H_2 meets $D_{H_1}(C)$. Since H_2 does not meet H_1, it must be contained in one side of H_1, so $H_2 \subseteq D_{H_1}(C)$. Simillarly $H_1 \subseteq D_{H_2}(C)$. So $D_{H_1}(C)$ and $D_{H_2}(C)$ are opposite (condition (ii) of being opposite).

Now suppose H_1 and H_2 meet. Choose unit normals r_1 and r_2 to $D_{H_j}(C)$. So $r_j^2 = 1$ and $D_{H_j}(C) = D(r_j)$ for $j = 1, 2$. Suppose $D(r_1)$ and $D(r_2)$ are not opposite. Then $\langle r_1, r_2 \rangle > 0$. Choose $x_j \in \mathrm{cl}(C)$ such that $\mathfrak{H}(x_j) = \{H_j\}$. Let $r = R_{r_1}(r_2) = r_2 - 2\langle r_1, r_2 \rangle r_1$. Then $\langle r, x_1 \rangle = \langle r_2, x_1 \rangle$ and $\langle r, x_2 \rangle = -2\langle r_1, r_2 \rangle \langle r_1, x_2 \rangle$ have strictly opposite signs. So x_1 and x_2 are strictly on opposite sides of $r^\perp = R_{H_1}(H_2)$. So $R_{H_1}(H_2)$ meets (x_1, x_2) which is contained in C by lemma 11.15. So $R_{H_1}(H_2)$ meets C. But this is a contradiction since $R_{H_1}(H_2) \in \mathfrak{H}$. □

14 Vinberg's algorithm

For this section, maintain the setup of 13.1. Vinberg's algorithm gives a method for finding a chamber for the hyperbolic reflection group W.

14.1. Vinberg's algorithm: Fix $x_0 \in \mathrm{cl}_{\mathfrak{H}}(\mathcal{H}^n)$, called *controlling vector*. Choose a chamber C_0 for the arrangement $\mathfrak{H}(x_0)$. Pick an enumeration $L_1, L_2, \cdots$ of the mirrors in $\mathfrak{H} \setminus \mathfrak{H}(x_0)$ such that $d_{x_0}(L_1) \leq d_{x_0}(L_2) \leq \cdots$ (here d_{x_0} denotes the hyperbolic or horocyclic distance from x_0). For a mirror L, define $D_L^+ = D_L(x_0)$ if $x_0 \notin L$ and $D_L^+ = D_L(C_0)$ if $x_0 \in L$. Inductively, define $\Delta_0 \subseteq \Delta_1 \subseteq \cdots \subseteq \mathfrak{H}$ as follows:

- Let Δ_0 be the set of walls of C_0.
- Suppose we have defined $\Delta_0, \cdots, \Delta_{n-1}$. If $D_{L_n}(x_0)$ and D_H^+ are opposite for each $H \in \Delta_{n-1}$ such that $d_{x_0}(H) < d_{x_0}(L_n)$, then say that *we accept* L_n and let $\Delta_n = \Delta_{n-1} \cup \{L_n\}$. Otherwise, we *reject* L_n and let $\Delta_n = \Delta_{n-1}$.

Let $\Delta = \cup_n \Delta_n$. One has $C_0 = \cap_{H \in \Delta_0} D_H(C_0)$ and $C = C_0 \cap (\cap_{H \in \Delta \setminus \Delta_0} D_H(x_0))$ is the unique chamber of $\mathfrak{H}$ such that $C \subseteq C_0$ and $x_0 \in \mathrm{cl}(C)$. The collection $\Delta = \mathsf{Wall}(C)$.

We need a couple of lemmas before we show that the above algorithm works.

14.2 Lemma. *Fix $x_0 \in \mathrm{cl}_{\mathfrak{H}}(\mathcal{H}^n)$ and a chamber C_0 of the arrangement $\mathfrak{H}(x_0)$.*
(a) Then there is a unique chamber C of $\mathfrak{H}$ such that $x_0 \in \mathrm{cl}(C)$ and $C \subseteq C_0$.
(b) A hyperplane $H \in \mathfrak{H}(x_0)$ is a wall of C if and only if it is a wall of C_0.

Proof. (a) Let B be a $\mathfrak{H}$-small neighborhood of x_0. Then $B \cap C_0$ is non-empty and does not meet any mirror of $\mathfrak{H}$, so must be contained in some chamber C of $\mathfrak{H}$. Then $C \subseteq C_0$ and $x_0 \in \mathrm{cl}(C)$.

Suppose C_1 and C_2 are chambers of $\mathfrak{H}$ such that $C_j \subseteq C_0$ and $x_0 \in \mathrm{cl}(C_j)$. By 13.8, there exists $g \in W_{x_0}$ such that $gC_1 = C_2$. Since C_1 and C_2 are contained in C_0, it follows that $gC_0 = C_0$. But W_{x_0} is the reflection group of the arrangement $\mathfrak{H}(x_0)$, so it is simply transitive on the chambers of $\mathfrak{H}(x_0)$; hence $gC_0 = C_0$ implies $g = \mathsf{id}$. So $C_1 = C_2$.

(b) Let $H \in \mathfrak{H}(x_0)$. Suppose H is a wall of C. By implication $(i) \implies (iii)$ of 11.13, there exists $x \in \mathrm{cl}(C)$ such that $\mathfrak{H}(x) = \{H\}$. Since $C \in C_0$, we have $x \in \mathrm{cl}(C_0)$. Now, the reverse implication $(iii) \implies (i)$ of lemma 11.13 tells us H is a wall of C_0. Convesely, suppose H is a wall of C_0. Pick $x \in \mathrm{cl}(C_0)$ such that the only element of $\mathfrak{H}(x_0)$ passing through x is H. Let B be a $\mathfrak{H}$-small ball around x_0. Let $y \in (x_0, x] \cap B$. Since $\mathrm{cl}(C_0)$ is convex, we have $y \in \mathrm{cl}(C_0)$. Let $L \in \mathfrak{H}(y)$. Since $y \in B$, we have $L \in \mathfrak{H}(x_0)$. Since L contains both x_0 and y, it contains the whole geodesic ray joining x_0 and y, so $x \in L$ and hence $L = H$. So $\mathfrak{H}(y) = \{H\}$. Let U be any open set containing y. Since $y \in \mathrm{cl}(C_0)$, the intersection $U \cap B \cap C_0$ is non-empty. In the proof of part (a), we saw that $B \cap C_0 \subseteq C$. So $U \cap C$ is non-empty. So $y \in \mathrm{cl}(C)$. So H is a wall of C. $\square$

14.3 Lemma. *Let B_1, B_2 be opposite half spaces in $\mathcal{H}^n$. Let $x_0 \in \mathrm{cl}_{\mathfrak{H}}(\mathcal{H}^n)$ such that $x_0 \in B_1 \cap \mathrm{cl}(B_2)$. Let x be the projection of x_0 onto ∂B_1. Then $x \in \mathrm{cl}(B_2)$. Further $x \in \partial B_2$ if and only if $x_0 \in \partial B_2$ and the unit normals to B_1 and B_2 are orthogonal.*

Proof. Suppose $x \notin \mathrm{cl}(B_2)$. Let $i = 1$ or 2. Let r_i be the unit normal to B_i. So $r_i^2 = 1$, $B_i = D(r_i)$, $\langle r_1, r_2 \rangle \leq 0$. Since $x_0 \in B_1 \cap \mathrm{cl}(B_2)$, we have $\langle x_0, r_1 \rangle > 0$ and $\langle x_0, r_2 \rangle \geq 0$. Let $y = x_0 - \langle x_0, r_1 \rangle r_1$. Recall that $x = y/\sqrt{-y^2}$ and the geodesic segment $[x_0, x]$ is parametrized by $\mathbb{P}((1-t)x_0 + ty)$ with $t \in [0, 1]$. Since $\partial B_2 = r_2^{\perp}$ seperates x and x_0, it must meet $[x_0, x]$. So there exists $t \in [0, 1)$ such that $\langle r_2, (1-t)x_0 + ty \rangle = 0$. This is equivalent to

$$\langle r_2, x_0 \rangle = t \langle r_1, x_0 \rangle \langle r_2, r_1 \rangle.$$

If $t \neq 0$, then we must have $\langle r_2, x_0 \rangle = 0 = \langle r_2, r_1 \rangle$, which forces $\langle r_2, y \rangle = 0$ and $x \in \partial B_2$ which is a contradiction. So we must have $t = 0$ and $\langle r_2, x_0 \rangle = 0$. But in this case, since $y \notin \mathrm{cl}(B_2)$, we get

$$0 > \langle r_2, y \rangle = \langle r_2, x_0 \rangle - \langle r_1, x_0 \rangle \langle r_2, r_1 \rangle = -\langle r_1, x_0 \rangle \langle r_2, r_1 \rangle,$$

which again forces $\langle r_1, r_2 \rangle > 0$, again a contradiction. Thus $x \in \mathrm{cl}(B_2)$. Now $x \in \partial B_2$ if and only if $\langle y, r_2 \rangle = 0$ which translates into $\langle r_2, x_0 \rangle = \langle r_1, x_0 \rangle \langle r_2, r_1 \rangle$. Looking at signs of the three terms, we find this equality can hold if and only if $\langle r_2, x_0 \rangle = 0 = \langle r_2, r_1 \rangle$. □

14.4 Lemma. *Let L and H be hyperplanes in $\mathcal{H}^n$. Let $\xi \in \mathbb{P}(V_{\leq})$ such that $\xi \notin L \cup H$. If $\mathsf{pr}_L(\xi) = \mathsf{pr}_H(\xi)$, then $L = H$.*

Proof. Choose $x \in V_{\leq}$ such that $\mathbb{P}(x) = \xi$. Choose r, s such that $r^2 = s^2 = 1$ and $L = r^{\perp} \cap \mathcal{H}^n$, $H = s^{\perp} \cap \mathcal{H}^n$. Now $\mathsf{pr}_L(\xi)$ and $\mathsf{pr}_H(\xi)$ are scalar multiples of $p_r = x - \langle r, x \rangle r$ and $p_s = x - \langle s, x \rangle s$ respectively. So $p_r = \lambda p_s$ for some non-zero scalar λ. Taking inner product of both sides with r and s yields $\langle r, x \rangle = \langle s, x \rangle \langle r, s \rangle$ and $\langle s, x \rangle = \langle r, x \rangle \langle s, r \rangle$. From these two equations, it follows that $\langle r, s \rangle^2 = 1$, since $\langle r, x \rangle$ and $\langle s, x \rangle$ are non-zero. Now the equation $\langle r, x \rangle = \langle s, x \rangle \langle r, s \rangle$ implies $\langle r, x \rangle^2 = \langle s, x \rangle^2$. It follows that $p_r^2 = x^2 - \langle r, x \rangle^2 = x^2 - \langle s, x \rangle^2 = p_s^2$, so $\lambda^2 = 1$. If $\lambda = -1$, then $p_r = -p_s$ and $p_r^2 = \langle p_r, x \rangle = -\langle p_s, x \rangle = -p_s^2$, which contradicts $p_r^2 = p_s^2 < 0$. Hence $\lambda = 1$ and $r = s$. □

14.5 Theorem. *Vinberg's algorithm 14.1 produces a chamber C for $\mathfrak{H}$.*

Proof. By lemma 14.2(a), there is a unique chamber C of $\mathfrak{H}$ such that $x_0 \in \mathrm{cl}(C)$. By induction n we shall prove the following proposition $\mathbf{P}_n$ for all $n \geq 0$:

$\mathbf{P}_n$: *One has $\Delta_n = \mathsf{Wall}(C) \cap (\mathfrak{H}(x_0) \cup \{L_1, \cdots, L_n\})$.*

Lemma 14.2(b) proves $\mathbf{P}_0$. By induction, assume $\mathbf{P}_{n-1}$. Write $L = L_n$. If L is rejected by the algorithm, then there exists $H \in \Delta_{n-1}$ such that $D_L(C)$ and $D_H(C)$ are not opposite and 13.9 implies that $L \notin \mathsf{Wall}(C)$, which proves $\mathbf{P}_n$.

Now suppose L is accepted by the algorithm. If possible, suppose L is not a wall of C. Let x be the projection of x_0 onto L.

Claim: If $H \in \mathsf{Wall}(C)$ and $x \notin D_H(C)$, then H contains the geodesic ray T' through x_0 and x

proof of claim: Let $H \in \mathsf{Wall}(C)$ such that $x \notin D_H(C)$. Then H meets $[x_0, x]$. So $d_{x_0}(H) \leq d_{x_0}(L)$. If $d_{x_0}(L) = d_{x_0}(H)$, then x is also the projection of x_0 on H, hence 14.4 implies $H = L$. $d_{x_0}(H) < d_{x_0}(L)$. So $H \in \mathfrak{H}(x_0) \cup \{L_1, \cdots, L_{n-1}\}$. By induction hypothesis $H \in \Delta_{n-1}$. Since L got accepted by the algorithm, the half spaces $D_H(C)$ and $D_L(C)$ must be opposite. Since $x_0 \in \mathrm{cl}(D_H(C)) \cap D_L(C)$, lemma 14.3 implies that $x \in \mathrm{cl}(D_H(C))$. Since $x \notin D_H(C)$, infact $x \in H$. So 14.3 further implies $x_0 \in H$. This proves the claim.

Let B be a $\mathfrak{H}$-small ball around x and consider the line segment $T = T' \cap B$. Let H be any wall of C. By the claim, either $x \in D_H(C)$ in which case $T \subseteq D_H(C)$, or else H contains T' in which case $T \subseteq H$. So $T \subseteq \mathrm{cl}(C)$. But this is absurd since one verifies that T meets both open half spaces boudned by $L \in \mathfrak{H}$. This contradiciton proves that if L_n is accepted by the algorithm, then $L_n \in \mathsf{Wall}(C)$. Hence $\mathbf{P}_n$ holds. ☐

14.6 Corollary. *Fix $x \in \mathrm{cl}_{\mathfrak{H}}(\mathcal{H}^n)$ and a chamber C for $\mathfrak{H}$ such that $x \in \mathrm{cl}(C)$. Let W_x be the stabilizer of x in W and let C_x be the chamber of W_x containing C. Then $\mathfrak{H}(x) \cap \mathsf{Wall}(C)$ is the set of walls of C_x.*

Proof. By 13.8, we know that W_x is generated by the reflections in the hyperplanes $\mathfrak{H}(x)$, so a chamber for W_x (resp W) is a chamber for the arrangement $\mathfrak{H}(x)$ (resp. $\mathfrak{H}$). Let C_x be the chamber for W_x containing C. By lemma 14.2(a), there is a unique chamber C' of $\mathfrak{H}$ such that $C' \subseteq C_x$ and $x \in \mathrm{cl}(C')$. But C is such a chamber, so $C' = C$. Now part (b) of lemma 14.2(b) implies that the walls of C_x are just the walls of C that pass through x. ☐

15 Group action on hyperbolic space

We collect a few basic facts about action of automorphism groups on hyperbolic space.

15.1 Lemma. *Let G be a group and H be a subgroup of finite index. Then there exists $N \subseteq H \subseteq G$ such that N is a normal subgroup of G of finite index.*

Proof. Let N be the kernel of the homomorphism $G \to \mathsf{Aut}(G/H)$ obtained from the left action of G on $G/H = \{gH : g \in G\}$. Since H is finite index in G, it follows that $\mathsf{Aut}(G/H)$ is a finite group, so N is a normal subgroup of finite index in G. Verify that $N \subseteq H$. □

For the rest of the section, let V be a real or complex vector space with the signature $(n, 1)$ (bilinear or hermitian) form $\langle \mid \rangle$. Let $X = P_-(V)$ be the real or complex hyperbolic space.

15.2 Lemma. *If G be a subgroup of $\mathsf{Aut}(V)$ that fixes a point in X, then G is finite.*

Proof. Let $x \in X$ such that $gx = x$ for all $g \in G$. Choose $v \in V_-$ such that $P(v) = x$. Then G fixes the line containing v, so G fixes $v^\perp$. Hence $G \subseteq \mathsf{Aut}(v^\perp)$ which is finite since $v^\perp$ is negative definite. □

The following is a fundamental result on linear group action. We shall not prove it.

15.3 Theorem (Selberg's lemma, 1960)**.** *A finitely generated linear group over a field of characteristic zero is virtually torsion free, that is, it contains a torsion free subgroup of finite index.*

The following corollary will be useful for us.

15.4 Corollary. *(a) Let G be a finitely generated subgroup of $\mathsf{Aut}(V)$. Then G contains a normal subgroup of finite index that acts freely on the hyperbolic space $X = P_-(V)$.*

(b) Assume further that G is a discrete subgroup of $\mathsf{Aut}(V)$. Then X/G is an orbifold (i.e. it locally looks like Euclidean space modulo finite groups).

Proof. (a) By Selberg's lemma G contains a torsion free subgroup H of finite index and H contains a finite index subgroup N that is normal in G. Since N is torsion free, lemma 15.2 implies that it acts freely on X.

(b) Let $K = G/N$. Since N is discrete and acts freely on X, it follows that N acts properly discontinuously on X, it follows that $Y = X/N$ is a manifold, so locally it looks like Euclidean space. One has $X/G = Y/K$, so it is a manifold modulo a finite group, hence an orbifold. □

16 Lattices

16.1 Definition. A *lattice* L is a $\mathbb{Z}$-module of finite rank with a symmetric bilinear form $\langle\ ,\ \rangle : L \times L \to \mathbb{Q}$. If the bilinear form takes values in $\mathbb{Z}$, then we say that L is *integral*. Define the *radical* of L, denoted $\mathsf{rad}(L)$, by

$$\mathsf{rad}(L) = \{x \in L \colon \langle x, y\rangle = 0 \ \text{ for all } \ y \in L\}.$$

If the bilinear form is *non-degenerate*, i.e. $\mathsf{rad}(L) \neq 0$, then we say L is singular; otherwise, we say that L is non-singular. Unless otherwise stated, we shall assume that our lattices are non-singular. Let L be an integral lattice. Say that L is *even* if the norm of every lattice vector is an even integer, otherwise, say that L is odd. Let $V = L \otimes \mathbb{R}$ be the underlying vector space of L. Define the dual lattice of L, denoted $L^\vee$, to be the set of all $v \in V$ such that $\langle v, L\rangle \subseteq \mathbb{Z}$. Say that L is *self-dual* if $L = L^\vee$.

Let $v_1, \cdots, v_k$ be a $\mathbb{Z}$-basis for a non-singular lattice L. Then $M = ((\langle v_i, v_j\rangle))$ is called a *gram matrix* of L. Suppose M has m positive eigenvalues and n negative eigenvalues, then we say L has signature (m, n). A lattice of signature $(m, 1)$ is called a Lorentzian lattice. The scalar $d(L) = \mathsf{det}(M)$ is called the *discriminant* of L. (One verifies that the pair (m, n) and the scalar $\mathsf{det}(M)$ does not depend on the choice of the gram matrix M). We state the basic properties of the discriminant; the proofs are left out as exercises. If L is a integral lattice then $L \subseteq L^\vee$ and one has

$$d(L) = [L^\vee : L] = d(L^\vee)^{-1}.$$

If $L \subseteq M$ are integral lattices, then

$$L \subseteq M \subseteq M^\vee \subseteq L^\vee \ \text{ and } \ [L : M] = [M^\vee : L^\vee].$$

If $L \subseteq \mathbb{R}^n$, then $d(L) = \mathsf{vol}(\mathbb{R}^n/L)^2$.

We define a few important lattices. Recall that $\mathbb{R}^{m,n}$ denotes the $(m+n)$ dimensional real vector space with the quadratic form

$$x^2 = -x_1^2 - \cdots - x_n^2 + x_{n+1}^2 + \cdots + x_{m+n}^2.$$

So $\mathbb{R}^m = \mathbb{R}^{m,0}$. We often define a lattice L by specifying L as a subset of $\mathbb{R}^{m,n}$ with the bilinear form induced from $\mathbb{R}^{m,n}$.

16.2 Example. Let m, n be non-negative integers. Define $I_{m,n} \subseteq \mathbb{R}^{m,n}$ to be the set of all vectors $x = (x_1, \cdots, x_{m+n})$ where each $x_j \in \mathbb{Z}$. So $I_{m,0}$ is just the usual integer lattice $\mathbb{Z}^m$. If $m - n \equiv 0 \bmod 8$, define $I\!I_{m,n} \subseteq \mathbb{R}^{m,n}$ by

$$I\!I_{m,n} = \{x \in \mathbb{R}^{m,n} \colon 2x_j \in \mathbb{Z}, (x_i - x_j) \in \mathbb{Z} \ \text{ for all } \ i, j, \sum x_j \equiv 0 \bmod 2\}.$$

The lattice $H = I\!I_{1,1}$ is called a *hyperbolic cell*. For $n \geq 1$, let

$$L(a_n) = \{(x_0, \cdots, x_n) \in \mathbb{Z}^{n+1} \colon \sum x_i = 0\}.$$

For $n \geq 3$, let

$$L(d_n) = \{(x_1, \cdots, x_n) \colon \mathbb{Z}^n \colon \sum x_i \equiv 0 \bmod 2\}.$$

Let $L(e_8) = II_{8,0}$. Let $L(e_7)$ be the orthogonal complement of any norm 2 vector in $L(e_8)$ and let $L(e_6)$ be the orthogonal complement of any $L(a_2)$ sitting inside $L(e_8)$. One verifies that all the above lattices are integral and all the ones other than $I_{m,n}$ are even.

16.3 Lemma. *Let m, n be non-negative integers such that $(m - n) \equiv 0 \bmod 8$. Then $II_{m,n}$ is a even self dual integral lattice of signature (m, n).*

Proof. The lattice $II_{m,n}$ has signature (m, n) since the lattice spans the vector space $\mathbb{R}^{m,n}$, One directly verifies that $II_{m,n}$ is even and integral if $(m - n) \equiv 0 \bmod 8$. Let $I_{m,n}$ be the sublattice of $\mathbb{R}^{m,n}$ consisting of all vectors that have integer coordinates. One easily checks that $I_{m,n}$ is self-dual. Let $I^+_{m,n}$ be the set of all vectors $x \in I_{m,n}$ such that $\sum_j x_j \equiv 0 \bmod 2$. Then $I^+_{m,n}$ is of index 2 in $I_{m,n}$, so $[(I^+_{m,n})^\vee : I_{m,n}] = 4$. One verifies that $II_{m,n}$ is spanned by $I^+_{m,n}$ and the vector $v = (\frac{1}{2}, \frac{1}{2}, \cdots, \frac{1}{2})$ and v has order 2 in the quotient $II_{m,n}/I^+_{m,n}$. So

$$[II_{m,n} : I^+_{m,n}] = [(I^+_{m,n})^\vee : II^\vee_{m,n}] = 2.$$

On the other hand

$$4 = [(I^+_{m,n})^\vee : I_{m,n}] = [(I^+_{m,n})^\vee : II^\vee_{m,n}][II^\vee_{m,n} : II_{m,n}][II_{m,n} : I^+_{m,n}].$$

It follows that $[II^\vee_{m,n} : II_{m,n}] = 1$. $\square$

For each $m, n \in \mathbb{Z}_>$ such that $(m - n) \equiv 0 \bmod 8$, then there is a unique even self-dual integral lattice of signature (m, n) (see [S]); the lattice $II_{m,n}$ is a concrete model for this lattice.

16.4 Lemma. *Let L be an even unimodular lattice. Let z be a primitive null vector of L. Then there exists $z_1 \in L$ such that $H = \mathsf{span}_{\mathbb{Z}}\{z, z_1\} \simeq II_{1,1}$. Then $N = H^\perp$ is even unimodular, $z^\perp \simeq N \oplus \mathbb{Z}z$ and $z^\perp/z \simeq N$.*

Proof. Since L is self dual, there exists $z_2 \in L$ such that $\langle z, z_2 \rangle = 1$. Let $z_1 = z_2 - (z_2^2/2)z$. Then $z_1^2 = 0$ and $\langle z, z_1 \rangle = 1$. So $H = \mathsf{span}_{\mathbb{Z}}\{z, z_1\} \simeq II_{1,1}$. If $x \in L$, then verify that $x - \langle x, z_1 \rangle z - \langle x, z \rangle z_1 \in H^\perp = N$, so $L = H \perp N$. Let $f \in N^\vee$. Then f extends to a functional on $N \perp H = L$ by letting $f(H) = 0$. Since L is self dual, there exists $v \in L$ such that $f(\cdot) = \langle v, \cdot \rangle$. If $h \in H$, then $\langle v, h \rangle = f(h) = 0$, so $v \in H^\perp = N$. Thus N is self-dual, hence a Niemeier lattice. Let $x \in z^\perp$. Then we can write $x = n + cz + c_1 z_1$ with $c, c_1 \in \mathbb{Z}$ and $n \in N$. One has $c_1 = \langle x, z \rangle = 0$, so $x \in N \oplus \mathbb{Z}z$. So $z^\perp = N \oplus z\mathbb{Z}$. It follows that $N \simeq z^\perp/z$. $\square$

16.5 Lemma. *Let $L = II_{8n+1,1}$. Then the following sets are in natural bijection:*

1. *the orbits of cusps of L under* $\mathsf{Aut}(L)$.

2. *the isomorphism classes of positive definite even unimodular lattices of dimension* $8n$.

Proof. Let ρ_1 and ρ_2 be cusps of L. By lemma 16.4, there exists hyperbolic cells H_j containing ρ_j such that $L = H_j \oplus H_j^\perp$ and $H_j^\perp \simeq \rho_j^\perp/\rho_j$ are positive even unimodular lattice of dimension $8n$. If $g\rho_1 = \rho_2$ for some $g \in \mathsf{Aut}(L)$, then g induces an isomorphism from $\rho_1^\perp/\rho_1$ to $\rho_2^\perp/\rho_2$. Conversely, if $\rho_1^\perp/\rho_1 \simeq \rho_2^\perp/\rho_2$, then pick any isomorphism from $H_1^\perp$ to $H_2^\perp$ and take an isomorphism from H_1 to H_2 that sends ρ_1 to ρ_2. Using the splitting $L = H_j \oplus H_j^\perp$, we get an element $g \in \mathsf{Aut}(L)$ that takes ρ_1 to ρ_2. $\square$

17 Simply laced finite root systems

We give a quick run through the basics of simply laced finite root systems.

17.1 Definition. Let $(U, \langle\ ,\ \rangle)$ be a positive definite real inner product space of dimension n. A finite subset Φ of U is called a *simply laced root system* of rank n in U if $U = \mathsf{span}_{\mathbb{Z}}(\Phi)$ and for all $u, v \in \Phi$, we have, $\langle u, v\rangle \in \mathbb{Z}$, $v^2 = 2$ and $R_v(\Phi) = \Phi$. (*Recall: R_v denotes reflection in v*). Let W be the group generated by $\{R_v \colon v \in \Phi\}$. The reflection group W is called the *Weyl group* of Φ.

If Φ_1 and Φ_2 are root systems in U_1 and U_2 respectively, then $\Phi = \Phi_1 \cup \Phi_2$ is a root system in $U_1 \perp U_2$. We say that Φ is the direct sum of Φ_1 and Φ_2 and we write $\Phi = \Phi_1 \perp \Phi_2$. A root system is called *irreducible* if it is not a direct sum of two non-empty root systems.

In this section, Φ would always denote a simply laced root system. Since $R_s(s) = -s \in \Phi$ for all $s \in \Phi$, we have $\Phi = -\Phi$.

Let L be an positive definite integral lattice. Then the norm 2 vectors in L form a simply laced root system, which will be denoted by Φ_L. The vectors of Φ_L are called the roots of L. We say that L is a root lattice if L is spanned by its roots. The lattice $\mathsf{span}_{\mathbb{Z}}(\Phi_L)$ is called the root sublattice of L. The reflections in the roots of L preserve L and generates the *reflection group* $\mathsf{Ref}(L)$ of L. Say that L is a *simply laced root lattice* if $L = \mathsf{span}_{\mathbb{Z}}(\Phi_L)$. If L is a simply laced root lattice then Φ_L is a simply laced root system.

17.2 Example. One verifies that $L(a_n)$, $L(d_n)$, $L(e_8)$,$L(e_7)$, $L(e_6)$ introdued in the previous section are simply laced root lattices. Let $\Phi(a_n)$, $\Phi(d_n)$, $\Phi(e_8)$, $\Phi(e_7)$, $\Phi(e_6)$ be the root systems of these lattices.

17.3. Configuration of two roots: Let $u, v \in \Phi$. If u and v are linearly dependent, then $u = \pm v$, since $u^2 = v^2$. If $|\langle u, v\rangle| = 2$, then $\mathsf{det}(\mathsf{gram}\{u, v\}) = 0$ which means u and v are linearly dependent (since U is positive definite) and hence $u = \pm v$. Since U is a positive definite inner product space, $\mathsf{det}(\mathsf{gram}\{u, v\}) \geq 0$. So $\langle u, v\rangle^2 \leq 4$. If u and v are non-proportional, then $|\langle u, v\rangle| \neq 2$, so $\langle u, v\rangle \in \{-1, 0, 1\}$. Finally, if $u \neq \pm v$ and $\langle u, v\rangle > 0$, then $\langle u, v\rangle = 1$, so $u - v = R_v(u)$ is a root.

17.4 Lemma. *(a) Let $V = U \otimes \mathbb{C}$. Then there exists a unique hermitian form H on V such that $H(u, u') = \langle u, u'\rangle$ for all $u, u' \in U$. The action of W on U extends uniquely to a $\mathbb{C}$-linear action on V. This action of W on V preserves the hermitian form H.*

(b) Let $s \in U \setminus \{0\}$ and consider $R_s \in W$ acting on V. If $x \in V$ and $R_s(x) = -x$, then $x \in s\mathbb{C}$.

(c) Let V_1 be a complex subspace of V. Let $s \in U \setminus \{0\}$ such that If $s \notin V_1$ and $R_s(V_1) = V_1$, then $s \in V_1^{\perp}$ (orthogonal complement with respect to H).

Proof. (a) Write $V = U \oplus iU$ and identify U inside V by $u \mapsto (u + i.0)$. Elements of V can be uniquely written as $(u + iu')$ with $u, u' \in U$. Define $H : V \times V \to \mathbb{C}$ by

$$H(u + iu', v + iv') = \langle u, v\rangle + \langle u', v'\rangle + i(\langle u', v\rangle - \langle u, v'\rangle)$$

for $u, u', v, v' \in U$. One verifies that H is a hermitian form on V such that $H(u, u') = \langle u, u' \rangle$ for all $u, u' \in U$. Let H is any hermitian form on V such that $H(u, u') = \langle u, u' \rangle$ for all $u, u' \in U$, then $H(u + iu', u + iu') = H(u, u) + H(u', u') + i(H(u', u) - H(u, u')) = \langle u, u \rangle + \langle u', u' \rangle$, since $\langle \ , \ \rangle$ is symmetric. Thus the H-norm of every element of V is determined by $\langle \ , \ \rangle$ and hence (by polarization) H is determined by $\langle \ , \ \rangle$.

(b) One has $-x = R_s(x) = x - 2H(x, s)s/s^2$, so $x = H(x, s)s/s^2$.

(c) Since R_s preserves V_1 it also preserves $V_1^{\perp}$. Write $s = s_1 + s_2$ where $s_1 \in V_1$ and $s_2 \in V_1^{\perp}$. Since $s \notin V_1$, we have $s_2 \neq 0$. Now $R_s(s_1) + R_s(s_2) = R_s(s) = -s_1 - s_2$. Since $R_s(s_1) \in V_1$ and $R_s(s_2) \in V_1^{\perp}$, it follows that $R_s(s_j) = -s_j$. Since $s_2 \neq 0$, part (b) implies that s_2 must be a non-zero multiple of s. Since $s_2 \in V_1^{\perp}$, we have $s \in V_1^{\perp}$. $\square$

17.5 Lemma. *Suppose Φ is an irreducible root system in U. Then $V = U \otimes \mathbb{C}$ is an irreducible representation of the Weyl group W. Upto scaling, $\langle \ , \ \rangle$ is the unique W-invariant positive definite bilinear form on U.*

Proof. The bilinear form $\langle \ , \ \rangle$ on U extends to a hermitian form H on $V = U \otimes \mathbb{C}$ and the $\langle \ , \ \rangle$-invariant action of W on U uniquely extends to a $\mathbb{C}$-linear H-invariant action of W on V. Suppose V is not irreducible. Then V has proper non-zero sub-representations V_1 and V_2 such that $V = V_1 \perp_H V_2$. Let $\Phi_j = \Phi \cap V_j$ and let $U_j = \mathsf{span}_{\mathbb{R}}(\Phi_j)$. Let $s \in \Phi \setminus V_1$. Then $R_s(V_1) = V_1$ implies $s \in V_1^{\perp_H} = V_2$, so $s \in \Phi_2$. So $\Phi = \Phi_1 \coprod \Phi_2$. If $\Phi_2 = \emptyset$, then $\mathsf{span}_{\mathbb{R}}(\Phi_1) = \mathsf{span}_{\mathbb{R}}(\Phi) = U$, so $V = \mathsf{span}_{\mathbb{C}}(\Phi_1) \subseteq V_1$, a contradiction. Thus Φ_1 and Φ_2 are non-empty. Since Φ spans U, it follows that U_1 and U_2 span U. Also since V_1 and V_2 are orthogonal with respect to H, the real spaces U_1 and U_2 are orthogonal with respect to $\langle \ , \ \rangle$, so $U = U_1 \perp U_2$ (orthogonal direct sum with respect to $\langle \ , \ \rangle$). It follows that Φ is an orthogonal direct sum of the root systems Φ_1 and Φ_2, contradicting the irreducibility of Φ.

Suppose B and B' are two non-degenerate W-invariant real bilinear forms on U. Then they extend to non-degenerate W-invariant $\mathbb{C}$-bilinear forms on V. Since V is an irreducible represenation of W, these $\mathbb{C}$-bilinar forms must agree upto scaling (by Schur's lemma), hence B and B' must agree upto scaling. $\square$

17.6 Lemma. *Define $B : U \times U \to \mathbb{R}$ by $B(x, y) = \sum_{r \in \Phi} \langle r, x \rangle \langle r, y \rangle$. Then B is a positive definite W-invariant bilinear form on U. So if Φ is an irreducible root system in U, then $cB(x, y) = \langle x, y \rangle$ for some non-zero scalar c.*

Proof. Follows from 17.5, once one verifies that B is a positive definite W-invariant bilinear form on U. $\square$

17.7 Definition. A subset P of Φ is called a *positive system* if P consists of all the roots on one side of a hyperplane in U not meeting any of the roots.

Fix a functional $l : U \to \mathbb{R}$ such that $l(v) \neq 0$ for all $v \in \Phi$. The definitions that follow in this paragraph are subject to this choice. The choice of l defines a positive system $\Phi_+ = \{v \in \Phi \colon l(v) > 0\}$. Let $\Phi_- = \Phi \setminus \Phi_+$. The elements of Φ_+ (resp. Φ_-) are called the set of *positive* (resp. *negative*) roots. A positive root is called *simple* if it can not be written as a sum of two positive roots. A *simple system* is a set of positive roots (with respect to some choice of positive system). Define a partial order $\leq$ on the set of roots as follows: $r \leq s$ if $(s-r)$ is a non-negative integer linear combination of the simple roots. Define the *Weyl vector* ρ to be $\rho = \frac{1}{2}\sum_{r\in\Phi_+} r$. Define the *height* of a root r to be $\mathsf{ht}(r) = \langle r, \rho\rangle$.

17.8 Lemma. *Let Δ be a simple system. Let $u, v \in \Delta$, $u \neq v$. Then $\langle u, v\rangle < 0$.*

Proof. Suppose $\langle u, v\rangle > 0$. Then 17.3 implies that $(u-v)$ is a root. So either $(u-v)$ or $(v-u)$ is a positive root. If $(u-v)$ is a positive root then $u = v+(u-v)$ contracdicts that u is simple. If $(v-u)$ is positive then $v = u+(v-u)$ contradicts that v is simple. □

17.9 Lemma. *Let Δ be a simple system. Then Δ is a basis of U. Each root can be written as an integer linear combination of Δ with all coefficients having the same sign (either all coefficients non-negative or all non-positive).*

Proof. Suppose there is a dependence relation among Δ. Then we can label the simple roots $s_1, \cdots, s_n$ such that this dependence relation has the form $\sum_{i=1}^r c_i s_i = \sum_{j=r+1}^n c_j s_j$ with all $c_i \geq 0$. Let $v = \sum_{i=1}^r c_i s_i$. Then $v^2 = \sum_{i=1}^r \sum_{j=r+1}^m c_i c_j \langle s_i, s_j\rangle \leq 0$, so $v = 0$. It follows that $0 = l(v) = \sum_{i=1}^r c_i l(s_i) = \sum_{i=r+1}^n c_i l(s_i)$. Since $l(s_i) > 0$ for all i, it follows that $c_i = 0$ for all i. Thus the simple roots are linearly independent.

It remains to show that each positive root r can be written as $r = \sum_{r\in\Delta} c_r r$ with each $c_r \in \mathbb{Z}_{\geq 0}$. Suppose not. Then there is a positive root r with smallest value of $l(r)$ such that r can not be written as a non-negative integer linear combination of simple roots. In particular r is not simple, so we can write $r = r_1 + r_2$ for two positive roots r_1 and r_2. Then $0 < l(r_j) < l(r)$, so r_1 and r_2 can be written as non-negative integer linear combination of the simple roots, so the same is true for r as well. □

17.10 Lemma. *Let Δ be a simple system. Let $s \in \Delta$. Let Δ_s be the set of $r \in \Delta$ such that $\langle r, s\rangle = -1$. Let $u \in \Phi$. Write $u = \sum_{r\in\Delta} n_r r$.*

(a) Then $R_s(u) = \left(\sum_{r\in\Delta_s} n_r - n_s\right)s + \sum_{r\in\Delta_s} n_r r + \sum_{r\in\Delta\setminus(\Delta_s\cup\{s\})} n_r r$.

(b) Suppose $u \in \Phi_+ \setminus \{s\}$. Then $\sum_{r\in\Delta_s} n_r \geq n_s$ and $R_s(u) \in \Phi_+$.

(c) One has $R_s(\Phi_+) = (\Phi_+ \setminus \{s\}) \cup \{-s\}$.

(d) One has $R_s(\rho) = \rho - s$.

(e) If $v \in \Phi_+$, then $\langle v, \rho\rangle \geq 1$. One has $\langle v, \rho\rangle = 1$ if and only if v is a simple root. So the simple roots are precisely the roots of height 1.

(f) If $s = \sum_{r\in\Phi_+} m_r r$ with $m_r \geq 0$ for all r, then $m_s = 1$ and $m_r = 0$ if $r \neq s$. Simple roots are the positive roots that cannot be written as a non-trivial non-negative integer linear combination of the positive roots.

Proof. (a) Part (a) is a direct computation. (b) Consider the expression for $R_s(u)$ in part (a). Since $u \neq s$, one of the terms of the second or the third sum is non-zero. When we write a root as a linear combination of simple roots, all coefficients must have the same sign. It follows that $R_s(u)$ is a positive root and $(\sum_{r\in\Delta_s} n_r - n_s) \geq 0$. This proves (b). Part (c) follows from part (b). Part (c) implies that $R_s(2\rho) = R_s(\sum_{r\in\Phi_+} r) = \sum_{r\in(\Phi_+\setminus\{s\})\cup\{-s\}} r = 2\rho - 2s$. So (d) holds. Part (d) implies that $\langle \rho, s\rangle = 1$ for each simple root. If v is any positive root, by 17.9 we can write $v = \sum_{r\in\Delta} n_r r$ with each $n_r \in \mathbb{Z}_{\geq}$. So $\langle v, \rho\rangle = \sum_{r\in\Delta} n_r \geq 1$. If $\langle v, \rho\rangle = 1$, then $n_r = 1$ for some $r \in \Delta$ and $n_{r'} = 0$ for all $r' \in \Delta \setminus \{r\}$. So $v = r$ is a simple root. This proves (e). Finally, suppose $s = \sum_{r\in\Phi_+} m_r r$ with all $m_r \geq 0$. Then considering height of both sides, we find that we must have $m_s = 1$ and $m_r = 0$ for all $r \neq s$. Hence (f) holds. □

17.11 Theorem. *(a) Let $\Delta \subseteq \Phi$. The following are equivalent:*

(i) Δ is a simple system.

(ii) Δ is a linearly independent set and each root can be written as an integer linear combination of the elements of Δ with all coefficients non-negative or all coefficients non-positive.

(b) Suppose Δ is a simple system. Let Φ' be the set of roots that can be written as a non-negative integer linear combination of elements of Δ. Then Φ' is a positive system and Δ is the corresponding system of simple roots.

Proof. Lemma 17.9 implies (i) implies (ii). Suppose Δ is a set of roots satisfying (ii). Since Δ is a basis of U, there exists a functional $l : U \to \mathbb{R}$ such that $l(r) = 1$ for all $r \in \Delta$. Then $l(r) \neq 0$ for all $r \in \Phi$. Let Φ_+ be the positive system with respect to l. The condition on Δ implies that Φ_+ consists precisely of the roots that can be written as non-negative linear combination of Δ, in particular $\Delta \subseteq \Phi_+$. Let r be a simple root with respect to the positive system Φ_+. Write $r = \sum_{s\in\Delta} n_s s$ with all n_s having the same sign. Since each $s \in \Delta$ is a positive root and r is simple, lemma 17.10(f) implies that $n_s = 1$ for some s and $n_r = 0$ for $r \neq s$, so $r = s$. So Δ contains each simple root with respect to the positive system Φ_+. Since there are $\mathsf{dim}(U) = |\Delta|$ many simple roots, Δ is the set of simple roots with respect to the positive system Φ_+. This proves part (a) and part (b) together. □

17.12 Lemma. *Let U_1 and U_2 be positive definite real inner product spaces. Let A_j be a spanning subsets of U_j for $j = 1, 2$. Let $f : A_1 \to A_2$ be a bijection preserving inner products. Then f extends to a linear isometry from U_1 to U_2.*

In particular, if Φ_1 and Φ_2 are root systems in U_1 and U_2 and $f : \Phi_1 \to \Phi_2$ is an isomorphism of root systems (i.e. f is a bijection that preserves the inner products between roots), then f extends to a linear isometry from U_1 to U_2.

Proof. Let $\Delta \subseteq A_1$ be a basis of U_1. Let $x \in A_1$ such that $x = \sum_{s\in\Delta} c_s s$. Let $a \in A_1$. Then

$$\langle \sum_{s\in\Delta} c_s f(s), f(a)\rangle = \sum_{s\in\Delta} c_s \langle f(s), f(a)\rangle = \sum_{s\in\Delta} c_s \langle s, a\rangle = \langle x, a\rangle = \langle f(x), f(a)\rangle.$$

So the functional $\langle \sum_{s\in\Delta} c_s f(s) - f(x), \ \rangle$ vanishes on $f(A_1) = A_2$ which spans U_2, so $\sum_{s\in\Delta} c_s f(s) - f(x) = 0$. So f agrees with the linear extension of $f|_\Delta$. Since A_2 spans U_2, the linear extension of f is onto U_2. If $f(x) = 0$, then $0 = \langle f(x), f(x)\rangle = \langle x, x\rangle$, so $x = 0$, so f is injective. □

17.13 Definition. Let $\mathfrak{H} = \{r^\perp : r \in \Phi\}$. Note that if Φ_+ is any positive system then $\mathfrak{H} = \{r^\perp : r \in \Phi_+\}$. Lemma 17.10(e) implies that ρ is not on any hyperplane of $\mathfrak{H}$. Let C be a chamber containing ρ. This chamber C is called *Weyl chamber.*

17.14 Theorem. *Let Φ_+ be a positive system, Δ be the corresponding simple system and ρ be the Weyl vector. Let $S = \{R_s : s \in \Delta\}$ be the set of simple reflections.*

(a) One has $C = D_{\mathfrak{H}}(\rho) = \{x \in U : \langle x, r\rangle > 0 \ \forall r \in \Phi_+\}$.

(b) One has $C = \cap_{s\in\Delta} D_{s^\perp}(\rho)$.

(c) The walls of the weyl chamber C are the hyperplanes orthogonal to the simple roots. The pair (W, S) is a Coxeter system. In particular, the reflections in the simple root generate the Weyl group.

Proof. For each $r \in \Phi_+$, we have $\langle r, \rho\rangle > 0$. Hence part (a). If $\langle x, s\rangle > 0$ for all simple root s, then 17.9 impli $\langle x, r\rangle > 0$ for all $r \in \Phi_+$. So part (b) holds. Now [B] prop 9, p 66 implies that each wall of C is a hyperplane orthogonal to a simple root. Finally, fix any simple root s. Consider the vector $R_s(\rho) = \rho - s$. Note that $\langle \rho - s, s\rangle < 0$ while $\langle \rho - s, r\rangle \geq 1$ for all simple roots $r \neq s$ since $\langle s, r\rangle \leq 0$. So $\cap_{r\in\Delta\setminus\{s\}} D_{r^\perp}(\rho)$ properly contains C. So 11.13 implies that $s^\perp$ is a wall of C. From [B] theorem 1, p 78, it follows that (W, S) is a Coxeter system. □

17.15 Theorem. *There is a natural one to one correspondence between the following collections: (a) The set of chambers of $\mathfrak{H}$. (b) The set of positive syetems. (c) The set of simple systems. The Weyl group acts simply transitively on each of these collections.*

Proof. A positive system Φ_+ determines a simple system Δ by defintion of simple roots. Given the simple system Δ, we can recover the positive system Φ_+ as the set of roots that can be written as non-negative integer linear combination of Δ, see 17.11(b). A positive system Φ_+ determines a chamber $C = \{x \in U : \langle x, r\rangle > 0 \ \forall r \in \Phi_+\}$. Conversely, given a chamber C, pick any $x \in C$, then $\langle x, r\rangle \neq 0$ for all $r \in \Phi$, so the functional $\langle x, \cdot\rangle$ determines a positive system Φ_+. These correspondences are inverse of each other and sets up bijection between the collections (a), (b) and (c). Clearly the bijections are all compatible with W. and we know from [B] that W acts simply transitively on the set of chambers. So W acts simply transitively on each set. □

17.16 Definition. We define a graph, called the *Dynkin diagram* of Φ, denoted by $\mathsf{Dynkin}(\Phi)$. The vertices of $\mathsf{Dynkin}(\Phi)$ correspond to a set of simple roots and two simple roots r and s are joined by an edge if and only $\langle r, s\rangle = -1$. Since the simple systems are all conjugate under W, the Dynkin diagram of Φ does not depend on the choice of the simple system.

Let δ be a Dynkin diagram with n vertices. A set of vectors $\{v_1, \cdots, v_n\}$ in a vector space is said to form the Dynkin diagram δ if $v_i^2 = 2$ for all i and we can label the vertices of δ with $v_1, \cdots, v_n$ such that $\langle v_i, v_j\rangle = -1$ whenever there is an edge between the vertices labeled by v_i and v_j and $\langle v_i, v_j\rangle = 0$ otherwise.

Often we shall use the same symbol to denote a Dynkin diagram and a set of simple roots forming the Dynkin diagram.

17.17 Lemma. *(a) Let $s, x \in U \setminus \{0\}$. If $R_s(x) = -x$, then $x \in s\mathbb{R}$.*

(b) Let U_1 be a subspace of U. If $s \in \Phi \setminus U_1$ and $R_s(U_1) = U_1$, then $s \in U_1^{\perp}$.

(c) Let U be an orthogonal direct sum of subspaces $U_1, \cdots, U_m$ with each U_j stable under W. Let $\Phi_j = U_j \cap \Phi$. Then Φ is an orthogonal direct sum of the root systems $\Phi_1, \cdots, \Phi_m$.

Proof. The proof of (a) and (b) are same as proof of part (b) and (c) of lemma 17.4. For part (c), Suppose $U = U_1 \perp U_2$, and $WU_j = U_j$. Let $\Phi_j = \Phi \cap U_j$. Let $s \in \Phi \setminus U_1$. Then $R_s(U_1) = U_1$ implies that $s \in U_2$. Thus $\Phi = \Phi_1 \cup \Phi_2$. This proves part (c) for $m = 2$. The general result follows by induction. $\square$

17.18 Theorem. *Let $\Delta = \mathsf{Dynkin}(\Phi)$.*

(a) If $\Phi = \Phi_1 \perp \cdots \perp \Phi_m$, and $\Delta_j = \mathsf{Dynkin}(\Phi_j)$, then $\Delta = \coprod_j \Delta_j$.

(b) Conversely, if $\Delta = \coprod_j \Delta_j$, then each Δ_j is a simple system for a sub root system Φ_j of Φ and $\Phi = \Phi_1 \perp \cdots \perp \Phi_m$.

(c) In particular, Φ is irreducible if and only if Δ is connected.

Proof. (a) Let Δ_j be a simple system forming $\mathsf{Dynkin}(\Phi_j)$. The implication $(ii) \implies (i)$ of theorem 17.11 implies that $\coprod_j \Delta_j$ is a simple system for Φ. So $\coprod_j \Delta_j$ is a Dynkin diagram of Φ.

(b) Suppose Δ is a disjoint union of Δ_1 and Δ_2. Let $U_j = \mathsf{span}_{\mathbb{R}}(\Delta_j)$ and $\Phi_j = U_j \cap \Phi$. Since the Weyl group W is generated by simple reflections, each U_j is W-stable. Lemma 17.17(c) implies that Φ is an orthogonal direct sum of Φ_1 and Φ_2. Let $r \in \Phi$. We can uniquely write $r = \sum_{s \in \Delta_1} n_s s + \sum_{s \in \Delta_2} n_s s$ with each $n_j \in \mathbb{Z}$ having the same sign. The first sum is in U_1 and the second is in U_2. So if $r \in \Phi_1$, then the second sum must be 0. It follows that Δ_1 is a simple system for Φ_1 and similarly Δ_2 is a simple system of Φ_2. This proves (b) for $m = 2$. The general case follows by induction. part (c) follows from part (a) and (b). $\square$

17.19 Lemma. *If Φ is irreducible, then W acts transitively on Φ.*

Proof. Fix a simple system Δ and the corresponding Weyl chamber C. If $s, s' \in \Delta$ are connected in the Dynkin diagram, then one verifies that $R_s R_s'(s) = s'$. Since Δ is connected, all the simple roots are conjugate under W. Pick $r \in \Phi$. By results of [B], the hyperplane $r^{\perp}$ is a wall of some chamber. Since W acts transitively on the set of chambers, $r^{\perp}$ conjugate to some wall of C, hence r is conjugate to a simple root. $\square$

17.20 Theorem. *(a) The W orbit of a root r is the irreducible component of Φ containing r.*

(b) Φ can be uniquely decomposed as a orthogonal direct sum of irreducible root systems.

(c) Let $\Phi = \Phi_1 \perp \cdots \perp \Phi_m$ be the irreducible decomposition of Φ. Let Δ_j be the Dynkin diagram of Φ_j. Then each Δ_j is connected and the Dynkin diaram

of Φ is the disjoint union of Δ_j's. Conversely, let $\Delta_1, \cdots, \Delta_m$ be the connected components of the Dynkin diagram of Φ. Then each Δ_j is a simple system for an irreducible sub-root system Φ_j of Φ and $\Phi = \Phi_1 \perp \cdots \perp \Phi_m$.

Proof. If Φ is not irreducible then it can be written as an orthogonal direct sum of two sub-root systems. By induction, we have a decomposition $\Phi = \Phi_1 \perp \cdots \perp \Phi_m$ with each Φ_j irreducible. Let W_j be the Weyl group of Φ_j. If $s \in \Phi$, then $s \in \Phi_j$ or else $s \perp \Phi_j$, so W preserves Φ_j for each j. Let $r \in \Phi$. Then $r \in \Phi_j$ for some j. Then $Wr \subseteq \Phi_j$. On the other hand 17.19 implies that $W_j r = \Phi_j$. So $Wr = \Phi_j$. So the irreducible Φ_j's that occur in any decomposition of Φ are nothing but the orbits of roots under W. This proves part (a) and (b). Part (c) now follows from this and 17.18. □

17.21 Theorem. *The irreducible simply laced root systems are classified by the Dynkin diagrams a_n (with $n \geq 1$), d_n (with $n \geq 3$), e_6, e_7, e_8.*

Proof. One verifies that the root systems listed in 17.2 have the Dynkin diagrams listed in the theorem. For each of these diagrams, one defines the the affine diagram $\tilde{\Delta}$. Each $\tilde{\Delta}$ admits a numbering $\{n_i \colon i \in \tilde{\Delta}\}$ with the property that $\sum_{j \colon (i,j) \in \mathsf{Edge}(\tilde{\Delta})} n_t = 2n_i > 0$ for all $i \in \tilde{\Delta}$; such a numbering is called a balanced numbering. Let I be a Dynkin diagram of a simply laced irreducible root system. Let $\{s_i \colon i \in I\}$ a set of simple roots labeling the Dynkin diagram I. If I contains an affine diagram $\tilde{\Delta}$, then consider $v = \sum_{i \in \tilde{\Delta}} n_i s_i$. Then $v^2 = 0$, so $v = 0$, which contradicts linear independence of the simple roots. Thus I cannot contain an affine diagram. Now the theorem follows from the lemma below. □

17.22 Lemma (graph theory lemma for *ADE*)**.** *Any connected graph either contains an affine diagram or is a spherical diagram.*

A graph that properly contains an affine diagram is called an indefinite diagram. Let D be a set of norm 2 vectors in a real inner product space such that $\langle s, s' \rangle \in \{0, -1\}$ if s, s' are distinct elements of D. Consider the graph with vertex set is D and edges corresponding to pairs (s, s') such that $\langle s, s' \rangle = -1$. We call this the (simply laced) Dynkin diagram of the set D. Write $L = \mathbb{Z}D$.

17.23 Lemma. *(a) If D is a connected finite type diagram, then L is positive definite root lattice and D is a simple system for the root system $L(2)$.*

(b) If D is a connected affine diagram, then L is positive definite and affine.

(c) If D is indefinite, then L is indefinite.

proof of (c). Let A be an affine diagram contained in D. Let $v = D - A$ that is connected to A. Write $\{s_1, \cdots, s_k\}$ be the vertices of A. Let $\{s_1, \cdots, s_t\}$ be the vertices that are connected to v. Let $n_1, \cdots, n_k$ be the balanced numbering on A. Write $u = n_1 s_1 + \cdots + n_k s_k$. Then $\langle u, s_j \rangle = 0$ for all j, so $u^2 = 0$. Also $\langle v, u \rangle = -\sum_{j=1}^{t} n_j \in \mathbb{Z}_{<0}$. So $(v + 2u)^2 = 2 + 4\langle u, v \rangle < 0$. So L contains both positive and negative norm vectors. □

17.24 Lemma. *Let L be a simply laced root lattice. Let $\Phi_L = \Phi_1 \perp \cdots \perp \Phi_m$. Let $L_j = \mathsf{span}_{\mathbb{Z}}(\Phi_j)$. Then L_j is a root sub-lattice of L with root system Φ_j and $L = L_1 \perp \cdots \perp L_m$.*

Proof. One has $L_j = \mathsf{span}_{\mathbb{Z}}(\Phi_j) \subseteq \mathsf{span}_{\mathbb{Z}}(\Phi_L) = L$, so L_j is a sublattice of L. Since L_j is spanned by the it roots, L_j is a root lattice. Since $\Phi_1, \cdots, \Phi_m$ are mutually orthogonal subsets in V, one has $L_1 \perp \cdots \perp L_m \subseteq L$. Finally, $L_1 \perp \cdots \perp L_m$ contains Φ_L which spans L, so $L_1 \perp \cdots \perp L_m = L$. $\square$

17.25 Corollary (classification of simply laced root lattices)**.** *Each simply laced root lattice is an orthogonal direct sum of the root lattices $L(a_n)$, $L(d_n)$, $L(e_6)$, $L(e_7)$, $L(e_8)$. So there is a natural bijection between the set of simply laced root lattices and the simply laced root systems.*

Proof. Let L_1 and L_2 be simply laced root lattices with root systems Φ_1 and Φ_2. Let $U_j = L_j \otimes_{\mathbb{Z}} \mathbb{R}$. Suppose $f : \Phi_1 \to \Phi_2$ is an isomorphism of root systems. Then f extends to a linear map from U_1 to U_2. Since Φ_j spans L_j, we have $f(L_1) = L_2$, so f defines an isometry from L_1 to L_2. Thus we see that a simply laced root lattice is determined by its root system. Now the corollary follows from 17.24 and the classification of simply laced root systems. $\square$

17.26 Theorem. *Assume that Φ is irreducible. Fix (Φ_+, Δ) and recall this determines a partial order on Φ. Then there exists a unique root $s_{\max}$, called the highest root, such that $s_{\max} \geq r$ for all $r \in \Phi$.*

Proof. Assume $\Delta \neq a_1$. Let r be a root. Let $s_{\max} = \sum_{r \in \Delta} n_r r$ be a root which is maximal with respect to the patial order on the set of roots. If $r \in \Phi_-$, then $-r > r$, so r is not maximal. So $s_{\max} \in \Phi_+$. Write $s_{\max} = \sum_{s \in \Delta} n_s s$. If possible suppose $n_s = 0$ for some $s \in \Delta$. Since the Dynkin diagram Δ is connected, there exists $r, s \in \Delta$ such that $n_s = 0$, $n_r \neq 0$ and $\langle r, s \rangle = -1$, for otherwise, $\{r \in \Delta \colon n_r = 0\} \cup \{r \in \Delta \colon n_r \neq 0\}$ would be a disconnection of Δ. Part (a) of 17.10 implies that $R_s(s_{\max}) > s_{\max}$ contradicting the maximality of $s_{\max}$. So $n_r > 0$ for all r.

Since Δ is connected and $\Delta \neq a_1$, a simple root cannot be maximal. If $s \in \Delta$ and $\langle s, s_{\max} \rangle < 0$, then, since $s_{max} \neq -s$, we must have $\langle s, s_{\max} \rangle = -1$, which implies $R_s(s_{\max}) = s_{max} + s$ is a root, contradicting the maximality of $s_{\max}$. So $\langle s_{\max}, s \rangle \geq 0$ for each $s \in \Delta$. We must have $\langle s_{\max}, s \rangle > 0$ for some $s \in \Delta$ since the simple roots form a basis of the vector space.

Now if s' is another root of maximal height then we also have $\langle s', s \rangle \geq 0$ for all $s \in \Delta$. So $\langle s_{\max}, s' \rangle = \sum_{r \in \Delta} n_s \langle s, s' \rangle > 0$. If $s_{\max} \neq s'$, then $(s_{\max} - s')$ would be a root. By theorem 17.11, this would imply that either $s_{\max} > s'$ or $s' > s_{\max}$ contradicting the maximality of $s_{\max}$ and s'. Hence there is a unique highest root. $\square$

The following lemma is from [Miyamoto]. We reproduce the proof since the proof in [Miyamoto] has typos.

17.27 Lemma. *Let N be a simply laced root lattice. Let $v \in N$ such that $\langle v, \Phi_N \rangle \subseteq \{-1, 0, 1\}$. Then $v = 0$.*

Proof. Choose a simple system $\{e_1, \cdots, e_n\}$ for Φ_N. Write $v = \sum_{j=1}^n a_j e_j$ with $a_j \in \mathbb{Z}_{\geq 0}$ for $j \leq t$ and $a_j \in \mathbb{Z}_{\leq 0}$ for $j > t$. By changing v to its negative if necessary, we may without loss assume that $a_1 > 0$, that is, $t \geq 1$.

Write $v^+ = \sum_{j=1}^t a_j e_j$, $v^- = v - v^+$ and $|v^+| = \sum_{j=1}^t a_j$. Write $u \gg w$ if $(u-w)$ is a positive linear combination of the simple roots. Suppose there exists v contradicting the statement of the lemma. Choose such a v with minimal value of $|v^+|$. Let w be a highest root such that $w \ll v^+$. In other words w is a root of the form $\sum_{j=1}^t c_j e_j$ with $0 \leq c_j \leq a_j$ for all $j = 1, \cdots, t$ and with maximal possible value of $\sum_j c_j$ among these roots. Then $\langle v, w\rangle \in \{-1, 0, 1\}$. If $k > t$, then $\langle w, e_k\rangle = \sum_{j=1}^t c_j \langle e_j, e_k\rangle \leq 0$, since the inner products between disrtinct simple roots are non-positive. So $\langle w, v^-\rangle = \sum_{k=t+1}^n a_k \langle w, e_k\rangle \geq 0$. It follows that

$$0 > \langle v, w\rangle - 2 = \langle w, v - w\rangle = \sum_{j=1}^t (a_j - c_j)\langle w, e_j\rangle + \langle w, v^-\rangle.$$

and hence $0 > \sum_{j=1}^t (a_j - c_j)\langle w, e_j\rangle$. So there exists some i with $1 \leq i \leq t$ such that $\langle w, e_i\rangle < 0$ and $(a_i - c_i) > 0$. Since w and e_i are roots, we have $\langle w, e_i\rangle = -1$. It follows that $(e_i + w) = R_{e_i}(w) = e_i + \sum_{j=1}^t c_j e_j$ is a root satisfying $v^+ \gg (e_i + w) \gg w$, contradicting the maximality of w. $\square$

18 Affine reflection groups

In this section we study the affine reflection groups of simply laced root lattices.

18.1. Affine isometries: Let V be a real positive definite finite dimensional inner product space. A map $T : V \to V$ is called an *affine transformation* if $x \mapsto (T(x) - T(0))$ is a linear transformation. Let $\mathsf{Aff}(V)$ be the group of affine transformations of V.

A coset H of a codimension one subspace of V is called an *affine hyperplane* in V. For $v \in V$ and $k \in \mathbb{R}$, define the affine hyperplane

$$H(v, k) := \{x \in V : \langle x, v\rangle = k\}. \tag{3}$$

We write $H(v, 0) = H(v)$. The orthogonal reflection R_H in an affine hyperplane H is called an *affine reflection* and H is called the mirror of this reflecton. So $R_{H(v)} = R_v$. Note the the affine reflection in H is the unique isometry of V of order 2 that fixes H. For $v \in V$, let $T_v : V \to V$ defined by

$$T_v(x) = v + x$$

be the translation by v. If S is a subgroup of V, then let $T(S) = \{T_v : v \in S\}$. Affine reflections and translations are examples of affine transformations.

18.2 Lemma. *Let S be an additive subgroup of V. Let G be a group of automorphisms of V such that G preserves S. Then G normalizes $T(S)$ and $G \cap T(S) = \{\mathsf{id}\}$. So the subgroup of $\mathsf{Aff}(V)$ generated by G and $T(S)$ is isomorphic to the semidirect product $S \rtimes G$.*

Proof. If $g : V \to V$ is any invertible map then one verifies that $gT_vg^{-1} = T_{gv}$. So if G preserves $S \subseteq V$, then G normalizes $T(S)$. Also, $G \cap T(S) = \{\mathsf{id}\}$ since nontrivial translations do not fix 0. So the group generated by G and $T(S)$ in $\mathsf{Aff}(V)$ is a semidirect product. Finally note that $T(S) \simeq S$ and the conjugation action of G on $T(S)$ correspond to the usual action of G on S. ☐

18.3. Affine reflections: Let $v \in V$ and $v^2 = 2$. Let R_v be the orthogonal reflecton in v. For $k \in \mathbb{Z}$, define

$$R_{v,k} = T_{kv} \circ R_v = R_v \circ T_{-kv}.$$

Suppose $x \in V$ such that $\langle x, v\rangle = k$. Then

$$\langle R_{v,k}(x), v\rangle = \langle T_{-kv}x, -v\rangle = \langle x - kv, -v\rangle = -\langle x, v\rangle + kv^2 = -k + 2k = k.$$

So $R_{v,k}$ fixes the hyperplane $H_{v,k}$ and

$$R^2_{v,k} = T_{kv}R_vT_{kv}R_v^{-1} = T_{kv}T_{R_v(kv)} = T_{kv}T_{-kv} = \mathsf{id}\,.$$

So $R_{v,k}$ is the affine reflection in the hyperplane $H(v, k)$.

18.4. Simply laced affine reflection groups: Let K be a simply laced root lattice with root system Φ and Weyl group W. Let $V = K \otimes \mathbb{R}$. Let $\mathfrak{H}$ be the set of mirrors of W and and let

$$\mathfrak{H}^{\text{aff}} = \{H(v, k) \colon v \in \Phi, k \in \mathbb{Z}\}.$$

The affine reflection group W^{aff} of Φ (or K) is defined to be the group generated by the affine reflections in the hyperplanes $\mathfrak{H}^{\text{aff}}$. Sometimes we write $W^{\text{aff}} = \mathsf{AR}(K)$. Since $R_{v,k} = T_{kv} \circ R_v$, the affine reflection group W^{aff} is the subgroup of $\mathsf{Aff}(V)$ generated by the finite reflection group W and the translations $T(K)$. Lemma 18.2 implies that $W^{\text{aff}} = T(K) \rtimes W \simeq K \rtimes W$. So we have a short exact sequence

$$1 \to K \to W^{\text{aff}} \to W \to 1.$$

The projection $W^{\text{aff}} \to W$ is given by $g \mapsto \bar{g} = T_{-g(0)} \circ g$. So every $g \in W^{\text{aff}}$ can be uniquely written as $g = T_{g(0)} \circ \bar{g}$ with $\bar{g} \in W$. From the description $W^{\text{aff}} \simeq K \rtimes W$ one finds easily that $\mathfrak{H}^{\text{aff}}$ is stable under W^{aff} and W^{aff} acts properly discontinuously on V. So W^{aff} is a discrete reflection group acting on V and $\mathfrak{H}^{\text{aff}}$ is the set of mirrors of W^{aff}.

18.5 Theorem. *Let (K, Φ, W, V) be as in 18.4. Assume Φ is irreducible. Fix a positive system Φ_+ for Φ. Let $s_1, \cdots, s_n$ be the simple roots, let C be the Weyl chamber, and let $s_{\max}$ be the highest root for Φ. Then*

$$C^{\text{aff}} = C \cap D_{H(s_{\max},1)}(0) = C \cap \{x \in V \colon \langle x, s_{\max}\rangle < 1\}$$

is the unique chamber of W^{aff} such that $C^{\text{aff}} \subseteq C$ and $0 \in \text{cl}(C^{\text{aff}})$. The walls of C^{aff} are $H(s_1), \cdots, H(s_n)$, $H(s_{\max}, 1)$.

Proof. Let C_* be the chamber of W^{aff} containing $\epsilon\rho$ where ϵ be a small positive real and ρ is the Weyl vector. If $r \in \Phi_+$, verify that

$$\cap_{n\in\mathbb{Z}} D_{H(r,n)}(\epsilon\rho) = \{x \in V : 0 < \langle x, r\rangle < 1\}.$$

Since $\mathfrak{H}^{\text{aff}} = \{H(r, n) \colon r \in \Phi_+, n \in \mathbb{Z}\}$, it follows that

$$C_* = \cap_{r\in\Phi_+}\{x \in V : 0 < \langle x, r\rangle < 1\}.$$

Since C^{aff} is the intersection of a set of half spaces bounded by hyperplanes in $\mathfrak{H}^{\text{aff}}$, the set C^{aff} is contained in a chamber of W^{aff}. Let $x \in C^{\text{aff}}$ and $r \in \Phi_+$. Since $\langle x, s_j\rangle > 0$ for $j = 1, \cdots, n$, we have $\langle x, r\rangle > 0$. By 17.26, $(s_{\max} - r)$ is a non-negative integer linear combination of simple roots, so $0 < \langle x, s_{\max} - r\rangle$, which implies $\langle x, r\rangle < \langle x, s_{\max}\rangle < 1$. It follows that C^{aff} is contained in C_*. So C^{aff} is a chamber of W^{aff}. Finally note that 0 is not a limit point of $C \setminus C^{\text{aff}}$. So C^{aff} is the only chamber of W^{aff} that is contained in C and contains 0 in its closure. ∎

18.6 Theorem. *Let (K, Φ, W, V) be as in 18.4. The reflection group W^{aff} acts transitively on the chambers of W^{aff}.*

Proof. See [B]. ∎

18.7 Theorem. *Let (K, Φ, W, V) be as in 18.4. Assume W is irreducible. Let C^a be a chamber of W^{aff} such that $0 \in \text{cl}(C^a)$. Let $\mathfrak{H}_0$ be the walls of C^a that pass through 0. Then $C = \cap_{H \in \mathfrak{H}_0} D_H(C^a)$ is the unique chamber of W containing C^a. Further, C^a is the unique chamber of W^{aff} such that $C^a \subseteq C$ and $0 \in \text{cl}(C^a)$. There is a simple system $s_1, \cdots, s_n$ of W such that $\mathfrak{H}_0 = \{s_1^\perp, \cdots, s_n^\perp\}$. The walls of the chamber C^a are $H(s_{\max}, 1), s_1^\perp, \cdots, s_n^\perp$, where $s_{\max}$ is the higest root of W with respect to the simple system $s_1, \cdots, s_n$. One has*

$$C = \cap_{j=1}^n \{x \in V \colon \langle x, s_j \rangle > 0\} \text{ and } C^{\text{aff}} = C \cap \{x \in V : \langle x, s_{\max} \rangle < 1\}.$$

Proof. The chamber C^a does not meet any hyperplane of W^{aff}. In particular, C^a does not meet any hyperplane of W. So C^a is contained in a unique chamber C of W. By theorem 17.15, there exists a simple system $s_1, \cdots, s_n$ of W such that C is the Weyl chamber with respect to this simple system. In other words, the walls of C are $s_1^\perp, \cdots, s_n^\perp$ and

$$C = \cap_{i=1}^n D_{s_j^\perp}(C) = \cap_{i=1}^n D_{s_j^\perp}(C^a).$$

By theorem 18.5, there exists a unique chamber C^{aff} of W^{aff} such that $C^{\text{aff}} \subseteq C$ and $0 \in \text{cl}(C^{\text{aff}})$. So $C^a = C^{\text{aff}}$. Now 18.5 implies

$$C^{\text{aff}} = C \cap \{x \in V : \langle x, s_{\max} \rangle < 1\}$$

where $s_{\max}$ is the highest root of W with respect to the simple system $s_1, \cdots, s_n$ and the walls of C^{aff} are $H(s_{\max}, 1), s_1^\perp, \cdots, s_n^\perp$. So $\mathfrak{H}_0 = \{s_1^\perp, \cdots, s_n^\perp\}$. ∎

18.8 Corollary. *Let (K, Φ, W, V) be as in 18.4. Assume W is irreducible. Let C^a be a chamber of W^{aff}. Then there exists a $v, s_0, s_1, \cdots, s_n \in K$ such that $s_1, \cdots, s_n$ is a simple system for K, s_0 is the corresponding highest root and the walls of C^a obtained by trainslating $H(s_0, 1)$, $H(s_1), \cdots, H(s_n)$ by the translation T_v.*

Proof. By [B], the affine reflection group W^{aff} acts transitively on the set of chambers, so there exists $g \in W^{\text{aff}}$ such that $0 \in \text{cl}(gC^a)$. We can write $g = g_1 T_{-v}$ for some $g_1 \in W$ and some $v \in K$. Since $g_1 0 = 0$, it follows that $0 \in \text{cl}(T_{-v} C^a)$. In other words, there is a chamber C^0 of W^{aff} such that $0 \in \text{cl}(C^0)$ and $C^a = T_v(C^0)$. The previous lemma gives us a description of the chambers of W^{aff} whose closure contain 0. ∎

18.9 Lemma. *Let K be a simply laced root lattice and let Φ be the root system of K. Let $\Phi = \Phi_1 \perp \cdots \perp \Phi_m$ be the irreducible decomposition of Φ and $K = K_1 \perp \cdots \perp K_m$ be the corresponding orthogonal decomposition of K (see 17.24). Then $\mathsf{AR}(K) \simeq \mathsf{AR}(K_1) \times \cdots \times \mathsf{AR}(K_m)$. Let C_j be a chamber for the affine reflection group of Φ_j acting on $K_j \otimes \mathbb{R}$. then $C_1 \times \cdots \times C_m$ is a chamber for the affine reflection group of Φ acting on $K \otimes \mathbb{R}$.*

Proof. We argue for $m = 2$. The general argument is identical. Suppose $K = K_1 \perp K_2$. One verifies that the natural map $\mathsf{AR}(K_1) \times \mathsf{AR}(K_2) \to \mathsf{AR}(K_1 \perp K_2)$ is onto since all the reflections and translations of $\mathsf{AR}(K_1 \perp K_2)$ are in the image of the map. So the actions of $\mathsf{AR}(K_1) \times \mathsf{AR}(K_2)$ and $\mathsf{AR}(K_1 \perp K_2)$ on $K_1 \perp K_2$ are isomorphic. It follows that if C_j is a chamber of $\mathsf{AR}(K_j)$ acting on $K_j \otimes \mathbb{R}$, then $C_1 \times C_2$ is a chamber of $\mathsf{AR}(K)$ acting on $K \otimes \mathbb{R}$. $\square$

18.10 Definition (Affine Dynkin diagrams)**.** Let (K, Φ, W) be as in 18.4. The *affine Dynkin diagram* of type Φ is, by definition the diagram obtained by adding the lowest root (negative of the highest root) to the set of simple roots. By lemma 18.8, the walls of a chamber of $\mathsf{AR}(K)$ acting on $K \otimes \mathbb{R}$ correspond to a set of roots $s_0, \cdots, s_n$ where $\{s_0, \cdots, s_n\}$ is a set of simple roots and s_0 is the corresponding lowest root. So the walls of a chamber of W^{aff} correspond to the vertices of the affine Dynkin diagram.

Let K be a lattice with simply laced root system Φ, not necessarily irreducible. Let $\Phi_1, \cdots, \Phi_k$ be the irreducible components of Φ. Then the affine Dynkin diagram of Φ (or K) is, by definition, obtained by changing all components of the Dynkin diagram of Φ to the corresponding affine diagrams, or in other words, by adding the lowest root to a simple system for Φ_j for each j. Lemma 18.9 imply that the vertices of the affine diagram of K correspond bijectively to the walls of a chamber of $\mathsf{AR}(K)$ acting on $K \otimes \mathbb{R}$

For the next two lemmas assume the following setup. Let K be a simply laced root lattice and Φ be the root system of K. Let W (resp. W^{aff}) be the reflection group (resp. affine reflection group) of K. Let $\mathcal{H}$ (resp. $\mathcal{H}^{\text{aff}}$) be the set of half spaces in $K \otimes \mathbb{R}$ bounded by the mirrors of W (resp. W^{aff}). Consider the singular lattice $K^0 = K \oplus \mathbb{Z}$ where $\mathbb{Z}$ spans the one dimensional radical of K^0. The roots of K^0 are $\Phi^0 = \{(v, k) \colon v \in \Phi, k \in \mathbb{Z}\}$. Let W^0 be the reflection group of the singular lattice K^0.

18.11 Lemma. *(a) The intersection of all the mirrors of W is $\{0\}$.*
(b) W^{aff} acts faithfully on $\mathcal{H}^{\text{aff}}$.

Proof. (a) Let Φ_1 (resp. Φ_2) be root systems in V_1 and V_2. Let A_j be the intersection of the mirrors of V_j. Then the intersection of the mirrors of the root system $\Phi_1 \perp \Phi_2$ in $V_1 \perp V_2$ is $(A_1 \times V_2) \cap (V_1 \times A_2) = A_1 \times A_2$. So if the intersection of the mirrors of Φ_1 and Φ_2 is $\{0\}$, then the same holds for $\Phi_1 \perp \Phi_2$. Since Φ is an orthogonal direct sum of irreducible simply laced root systems, it suffices to check part (a) when Φ is irreducible and this can be done directly using the classification.

(b) Suppose $g \in W^{\text{aff}}$ fixes all the half spaces in $\mathcal{H}^{\text{aff}}$. In particular, g setwise fixes all the mirrors of W^{aff} through the origin, i.e. all the mirrors of W. By part (a), the intersection of all these mirrors is $\{0\}$, so g fixes 0, hence $g \in W$. From 17.15, we know that W acts simply transitively $\mathcal{H}$, so we must have $g = \mathsf{id}$. $\square$

18.12 Lemma. *There is an isomorphism of permutation actions between $(W^{\text{aff}}, \mathcal{H}^{\text{aff}})$ and (W^0, Φ^0). In particular, there is an isomorphism $W^0 \simeq W^{\text{aff}}$*

such that the reflection $R_{(v,k)}$ of the singular lattice corresponds to the affine reflection $R_{v,k}$.

Proof. Since W^0 acts faithfully on $K^0 = \mathsf{span}(\Phi^0)$, the group must also act faithfully on Φ^0, that is, we may identify W^0 as a subgroup of $\mathsf{Aut}(\Phi^0)$ (the set of bijefctons of Φ^0). By Lemma 18.11, we may identify W^{aff} as a subgroup of $\mathsf{Aut}(\mathcal{H}^{\text{aff}})$. Define

$$\alpha : \Phi^0 \to \mathcal{H}^{\text{aff}} \text{ given by } (v,k) \mapsto \alpha(v,k) := \{x \in V : \langle x, v\rangle > k\}.$$

Each half space in $\mathcal{H}^{\text{aff}}$ is bounded by some mirror $H_{v,k}$. Observe that the two half spaces bounded by $H_{v,k}$ are $\alpha(v,k)$ and $\alpha(-v,-k) = \{x \in V : \langle x, v\rangle < k\}$, so α is onto. If $\alpha(v,k) = \alpha(v',k')$, then the corresponding bounding hyperplanes must be equal, that is $H_{v,k} = H_{v',k'}$. Verify that this can happen if and only if $(v',k') = \pm(v,k)$. Since $\alpha(v,k)$ and $\alpha(-v,-k)$ are distinct, α is also one to one. The bijection $\alpha : \Phi^0 \to \mathcal{H}^{\text{aff}}$ induces a group isomorphism $\alpha_* : \mathsf{Aut}(\Phi^0) \to \mathsf{Aut}(\mathcal{H}^{\text{aff}})$ such that $\alpha(gr) = \alpha_*(g)\alpha(r)$ for all $g \in \mathsf{Aut}(\Phi^0)$ and $r \in \Phi^0$. We shall verify that

$$\alpha(R_{(u,j)}(v,k)) = R_{u,j}(\alpha(v,k)) \text{ for all } (u,j),(v,k) \in \Phi^0. \tag{4}$$

Note that $x \in R_{u,j}(\alpha(v,k))$ if and only if $R_{u,j}^{-1}(x) \in \alpha(v,k)$, that is $\langle R_{u,j}^{-1}(x), v\rangle > k$. We compute

$$\langle R_{u,j}^{-1}x, v\rangle = \langle T_{-ju}x, R_u v\rangle = \langle x - ju, R_u v\rangle = \langle x, R_u v\rangle + j\langle u, v\rangle.$$

It follows that $x \in R_{u,j}(\alpha(v,k))$ if and only if $\langle x, R_u v\rangle + j\langle u,v\rangle > k$, or in other words $x \in \alpha(R_u(v), k - j\langle u,v\rangle) = \alpha(R_{(u,j)}(v,k))$. This proves equation (4). This equation amounts to saying that under α_*, the reflection $R_{(u,j)}$ acting on Φ^0 corresponds to the $R_{u,j}$ acting on $\mathcal{H}^{\text{aff}}$. So α_* induces an isomorphism between the groups $W^0 = \langle R_{(u,j)} : (u,j) \in \Phi^0\rangle$ and $W^{\text{aff}} = \langle R_{u,j} : u \in \Phi, j \in \mathbb{Z}\rangle$.

$\square$

19 Mirror arrangements at cusps

Maintain the setup of 13.1. Let W be a hyperbolic reflection group in $\mathcal{H}^n$ with mirror arrangement $\mathfrak{H}$. Let $v \in C^+$ be a cusp of $\mathfrak{H}$ (recall: C^+ is the positive light cone). Let W_v be the stabilizer of v in W. By theorem 13.8, we know that W_v is the subgroup of W generated by the reflections in the hyperplanes $\mathfrak{H}_v$. For $t \geq 0$, the sets

$$\partial B_t(v) = \{x \in \mathbb{R}^{n,1} \colon x^2 = -1, \langle x, v\rangle = -t\}$$

are the horospheres areound v. Fix a null vector v' such that $\langle v, v'\rangle = -1$. Let $u \mapsto [u]$ be the projection from $v^\perp \to v^\perp/v$. Recall the map $j : \mathcal{H}^n \to v^\perp$ and $J : \mathcal{H}^n \to v^\perp/v$ defined by

$$j(x) = \langle x, v\rangle^{-1} x + v' \quad\text{and}\quad J(x) = [j(x)].$$

19.1 Lemma. *(a) For each $t \geq 0$, the map $J : \partial B_t(v) \to v^\perp/v$ is an isomorphism.*

(b) Let H be a hyperplane in $\mathcal{H}^n$ through v. Choose $s \in V$ such that $s^2 = 2$ and $s^\perp = H$. Then $J(H)$ is equal to the affine hyperplane $H([s], \langle v', s\rangle)$ (see equation (3)*) in $v^\perp/v$. The map j takes the two sides of H onto the two sides of $J(H)$; in particular, $J^{-1}(J(H)) = H$. The reflection in H corresponds to reflection in $J(H)$ under this correspondence.*

Proof. (a) Identify $\mathsf{span}\{v, v'\}^\perp$ with $v^\perp/v$. Define

$$f_t : \mathsf{span}\{v, v'\}^\perp \to \mathbb{R}^{n,1} \quad\text{by}\quad f_t(u) = t((u^2 + t^{-2})\tfrac{v}{2} + v' - u).$$

Verify that $\langle f_t(u), v\rangle = -t$ and $f_t(v)^2 = -1$, so $f_t(u) \in \partial B_t(v)$. One verifies that f_t and $J|_{\partial B_t(v)}$ are mutual inverses.

(b) Let $c = \langle v', s\rangle$. Let $x \in \mathcal{H}^n$. Then

$$\langle J(x), [s]\rangle = \langle j(x), s\rangle = \langle x, s\rangle\langle x, v\rangle^{-1} + c.$$

So $x \in H$ if and only if $\langle x, s\rangle = 0$ if and only if $\langle J(x), [s]\rangle = c$. Since $\langle x, v\rangle < 0$ for all $x \in \mathcal{H}^n$, It follows that, under J, the hyperplane H maps to the affine hyperplane $H([s], c)$. and the two sides of H map to the two sides of $H([s], c)$. By part (a), we know that J is onto, hence $J(H)$ is equal to the affine hyperplane $H([s], c)$ and the two sides of H map onto the two sides of $H([s], c)$.

Let $u \in v^\perp/v$. Let $x_1, x_2 \in \mathcal{H}^n$ such that $u = J(x_1) = J(x_2)$. So $\frac{x_1}{\langle x_1, v\rangle} - \frac{x_2}{\langle x_2, v\rangle} \in \mathbb{R}v$. Since $\langle s, v\rangle = 0$, we have $\frac{\langle x_1, s\rangle}{\langle x_1, v\rangle} - \frac{\langle x_2, s\rangle}{\langle x_2, v\rangle} = 0$. Then

$$j(R_s(x_i)) = j(x_i - \langle x_i, s\rangle s) = \tfrac{x_i - \langle x_i, s\rangle s}{\langle x_i, v\rangle} + v' = j(x_i) - \tfrac{\langle x_i, s\rangle}{\langle x_i, v\rangle} s.$$

So $J(R_s(x_1)) = J(R_s(x_2))$. So there is a well defined map $\phi : v^\perp/v \to v^\perp/v$ given by $\phi(u) = JR_sJ^{-1}(u)$. One computes

$$R_{H([s],c)}(u) = u - \langle u, s\rangle[s] + c[s] = u - \langle \tfrac{x_1}{\langle x_1, v\rangle} + v', s\rangle[s] + c[s] = u - \tfrac{\langle x_i, s\rangle}{\langle x_i, v\rangle}[s].$$

So $R_{H([s],c)}(u) = JR_s(x_1) = JR_sJ^{-1}(u) = \phi(u)$, hence $\phi = R_{H([s],c)}$. ☐

19.2 Definition. Consider the affine hyperplane arrangement

$$\mathfrak{H}_v^{\mathsf{aff}} = \{J(H) \colon H \in \mathfrak{H}_v\}.$$

in $v^\perp/v$. Let W_v^{aff} be the subgroup of $\mathsf{Affine}(v^\perp/v)$ generated by the reflections in the hyperplanes of $\mathfrak{H}_v^{\mathsf{aff}}$. Let $H \in \mathfrak{H}_v$ and H^+ and H^- be the two sides of H. Then 19.1 implies $J(H)$ is a hyperplane in $\mathfrak{H}_v^{\mathsf{aff}}$ and $J(H^+)$ and $J(H^-)$ are the two sides of $J(H)$ and $J^{-1}J(H^\pm) = H^\pm$. So if $x \in \mathcal{H}^n \setminus H$, then

$$J(D_H(x)) = D_{J(H)}(J(x)) \text{ and } J^{-1}J(D_H(x)) = D_H(x).$$

So J sets up a bijection between the hyperplanes and half space of $\mathfrak{H}_v$ with the hyperplanes and half spaces of $\mathfrak{H}_v^{\mathsf{aff}}$. fi By 19.1(a), we have an isomorphism $J : \partial B_t(v) \to v^\perp/v$. Let $c_J(g) = JgJ^{-1}$. Part (b) of 19.1 shows that $c_J(W_v) = W_v^{\mathsf{aff}}$. Thus, for each $t \geq 0$,

$$(J, c_J) : (\partial B_t(v), W_v) \to (v^\perp/v, W_v^{\mathsf{aff}})$$

is an isomorphism in the category of permutation actions with inverse f_t.

Let K be a compact subset of $v^\perp/v$. Fix $t > 0$. Then $f_t(K)$ is a compact subset of $\partial B_t(v)$. So $A = \{g \in W_v \colon gf_t(K) \cap f_t(K) \neq \emptyset\}$ is a finite set, which implies $\{\sigma \in W_v^{\mathsf{aff}} \colon \sigma K \cap K \neq \emptyset\} = c_J(A)$ is finite. So W_v^{aff} is a discrete affine reflection group acting on $v^\perp/v$.

19.3 Lemma. *(a) The map J sets up a bijection between the facets of W_v acting on $\mathcal{H}^n$, and the facets of W_v^{aff} acting on $v^\perp/v$.*

(b) If F is a facet of $\mathfrak{H}_v$, then $J^{-1}J(F) = F$.

(c) If F is a facet of $\mathfrak{H}_v$, then $J(\mathrm{cl}(F)) = \mathrm{cl}(J(F))$.

(d) If C is a chamber of W_v, then $J(C)$ is a chamber of W_v^{aff} and J sets up a bijection between the walls of C and the walls of $J(C)$.

Proof. (a) Suppose $x, y \in \mathcal{H}^n$ belong to the same facet of $\mathfrak{H}_v$. Let L be any hyperplane of $\mathfrak{H}_v^{\mathsf{aff}}$. Then $L = J(H)$ for some $H \in \mathfrak{H}_v$. Either x and y both belong to H in which case $J(x)$ and $J(y)$ both belong to L, or else x and y are strictly on the same side of H, in which case $J(x)$ and $J(y)$ are strictly on the same side of L. So $J(x)$ and $J(y)$ belong to the same facet of $\mathfrak{H}_v^{\mathsf{aff}}$. It follows that each facet of $\mathfrak{H}_v$ maps into a facet of $\mathfrak{H}_v^{\mathsf{aff}}$. Since the facets of $\mathfrak{H}_v$ (resp. $\mathfrak{H}_v^{\mathsf{aff}}$) form a partition of $\mathcal{H}^n$ (resp. $v^\perp/v$) and the map J is onto, it follows that the image of a facet of $\mathfrak{H}_v$ under J must be equal to a facet of $\mathfrak{H}_v^{\mathsf{aff}}$. This proves part (a). part (b) follows from part (a).

(c) Since J is continuous $J(\mathrm{cl}(F)) \subseteq \mathrm{cl}(J(F))$. Fix $t > 0$. Recall that $J : \partial B_t(v) \to v^\perp/v$ is an isomorphism with an continuous inverse f. So if $x \in \mathrm{cl}(J(F))$, then $y = f(x) \in f(\mathrm{cl}(J(F)) = \mathrm{cl}(f(J(F)) \subseteq \mathrm{cl}(J^{-1}J(F)) \subseteq \mathrm{cl}(F)$, so $x = J(y) \in J(\mathrm{cl}(F))$. This proves part (c).

(d) Let $x \in \mathcal{H}^n$. Clearly, J sets up a bijection between the hyperplanes of $\mathfrak{H}_v$ through x and the hyperplanes of $\mathfrak{H}_v^{\mathsf{aff}}$ through $J(x)$. Let H be a wall of C. Then there exist $x \in \mathrm{cl}(C) \cap H$ such that H is the only mirror of $\mathfrak{H}_v$ through x. Then $J(x) \in J(\mathrm{cl}(C)) \cap J(H) = \mathrm{cl}(J(C)) \cap J(H)$ and $J(H)$ is the only mirror of

$\mathfrak{H}_v^{\mathsf{aff}}$ through $J(x)$. So $J(H)$ is a wall of $J(C)$. Conversely, suppose $H \in \mathfrak{H}_v$ such that $J(H)$ is a wall of $J(C)$. Then there exists $y \in \mathrm{cl}(J(C)) \cap J(H)$ such that $J(H)$ is the only mirror of $\mathfrak{H}_v^{\mathsf{aff}}$ through y. Fix $t > 0$ and consider the inverse f of $J|_{\partial B_t(v)}$. Then $x = f(y) \in f(\mathrm{cl}(J(C)) \subseteq \mathrm{cl}(f(J(C))) \subseteq \mathrm{cl}(C)$. Since $J(H)$ is the only wall of $\mathfrak{H}_v^{\mathsf{aff}}$ through $f(x) = y$, it follows that H is the only wall of $\mathfrak{H}_v$ through x. So H is a wall of C. $\square$

20 Reflection groups of Lorentizian lattices

20.1 Definition. A *lattice* L is a $\mathbb{Z}$-module of finite rank with a non-degenerate $\mathbb{Q}$-valued bilinear form $\langle\ ,\ \rangle$. Say that L is an *integral*, if the bilinear form taked values in $\mathbb{Z}$. A *root* of L is a positive norm vector s such that $R_s \in \mathsf{Aut}(L)$. The reflections in the roots of L generates the *reflection group* $\mathsf{Ref}(L)$ of L. Say that L is a *root lattice* is L is spanned by its roots. Say that L is *Lorentzian* if L has signature $(n, 1)$.

We are interested in studying the action of discrete groups of the form $\mathsf{Aut}(L)$ or $\mathsf{Ref}(L)$ on the hyperbolic space $\mathbb{P}_-(L \otimes \mathbb{R})$. The set $\{x \in L \otimes \mathbb{R} \colon x^2 = -1\}$ is a disjoint union of two sheets. Picking a hyperboloid model for $\mathbb{P}_-(L \otimes \mathbb{R})$ amounts to choosing one of the sheets. We write $\mathsf{Aut}^+(L)$ to be the index 2 subgroup of $\mathsf{Aut}(L)$ that preserves both sheets of the hyperboloid.

20.2 Lemma. *Let L be a Lorentzian lattice. Then $\mathsf{Aut}(L)$ acts properly discontinuously on the hyperbolic space $\mathbb{P}_-(L \otimes \mathbb{R})$.*

Proof. The bilinear form extends to $U = L \otimes \mathbb{Q}$. By [S] p. 30, theorem 1, there exists a basis $e_0, \cdots, e_n$ of U such that $\langle e_i, e_j \rangle = 0$. Since $\langle\ ,\ \rangle$ has signature $(n, 1)$, by re-indexing the e_j's and scaling them by positive integers if required, we may assume that $a_0 = -e_0^2, a_1 = e_1^2, \cdots, a_n = e_n^2$ are positive intgers

$$(x_0 e_0 + \cdots + x_n e_n)^2 = -a_0 x_0^2 + a_1 x_1^2 + \cdots + a_n x_n^2.$$

Now, $M = \sum_{j=0}^n \mathbb{Z} e_j$ is an lattice in U such that $L \otimes \mathbb{Q} = M \otimes \mathbb{Q} = U$. So $\mathsf{Aut}(L)$ and $\mathsf{Aut}(M)$ are commensurable in $\mathsf{Aut}(U)$. Thus it suffices to show that $\mathsf{Aut}(M)$ acts properly discontinuously on the hyperbolic space.

Identify $\mathbb{R}^{n,1}$ with $U \otimes \mathbb{R}$ by $(t_0, \cdots, t_n) \mapsto t_0 a_0^{-1/2} e_0 + \cdots + t_n a_n^{-1/2} e_n$. As a concrete hyperboloid model on which $\mathsf{Aut}^+(M)$ acts, we take

$$H^n = \{x \in U \otimes \mathbb{R} \colon \langle x, e_0 \rangle < 0, x^2 = -1\}.$$

Any compact set in H^n is contained ball around $y_0 = e_0/\sqrt{a_0}$. So it suffices to show that, given any $r > 0$, the ball $gB_r(y_0)$ intersects $B_r(y_0)$ only for finitely many $g \in \mathsf{Aut}(L)$. Since $g \in \mathsf{Aut}(L)$ acts on H^n by isometies, if $d(gy_0, y_0) > 2r$ then $gB_r(y_0) = B_r(gy_0)$ and $B_r(y_0)$ are disjoint. So we are reduced to showing that $\{g \in \mathsf{Aut}(L) \colon d(gy_0, y_0) < r\}$ is finite for all r. Fix $r > 0$. Let

$$S = \{x \in L \colon \langle x, e_0 \rangle < 0, d(x, e_0) < r, x^2 = -a_0\}.$$

If $g \in \mathsf{Aut}^+(M)$ and $d(ge_0, e_0) < r$, then $ge_0 \in S$. For each $x \in S$, the set $\{g \in \mathsf{Aut}^+(M) \colon ge_0 = x\}$ is a coset of the stabilizer of e_0 in $\mathsf{Aut}^+(M)$, and this stabilizer is finite since it preseves the positive definite lattice $e_0^\perp = \sum_{j=1}^n \mathbb{Z} e_j$. It remains to show S is a finite set. Let $x = \sum_{j=0}^n x_j e_j \in S$. Then $\langle x, e_0 \rangle < 0$ is equivalent to $x_0 > 0$. The condition $d(x, e_0) < r$ is equivalent to $\cosh^{-1}(-\langle x/\sqrt{a_0}, e_0/\sqrt{a_0} \rangle) < r$ or $x_0 < \cosh r$. So x_0 must be in the finite set $(0, \cosh r) \cap \mathbb{Z}$. Finally, $x^2 = -a_0$ implies $a_1 x_1^2 + \cdots + a_n x_n^2 = a_0 x_0^2 - a_0$. So for each fixed x_0, there are finitely many choices for $(x_1, \cdots, x_n)$. $\square$

Setup: Let L be a $\mathbb{Z}$-root lattice of signature $(n,1)$ with each root having norm 2. Let $V = L \otimes \mathbb{R} \simeq R^{n+1}$. Let H^n be the hyperbolic space of V. If $r \in V$ has positive norm, then let $H_r^{\pm} = \{x \in H^n \colon \pm \langle x, r\rangle > 0\}$ be the two sides of $r^{\perp}$. Let Φ be the set of roots of L and let W be the reflection group of L, the group generated by reflections in Φ. Then W modulo scalars act faithfully on H^n. Let $\mathcal{M}$ be the set of mirrors of W in H^n. The connected components of $H^n \setminus \mathcal{M}$ are called the chambers of W. The group W acts transitively on the chambers. Fix a chamber C. Then W is generated by reflections in the walls of C. Let Δ be the set of roots of L, one for each wall of C, such that $C = \cap_{r \in \Delta} H_r^{-}$. We say that Δ is the set of simple roots corresponding to the chamber Δ. A vector ρ is called a Weyl vector for C, if $\rho \in \bar{C}$ and $\langle r, \rho\rangle = -1$ for all $r \in \Delta$.

20.3 Lemma. *Let L be a Lorentzian integral $\mathbb{Z}$-lattice. If r is a root of L, then $r^2 \mid 4[L' : L]^2$.*

Proof. For all $x \in L$, we have we have $R_r(x) = x - 2r^{-2}\langle r, x\rangle r \in L$, so $2r^{-2}\langle r, x\rangle r \in L$. Since r is a primitive vector in L, we must have $\langle 2r/r^2, x\rangle \in \mathbb{Z}$ for all $x \in L$, so $2r/r^2 \in L' \subseteq d^{-1}L$, so $2dr/r^2 \in L$. Since L is a integral lattice, $(2dr/r^2)^2 = 4d^2/r^2 \in \mathbb{Z}$. ∎

21 The even self-dual lattice of signature (25,1)

21.1 Definition. An even self dual positive definite lattice of dimension 24 is called a *Niemeier lattice*. There is a unique Niemeier lattice with no roots (hence minimum norm 4), called the Leech lattice and denoted by Λ. The Leech lattice has covering radius $\sqrt{2}$. There are 23 other Niemeier lattices, each with a root system of rank 24. Let Φ be an irreducible simply laced root system of rank n. The *Coxeter number* of Φ is defined to be $|\Phi|/n$. The root systems of type a_n, d_n, e_6, e_7, e_8 have Coxeter numbers $n+1, 2n-2, 12, 18, 30$ respectively. Consider a simply laced root system of rank 24 such that each of its irreducible component has the same Coxeter number. There are 23 such root systems. They are:

$$a_1^{24},\ a_2^{12},\ a_3^{8},\ a_4^{6},\ a_6^{4},\ a_8^{3},\ a_{12}^{2},\ a_{24},\ a_5^4d_4,\ a_9^2d_6,\ a_{15}d_9,\ a_{11}d_7e_6,\ a_{17}e_7,$$
$$d_3^8,\ d_4^6,\ d_6^4,\ d_8^3,\ d_{12}^2,\ d_{24},\ d_{10}e_7^2,\ d_{16}e_8,\ e_6^4,\ e_8^3.$$

For each of these root systems, there is a unique Niemeier Lattice having that root system. In particular, a Niemeier lattice is characterized by its root system. If N is a Niemeier lattice then $N \perp H \simeq \mathit{II}_{25,1}$. For this section, we let $L = \mathit{II}_{25,1}$; the unique even self dual lattice of signature $(25,1)$. One has $L \subseteq V = \mathbb{R}^{25,1}$. The norm 2 vectors (or short vectors) of L, denoted $L(2)$, are the roots of L. Reflections in these roots generate the reflection group $R(L)$ of L.

21.2 Definition. Fix a Leech cusp ρ in L. Define the define the height of $r \in L$ (with respect to ρ) to be

$$\mathsf{ht}(r) = -\langle r, \rho\rangle.$$

We say $v \in \mathbb{R}^{25,1}$ is a positive vector if it has positive height. Choose a cusp ρ' such that $\langle \rho, \rho'\rangle = -1$. This lets us split off a hyperbolic cell $H = \mathbb{Z}\rho + \mathbb{Z}\rho'$ and get a decomposition $L = \Lambda \perp H$, where $\Lambda = H^{\perp}$ is the Leech lattice. This lets us write $v \in L$ in the form $v = \lambda + m\rho' + n\rho \in L$ in the form $v = (\lambda; m, n)$ where $\lambda \in \Lambda$, $m, n \in \mathbb{Z}$. We shall call it a Leech coordinate system for L. In this coordinate system $\rho = (0; 0, 1)$, $\rho' = (0; 1, 0)$ and $v^2 = \lambda^2 - 2mn$. The hyperboloid model of $\mathcal{H}^{25}$ consists of all positive vectors of norm -1. In the Leech coordinates, $\mathcal{H}^{25}$ consists of vectors $(\alpha; m, n)$ such that $\alpha \in \Lambda \otimes \mathbb{R}$, $\alpha^2 - 2mn = -1$ and $m > 0$. Given $z \in L(0)$, let

$$\mathsf{Lr}_k(z) = \{s \in L(2) \colon \langle r, z\rangle = -k\}.$$

So $\mathsf{Lr}_k(\rho)$ is the set of roots of height k. The roots $\mathsf{Lr}(\rho) = \mathsf{Lr}_1(\rho)$ of height one are called the *Leech roots* (with respect to ρ). The Leech roots are indexed by the vectors of the Leech lattice Λ. For each $\lambda \in \Lambda$, one has a Leech root

$$r_\lambda = (\lambda; 1, (\lambda^2/2 - 1)), \quad \text{where} \quad \lambda \in \Lambda.$$

21.3 Lemma. *Let $a = (\alpha; m, *) \in L \otimes \mathbb{R}$, $m = \mathsf{ht}(a) \neq 0$. Then*

$$\langle r_\lambda, a\rangle = m - \tfrac{m}{2}(\lambda - \tfrac{\alpha}{m})^2 + \tfrac{a^2}{2m}.$$

Proof. Write the expression for $\langle r_\lambda, a\rangle$ and complete squares. Let $a = (\alpha; m, n)$.

$$\langle r_\lambda, a\rangle = \langle \lambda, \alpha\rangle - m(\tfrac{\lambda^2}{2} - 1) - n = m - \tfrac{m}{2}(\lambda - \tfrac{\alpha}{m})^2 + \tfrac{\alpha^2}{2m} - n$$

□

21.4 Lemma. *If r_λ and r_μ are two distinct Leech roots, then $\langle r_\lambda, r_\mu\rangle \leq 0$.*

Proof. Using 21.3, with $a = r_\mu$, we get $\langle r_\lambda, r_\mu\rangle = 2 - \frac{1}{2}(\lambda - \mu)^2$. Since the Leech lattice has minimum norm 4, the lemma follows. □

The lemmas below shows that the Leech roots form a set of "simple roots" for the reflection group of L.

21.5 Theorem (height reduction)**.** *(a) Let $a = (\alpha; m, *)$ be a root of L with $\mathsf{ht}(a) = m > 1$. Then there exists a Leech root r_λ such that $\langle a, r_\lambda\rangle \in [1, m]$ and $0 < \mathsf{ht}(R_{r_\lambda}(a)) < m$.*

(b) $R(L)$ is generated by the reflections in the Leech roots.

(c) Every positive root of L can be written (not uniquely) as a non-negative integer linear combination of the Leech roots.

Proof. (a) Let r_λ be a Leech root. From lemma 21.3 one has

$$\langle r_\lambda, a\rangle = m - \tfrac{m}{2}(\lambda - \tfrac{\alpha}{m})^2 + \tfrac{1}{m}.$$

Since the covering Radius of the Leech lattice is $\sqrt{2}$, we can choose $\lambda \in \Lambda$ such that $(\lambda - \frac{\alpha}{m})^2 \in [0, 2]$. So $\frac{m}{2}(\lambda - \frac{\alpha}{m})^2 \in [0, m]$ and hence $m - \frac{m}{2}(\lambda - \frac{\alpha}{m})^2 \in [0, m]$. So $\langle r_\lambda, a\rangle \in [\frac{1}{m}, m + \frac{1}{m}]$. But $\langle r_\lambda, a\rangle \in \mathbb{Z}$ and $m > 1$. So $\langle r_\lambda, a\rangle \in [1, m]$. It follows that

$$\mathsf{ht}(R_{r_\lambda}(a)) = \mathsf{ht}(a - \langle r_\lambda, a\rangle r_\lambda) = m - \langle r_\lambda, a\rangle \in [0, m-1].$$

However there are no roots of height zero. So $\mathsf{ht}(R_{r_\lambda}(a)) \in [1, m-1]$. This proves part (a). Part (b) and (c) follows from part (a) by induction on the height of a positive root. □

21.6 Theorem (Conway)**.** *Let ρ be a Leech cusp in L. Assume the setup of 21.2. There is a unique Weyl chamber for $R(L)$ around ρ, given by*

$$C = \bigcap_{\lambda \in \Lambda} D_{r_\lambda^\perp}(\rho) = \{x \in \mathcal{H}^{25} \colon \langle x, r_\lambda\rangle < 0 \quad \textit{for every Leech root } \ r_\lambda\}.$$

Let $\mathrm{cl}(\mathcal{H}^{25})$ be the union of $\mathcal{H}^{25}$ and the cusps of $R(L)$ and $\partial C = \mathrm{cl}(C) \setminus C$ be the boundary of C in $\mathrm{cl}(\mathcal{H}^{25})$. One has a homeomorphism

$$f : \partial C \to (\Lambda \otimes \mathbb{R}) \cup \{\infty\} \ \ \textit{given by} \quad (\alpha; m, n) \mapsto \alpha/m.$$

The map f takes the Voronoi cell around a lattice vector $\lambda \in \Lambda$ to the wall $r_\lambda^\perp \cap \partial C$ and maps the cusps of C to the deep holes of Λ. The point ∞ maps to the Leech cusp of ∂C.

Proof. Since the mimimum norm of a non-zero vector in the Leech lattice is 4, there is no root through ρ. Apply Vinberg's algorithm with ρ as the controlling vector. The Leech roots are the roots whose mirrors are closest to ρ so all these mirrors are accepted. Let $r = (\alpha; m, n)$ be any root with $m > 1$. So $\langle r, \rho\rangle = -m$. Note that $D_{r^\perp}(\rho) = D(-r)$ and $D_{r_\lambda^\perp}(\rho) = D(-r_\lambda)$. So $D_{r^\perp}(\rho)$ and $D_{r_\lambda^\perp}(\rho)$ are opposite half spaces if and only if $0 \leq \langle -r, -r_\lambda\rangle = \langle r, r_\lambda\rangle$. From lemma 21.3, we have,

$$\langle r, r_\lambda\rangle = m - \tfrac{m}{2}(\lambda - \tfrac{\alpha}{m})^2 + \tfrac{1}{m}.$$

Since the Leech lattice has covering radius $\sqrt{2}$, there exists $\lambda \in \Lambda$ such that $(\lambda - \frac{\alpha}{m})^2 \leq 2$, hence there exists a Leech root r_λ such that $\langle r, r_\lambda\rangle \geq 1/m > 0$. Hence $D_{r^\perp}(\rho)$ and $D_{r_\lambda^\perp}(\rho)$ are not opposite. So the mirror $r^\perp$ is rejected by the algorithm. $\square$

21.7 Theorem (Borcherds)**.** *Let C be the Weyl chamber in 21.6. Let $\partial C = \mathrm{cl}(C) \setminus C$ be the boundary of C in $\mathrm{cl}(\mathcal{H}^{25})$. One has a homeomorphism*

$$f : \partial C \to (\Lambda \otimes \mathbb{R}) \cup \{\infty\} \ \textit{ given by } \quad (\alpha; m, n) \mapsto \alpha/m.$$

The map f takes the wall $r_\lambda^\perp \cap \partial C$ onto the Voronoi cell around a lattice vector $\lambda \in \Lambda$ and maps the cusps of C to the deep holes of Λ.

A deep hole $v \in \Lambda$ corresponds to the cusp $z = (v; 1, v^2/2) \in \partial C$. The mirror of a Leech root r_λ passes through z if and only if $(\lambda - v)^2 = 2$, that is, λ is a Leech vector closest to the deep hole v. The point ∞ corresponds to the Leech cusp $\rho \in \partial C$. There are no mirrors through ρ.

Proof. Define $g : \Lambda \otimes \mathbb{R} \to L \otimes \mathbb{R}$ by

$$g(v) = (v; 1, \tfrac{v^2}{2} + 1 - \tfrac{d(v,\Lambda)^2}{2}).$$

Since the Leech lattice has covering radius $\sqrt{2}$, we have $g(v)^2 = d(v, \Lambda)^2 - 2 \leq 0$ and $g(v)^2 = 0$ if and only if v is a deep hole of the Leech lattice. So g induces a map $\bar{g} : \Lambda \otimes \mathbb{R} \to \mathrm{cl}(\mathcal{H}^{25})$ given by $\bar{g}(v) = \mathbb{P}(g(v))$. For a Leech root r_λ, one computes

$$\langle r_\lambda, g(v)\rangle = \tfrac{1}{2}(d(v, \Lambda)^2 - (v - \lambda)^2).$$

So $\langle r_\lambda, g(v)\rangle \leq 0$ for all Leech root r_λ and $\langle r_\lambda, g(v)\rangle = 0$ if and only if v is in the voronoi cell around λ. So the image of g is contained in ∂C and g takes the voronoi cell around λ to the wall $r_\lambda^\perp \cap \partial C$.

Next we verify that $\bar{g}$ is the inverse of f. Clearly $f \circ \bar{g} = \mathsf{id}$. Conversely, take $a = (\alpha; m, n) \in \partial C \setminus \{\rho\}$. So $\alpha^2 - 2mn = -1$ and $m > 0$. For each $\mu \in \Lambda$, we have

$$0 \geq \langle a, r_\mu\rangle = m - \tfrac{m}{2}(\mu - \tfrac{\alpha}{m})^2 + \tfrac{(-1)}{2m}.$$

Since $a \in \partial C$, there exists $\lambda \in \Lambda$ such that $a \in r_\lambda^\perp$, so $\langle a, r_\lambda\rangle = 0$ which implies

$$\tfrac{m}{2}(\lambda - \tfrac{\alpha}{m})^2 = m + \tfrac{(-1)}{2m} \leq \tfrac{m}{2}(\mu - \tfrac{\alpha}{m})^2 \ \text{ for all } \ \mu \in \Lambda.$$

Since $m > 0$, we find that $\frac{\alpha}{m}$ belongs to the Voronoi cell around λ, that is, $d(\frac{\alpha}{m}, \Lambda) = d(\frac{\alpha}{m}, \lambda)$. It follows that

$$\bar{g}(\tfrac{\alpha}{m}) = \mathbb{P}(\tfrac{\alpha}{m}; 1, \tfrac{1}{2}(\tfrac{\alpha}{m})^2 + 1 - \tfrac{1}{2}d(\tfrac{\alpha}{m}, \Lambda)^2) = \mathbb{P}(\alpha; m, \tfrac{\alpha^2}{2m} + m - \tfrac{m}{2}(\tfrac{\alpha}{m} - \lambda)^2).$$

Finally, note that

$$\tfrac{\alpha^2}{2m} + m - \tfrac{m}{2}(\tfrac{\alpha}{m} - \lambda)^2 = \tfrac{\alpha^2}{2m} + \tfrac{1}{2m} = n.$$

This verifies $\bar{g}(f(a)) = a$ and hence $f : \partial C \setminus \{\rho\} \to \Lambda \otimes \mathbb{R}$ and $\bar{g}$ are mutual inverses. Finally note that as $v \to \infty$, $g(v)/v^2 \to (0, 0, 1/2)$ so $\bar{g}(v) \to \rho$. So $\bar{g}$ extends to $(\Lambda \otimes \mathbb{R}) \cup \infty$ by taking ∞ to the Leech cusp ρ of ∂C. □

21.8 Corollary. *Let C be the Weyl chamber in 21.6. Then $\rho = (0; 0, 1)$ is the unique Leech cusp in* $\mathrm{cl}(C)$*.*

Proof. Let $z = (v; m, *)$ represent a cusp in $\mathrm{cl}(C)$. If $m = 0$, then $z^2 = v^2 = 0$, so $v = 0$ and since z is primitive, it follows that $z = \rho$. Now suppose $m \neq 0$. Then $z = (v; m, v^2/2m)$. Under the isomorphism in 21.7, z corresponds to a deep hole v/m of the Leech lattice and if λ is any Leech vector closest to this deep hole, then the Leech mirror $r_\lambda^\perp$ passes through z. □

21.9 Definition (The Dynkin diagram at a cusp)**.** Continue fixing a Leech cusp ρ of L and let C be the unique Weyl chamber containing ρ. Let w be a primitive positive norm 0 vector of L such that $\langle w, r_\lambda \rangle \leq 0$ for all Leech root r_λ. Then w determines a cusp in the closure of C which we again denote by w. Assume $w^\perp$ contains a root. Then $w^\perp / w\mathbb{Z}$ is a Niemeier lattice. This Niemeier lattice has a rank 24 simply laced root system that is a disjoint union of ADE's. Let Δ_w be the set of Leech roots whose mirrors pass through the cusp w:

$$\Delta_w = Lr_1(\rho) \cap w^\perp$$

If r, s are two distinct Leech roots in Δ_w, then $r^\perp \cap s^\perp$ contains a null vector w, so $\mathsf{span}\{r, s\}$ is either positive definite or singular. The inner product $\langle r, s \rangle$ is 0 or -1 in the first case and -2 in the second case. Consider the graph with vertex set Δ_w and a simple edge corresponding to pairs $s, s' \in \Delta_w$ such that $\langle s, s' \rangle = -1$ and an edge marked by ∞ when $\langle s, s' \rangle = -2$ (the ∞ is to remind us that the reflections in these generate the infinite dihedral group). We call this graph the Dynkin diagram at w and again denote it by Δ_w. Note that the span M of the roots in Δ_w cannot be indefinite, because if it was then $M^\perp$ would be positive definite contradicting $w \in M^\perp$.

Choose a primitive null vector w_1 such that $\langle w, w_1 \rangle = -1$. Let $U = \mathbb{Z}w + \mathbb{Z}w_1$ and $N = U^\perp \cap L$, so that $L = N \perp U$. One verifies that $w^\perp = N + \mathbb{Z}w$ and the roots of $w^\perp$ are

$$w^\perp(2) = w^\perp \cap L(2) = N(2) + \mathbb{Z}w = H(2) + \mathbb{Z}w = (H + \mathbb{Z}w)(2).$$

Let $H = \mathsf{span}_{\mathbb{Z}}\, N(2)$ be the sublattice spanned by the roots of N. From [Venkov] we know that the root system $N(2)$ has full rank, that is, $|N/H| < \infty$.

21.10 Lemma. *If a connected component of Δ_w contains two vertices s, t joined by an edge marked with ∞, then it is the affine diagram A_1.*

Proof. This is basically saying any diagram properly containing an affine A_1 is going to be indefinite, which cannot occur in Δ_w More precisely, let $s, t \in \Delta_w$ such that $\langle s, t\rangle = -2$. If possible suppose $u \in \Delta_w$ be a node connected to at least s or t. So $\langle u, s\rangle, \langle u, t, \in\rangle\{0, -1, -2\}$ and at least of them is nonzero. One verifies that $\mathsf{gram}(s, t, u) < 0$, so $\mathsf{span}\{s, t, u\}$ contains both positive and negative vectors and so has hyperbolic signature which implies $\mathsf{span}\{s, t, u\}^\perp$ is positive definite, contradicting $w \in \{\, s, t, u\}^\perp$. $\square$

21.11 Theorem. *Assume the setup of 21.9.*

(a) Each positive root of $w^\perp$ is a non-negative integer linear combination of Δ_w. In particular the Leech roots orthogonal w span $w^\perp$.

(b) Each connected component of Δ_w is an affine diagram.

(c) The Leech roots in any component of Δ_w are linearly independent.

(d) Let $X = \{s_0, s_1, \cdots, s_k\}$ be a connected component of Δ_w and let $\{n_0, n_1, \cdots, n_k\}$ be the balanced numbering on X [2]. Then $n_0 s_0 + \cdots + n_k s_k = w$.

(e) Each component of the Dynkin diagram Δ_w has the same coxeter number $h = -\langle w, \rho\rangle$

(f) The Dynkin diagram Δ_w is the affine diagram of type $w^\perp/w\mathbb{Z}$. More precisely, if the root system of $w^\perp/w\mathbb{Z}$ is of type $x_1 \cup x_2 \cup \cdots$ where each $x_j \in \{a_n, d_n, e_6, e_7, e_8\}$, then Δ_w is the diagram $X_1 \cup X_2 \cup \cdots$ where X_j is the affine diagram corresponding to x_j.

Proof. (a) By definition, H is spanned by its roots. So $H + \mathbb{Z}w$ is spanned by its roots, i.e. the roots in $w^\perp$. Let v be a positive root in $w^\perp$. Then we can write $v = \sum_{s \in \mathsf{Lr}(\rho)} n_s s$ with all $n_s \geq 0$. One has $0 = \langle w, v\rangle = \sum_{s \in \mathsf{Lr}(\rho)} n_s \langle w, s\rangle$. It follows that if $\langle w, s\rangle \neq 0$, then $n_s = 0$.

(b) Let $X, X', \cdots$ be the connected components of Δ_w. Claim 1 implies $H + \mathbb{Z}w = \mathbb{Z}\Delta_w = \mathbb{Z}X \perp \mathbb{Z}X' \perp \cdots$. The roots of $H + \mathbb{Z}w$ are a disjoint union $(\mathbb{Z}X)(2) \cup (\mathbb{Z}X')(2) \cup \cdots$ and the roots in each part are orthogonal to all the other parts. If X was of finite type then $\mathbb{Z}X$ would a positive definite root lattice with finitely many roots and this finite set of roots would be orthogonal to all the other roots of $H + \mathbb{Z}w$, which is impossible, since given any root v of $H + \mathbb{Z}w$, there are infinitely many distinct roots $\{v + nw \colon n \in \mathbb{Z}\}$ that are not orthogonal to it. Since X is not a spherical diagram, X must contain an affine diagram. If X properly contains an affine diagram, then $\mathbb{Z}X$ (and hence $w^\perp$) would contain a negative norm vector, which is not possible. So X must be an affine diagram.

(c) Let $X = \{s_0, \cdots, s_k\}$ be a connected component of Δ_w with balanced numbering $n_0, \cdots, n_k$. Assume that s_0 is a level one vertex (also called an extending node), that is, $n_0 = 1$. Then $\{s_1, \cdots, s_k\}$ forms a simply laced finite type Dynkin diagram. So $\mathbb{Z}s_1 + \cdots + \mathbb{Z}s_k$ is a positive definite lattice and $s_1, \cdots, s_n$ are vectors lying in a half space of the Euclidean space $\mathbb{R}s_1 + \cdots +$

[2] Note that $(1, 1)$ is a balanced numbering on the affine diagram of type A_1.

$\mathbb{R}s_k$ (determined by the condition $\langle v, \rho \rangle < 0$) such that any two of them have non-positive inner product. So $\{s_1, \cdots, s_n\}$ are linearly independent. Finally $\sum_{j=0}^{n} n_j s_j$ has norm 0, so it is linearly independent of $\{s_1, \cdots, s_n\}$.

(d) Write $u = \sum_{j=0}^{n} n_j s_j$. Then u is orthogonal to X and also orthogonal to the roots in all the other components of Δ_w. So u belongs to and is is orthogonal to $H + \mathbb{Z}w$. So u is a scalar multiple of w. On the other hand Some $n_j = 1$ and the vectors $s_0, \cdots, s_n$ are linearly independent. So u is primitive. Hence $u = \pm w$. Finally note that u has positive height, since each s_j has height 1 and n_j's are non-negative. So $u = w$.

(e) Part (d) implies $\langle \rho, w \rangle = \sum_j n_j \langle \rho, s_j \rangle = -\sum_j n_j$. One knows that the sum of the balanced numbering on an affine diagram is the Coxeter number of the diagram.

(f) Write $\Delta_w = X_1 \cup X_2 \cup \cdots$, where X_1, X_2 are the connected components of Δ_w. Write $X_j = \{s_0^j, \cdots, s_{k_j}^j\}$ and let $\{n_0^j = 1, n_1^j, \cdots, n_{k_j}^j\}$ be the balanced numbering on X_j. Let $x_j = \{s_1^j, \cdots, s_{k_j}^j\}$ and $\delta_w = \cup_j x_j$. As before δ_w is linearly independent set and $K = \oplus_{v \in \delta_w} \mathbb{Z}v$ is a positive definite simply laced root lattice with Dynkin diagram Δ. Part (d) implies that each $s_0^j \in K + \mathbb{Z}w$. So part (a) implies that $K + \mathbb{Z}w = \sum_{v \in \Delta_w} \mathbb{Z}v = H + \mathbb{Z}w$. This forces $K \simeq H$. Hence δ_w is a Dynkin diagram of H and hence Δ_w is the affine diagram of type $w^{\perp}$. □

We want to describe the environment of the easiest to describe deep hole in the Leech lattice Λ, namely, the hole of type A_1^{24} and the corresponding cusp of the Weyl chamber C. First we recall the following basic fact about representation of elements of $\Lambda/2\Lambda$ by "short vectors". The terminology short vectors here is borrowed from [CS] and means vectors of norm less than or equal to 8.

21.12 Theorem (short vector representative in Leech)**.** *Each vector in Λ is congruent modulo 2Λ to a short vector. The only congruences modulo 2Λ among short vectors are the following:*

(a) two short vectors of different lengths are not congruent modulo 2Λ.

(b) Two shorts vectors of norm 4 or norm 6 are congruent if and only if they are equal up to a sign.

(c) The norm 8 vectors are partitioned into "orthogonal frames" of size 48 (i.e. an orthogonal basis for the underlying vector space and their negatives) such that two norm 8 vectors are congruent modulo 2Λ if and only if they belong to the same frame.

sketch of proof. The theta function $\theta_\Lambda(\tau) = 1 + |\Lambda(4)|q^2 + |\Lambda(6)|q^3 + \cdots$ is a modular form of weight 12, so $\theta_\Lambda(\tau) = aE_{12}(\tau) + b\Delta(\tau)$ for some scalars a, b. The first two q-coefficients of θ_Λ are 1 and 0 and this allows us to calculate a and b. Thus the number of lattice vectors in any shell of the Leech lattice (in fact, of any even positive self dual positive definite 24 dimensional $\mathbb{Z}$ lattice with no roots) is determined. Suppose u, v are two short vectors such that $u \neq \pm v$ and $u \equiv v \bmod 2\Lambda$. Then $(u \pm v) \in 2\Lambda - \{0\}$, so $16 \pm 2\langle u, v \rangle \geq (u \pm v)^2 \geq 16$. This forces $\langle u, v\rangle = 0$ and $u^2 = v^2 = 8$. This proves (a) and (b) and also that for a norm 8 vector v, the set $\Lambda(\leq 8) \cap (v + 2\Lambda)$ has at most 48 elements (consisting of a maximal orthogonal set and their negatives). So if there are N_8 congruence classes represented by norm 8 vectors, then $N_8 \geq |\Lambda(8)|/48$. So the number of classes represented by short vectors is

$$1 + |\Lambda(4)|/2 + |\Lambda(6)|/2 + N_8 \geq 1 + |\Lambda(4)|/2 + |\Lambda(6)|/2 + |\Lambda(8)|/48$$

But the number on the right hand side turns out to be $2^{24} = |\Lambda/2\Lambda|$. □

This theorem quickly implies that half of a norm 8 Leech vector λ in deep hole of type A_1^{24} and the vertices of this deep hole correspond to the orthogonal frame containing λ. More precisely we have the following:

21.13 Lemma. *Let $\lambda \in \Lambda(8)$. Then $d(\lambda/2, \Lambda) \geq \sqrt{2}$. So $\lambda/2$ is a deep hole. By the short vector representative theorem we have the frame*

$$\Lambda(\leq 8) \cap (\lambda + 2\Lambda) = \{\pm\lambda_1, \cdots, \pm\lambda_{24}\}$$

where $\{\lambda = \lambda_1, \cdots, \lambda_{24}\}$ is an orthogonal basis $\Lambda \otimes \mathbb{R}$ consisting of of norm 8 vectors. Then the vertices of the hole $\lambda/2$ are the vectors $(\lambda \pm \lambda_j)/2$.

Proof. First note that $\lambda_j \equiv \lambda \bmod 2\lambda$, so $(\pm\lambda_j + \lambda)/2 \in \Lambda$ for each j. Now, let $v \in \Lambda$ such that $(v - \lambda/2)^2 \leq 2$. Then $(2v - \lambda) \in \Lambda(\leq 8)$ and $(2v - \lambda) \equiv \lambda \bmod 2\Lambda$. So from the short vector representation theorem, $(2v - \lambda) = \pm\lambda_j$, so $v = (\lambda \pm \lambda_j)/2$ for some j. This proves $d(\lambda/2, \Lambda) \geq \sqrt{2}$ and the only Leech vectors at distance $\sqrt{2}$ from $\lambda/2$ are $(\lambda \pm \lambda_j)/2$. □

21.14 Lemma. *Suppose r_λ, r_μ are two Leech roots in Δ_w such that $\langle r_\lambda, r_\mu\rangle = -2$. Then $w = r_\lambda + r_\mu = (\lambda + \mu; 2, *)$ and $(\lambda + \mu)/2$ is a deep hole of Leech lattice of type A_1^{24}. There are 48 Leech roots $\{r_{\lambda_j}, -r_{\lambda_j} + w \colon j = 1, \cdots, 24\}$ whose mirrors pass through w and they form an A_1^{24} Dynkin diagram. These 48 Leech roots correspond to an "orthogonal frame" of norm 8 vectors in the Leech lattice representing a congruence class in $\Lambda/2\Lambda$.*

Proof. From the calculation in lemma **??**, we have $2 - \frac{1}{2}(\lambda - \mu)^2 = \langle r_\lambda, r_\mu\rangle = -2$, so $(\lambda - \mu)^2 = 8$. So $(\lambda - \mu)/2$ has norm 2, and hence does not belong to the Leech lattice. It follows that $(r_\lambda + r_\mu) = (\lambda + \mu; 2, *)$ is a primitive vector of L. The inner products imply $(r_\lambda + r_\mu)$ and w are two orthogonal null vector. It follows that $(r_\lambda + r_\mu)$ is a scalar multiple of w. Both are primitive vectors of L and has negative inner product with ρ (since $w \in \mathsf{cl}(C)$). So $w = r_\lambda + r_\mu$. Conway's theorem implies $\alpha = \frac{1}{2}(\lambda + \mu)$ is a deep hole of Λ. Note that $\alpha - \mu = (\lambda - \mu)/2$ is half of a norm 8 vector in the Leech Lattice, so it is a deep hole of type A_1^{24}. So α is a deep hole of type A_1^{24}. □

21.15 Remark. The lemma above says that one can have two Leech roots in Δ_2 with inner product -2 if and only if the cusp is of type A_1^{24}. In all other cases, the inner product between two distinct Leech roots in Δ_w is 0 or -1, so the affine Dynkin diagram at w is simply laced.

Assume the setup of 21.9. To continue the discussion of the cusps of the fundamental domain C begun in 21.9, first we need to show that there exists a Leech glue root of type w, which means a Leech root v such that $\langle v, w\rangle = -1$.

21.16 Lemma. *(a) Let r be a root of L such that $\langle r, w\rangle = -1$. Then there exists a Leech glue root v of type w such that $r - v \in \mathsf{span}_{\mathbb{Z}}\, w^{\perp}(2)$.*

(b) In particular, there exists a Leech glue root

Proof. (a) Using 21.5 we can write $r = \epsilon \sum_{s \in \mathsf{Lr}_1(\rho)} c_s s$ where each $c_s \in \mathbb{Z}_{\geq 0}$ and $\epsilon = \pm 1$. So

$$-1 = \langle r, w\rangle = \epsilon \sum_{s \in \mathsf{Lr}_1(\rho)} c_s \langle s, w\rangle.$$

It follows that $\epsilon = 1$ and there exists a single Leech root v such that $-\langle v, w\rangle = c_v = 1$ and for all Leech root $s \neq v$, either $c_s = 0$ or $\langle s, w\rangle = 0$. This proves part (a). (b) Part (b) follows from part (a) since there always exists a root r such that $\langle r, w\rangle = -1$. For example, we can take $r = w_1 - w$ where w_1 is any primitive null vector such that $\langle w, w_1\rangle = -1$. $\square$

21.17 Theorem (from Niemeier cusp to Leech cusp)**.** *(a) Assume the setup of 21.9. Let u be a Leech glue root of type w. Then $N = II_{25,1} \cap w^{\perp} \cap u^{\perp}$ is a Niemeier Lattice of type w. Let Δ be a connected component of Δ_w and let $\{n_v : v \in \Delta\}$ be the balanced numbering of the affine diagram Δ. then there exists a unique Leech root $s \in \Delta$ of level 1 (meaning $n_s = 1$) such that $\langle u, s\rangle = -1$ and $\langle u, s'\rangle = 0$ for all $s' \in \Delta - \{s\}$. In other words, u is connected to a unique level one vertex of the affine diagram Δ. Further $\{v \in \Delta_w : \langle u, w\rangle = 0\} = \Delta_w \cap N$ is a simple system for the the root system of N.*

(b) Let $-\rho_N$ be the Weyl vector for the simple system $\Delta_w \cap N$. Then one has $\rho = \rho_N + hw_1 + (h+1)w$ where $w_1 = (u + w)$.

Proof. (a) Note that $(u+w)$ is a null vector such that $\langle u+w, w\rangle = -1$. So Since $\mathbb{Z}w + \mathbb{Z}u \simeq II_{1,1}$ the orthogonal compliment N is self-dual, hence is a Niemeier lattice of type w (by definition).

Theorem 21.11 implies that $w = \sum_{s \in \Delta} n_s s$. So

$$-1 = \langle u, w\rangle = \sum_{s \in \Delta} n_s \langle u, s\rangle.$$

Since each $\langle w, s\rangle \in \mathbb{Z}_{\leq 0}$ and each $n_s \in \mathbb{Z}_{>0}$, it follows that $\langle u, s\rangle = -1$ for a unique $s \in \Delta$ such that $n_s = 1$ and $\langle u, s'\rangle = 0$ for all $s' \in \Delta - \{s\}$. Removing a level one vertex from an affine diagram gives the corresponding spherical diagram. Since $\{v \in \Delta_w : \langle u, w\rangle = 0\} = \Delta_w \cap N$ omits one level one vertex from each connected component of Δ_w, it is a union of the corresponding spherical diagrams, hence it is the spherical diagram of type N. So the roots system generated by $(\Delta_w \cap N)$ is contained in the root system of N but is already is root system of type N. So $\Delta_w \cap N$ must be a simple system of for N.

(b) We have $\langle \rho, w\rangle = -h$ and hence $\langle rho, w_1\rangle = \langle \rho, w + u\rangle = -h - 1$. So $\rho_* = \rho + hw_1 + (h+1)w$ is orthogonal to w and w_1. So $\rho_* \in N$. Now

$\langle \rho_*, s\rangle = \langle \rho, s\rangle = -1$ for all $s \in \Delta_w \cap N$. Since the root system of N has full rank, these inner product conditions determine ρ_*. Specifically, ρ_* must be negative of the Weyl vector for the simple system of N. $\square$

21.18 Remark. Write $w_1 = u + w$. Then $H = \mathsf{gram}\{w, w_1\} = \left(\begin{smallmatrix} 0 & -1 \\ -1 & 0 \end{smallmatrix}\right)$ forms a hyperbolic cell and $N = H^\perp \cap L$. Given $v \in V$, unique there exists $v_N \in N$ and $m, n \in \mathbb{Z}$ such that $v = v_N + mw_1 + nw$. We write $v = (v_N; m, n)_N$. This is called the $N \perp H$ coordinate system of L.

21.19 Lemma. *Let N be a Niemeier lattice with roots. Identify $L = N \oplus H$ where $H = \mathsf{span}\{z, z'\}$ is a hyperbolic cell and z, z' are cusps of type N with $\langle z, z'\rangle = -1$.*

(a) Then the action of $R(L)_z$ on a horoshpere around z is isomorphic to action of $\mathsf{AR}(N)$ on $N \otimes \mathbb{R}$.

(b) Let Δ be a connected component of $\mathsf{Dynkin}(N)$ and let $\Delta^{\text{aff}} = \Delta \cup \{0\}$. Let $\{s_i \colon i \in \Delta\}$ be a simple system for Δ, let $s_{\max,\Delta}$ be the highest root and let $s_0 = -s_{\max,\Delta} + z$. So $\sum_{i \in \Delta^{\text{aff}}} n_i s_i = z$ where $\{n_i \colon i \in \Delta^{\text{aff}}\}$ is the balanced numbering on Δ^{aff}. Then there exists a unique chamber C for $R(L)$ such that $z \in \text{cl}(C)$ and the walls of C passing through z are the mirrors $s_j^\perp$ as j varies over Δ^{aff} and as Δ varies over the connected components of $\mathsf{Dynkin}(N)$.

Proof. (a) The mirrors of $R(L)$ through z are the hyperplanes orthogonal to the roots $\{r + nz \colon r \in \Phi_N, n \in \mathbb{Z}\}$. Part (a) follows from the correspondence defined in 19.2 using the map $J : \mathcal{H}^{25} \to N$ once we observe that $\langle r + nz, z'\rangle = -n$. So the hyperplane $(r + nz)^\perp$ (resp. reflection in $(r + nz)^\perp$) corresponds under J to the affine hyperplane $H([r], -n)$ (resp. reflection in $H([r], -n)$).

(b) There is a chamber C_0 of $\mathsf{Aff}(N)$ acting on $N \otimes \mathbb{R}$ whose walls are precisely $H(s_j, 0)$ with $j \in \Delta$ and $H(s_{\max,\Delta}, 1)$ as Δ varies over the conencted components of $\mathsf{Dynkin}(N)$. The results of section 19 imply that $J^{-1}(C_0)$ is a chamber for $R(L)_z$ and the walls of C_0 correspond one to one with the walls of $J^{-1}(C_0)$ under the map J. By the calculation in part (a), we know that for $i \in \Delta$, the mirrors $(s_i)^\perp$ of $R(L)$ corresponds to the affine hyperplane $H(s_i, 0)$ and the mirror $s_0^\perp = (s_{\max} - z)^\perp$ corresponds to the mirror $H(s_{\max,\Delta}, 1)$. So as Δ varies over the connected components of $\mathsf{Dynkin}(N)$, the mirrors $\{s_i^\perp \colon i \in \Delta^{\text{aff}}\}$ form the walls of a chamber $J^{-1}(C_0)$ for $R(L)_z$. Applying Vinberg's algorithm with controlling vector z, we find that there is a unique chamber C of $R(L)$ contained in $J^{-1}(C_0)$ containing z in its closure and the walls of C through z are precisely the walls of $J^{-1}(C_0)$. $\square$

21.20 Remark (Probably true and probably in Borcherds thesis in a similar form. need to check details). The above lemma can be rephrased as follows: Let N be a Niemeier lattice. Let w be a cusp of L of type N and let Δ_0 be a set of roots of L in $w^\perp$ that form an affine Dynkin diagram of type N. Then there exists a unique chamber C of $R(L)$ satisfying the conditions that w is in the closure of C and that the walls of C passing through w are precisely $\{s^\perp \colon s \in \Delta_0\}$. Let ρ be the Leech cusp of C. We fix ρ and C and this defines Leech roots and Leech glue roots of type w.

21.21 Theorem (holy construction)**.** *Let $(N, w, \Delta_0, C, \rho)$ be as in the above remark. Let u be a root such that $\langle u, w\rangle = -1$ and that connects to an unique level one vertex of each component of Δ_0* [3]*. Then $u^\perp$ is a wall of C, that is, u is actually a Leech glue root (need to check details for this statement). Let $w_1 = (u + w)$, so that $w^2 = w_1^2 = 0$ and $\langle w, w_1\rangle = -1$. So $L \cap \{w, w_1\}^\perp$ is a Niemeier lattice of type N. Write $N = L \cap \{w, w_1\}^\perp$. Then $\Delta \cap N$ is a simple system for N. Let $-\rho_N$ be the Weyl vector for this simple system. Then $\rho = \rho_N + hw_1 + (h + 1)w$.*

sketch of proof. We can apply Vinberg algorithm to construct the chamber C starting with the controlling vector w and the initial set of mirrors $\{s^\perp : s \in \Delta\}$ and the fundamental domain $C_0 = \cap\{v \in \mathcal{H}^{25} : \langle v, s\rangle \leq 0 \text{ for all } s \in \Delta_0\}$ for the affine reflection group $R(L)_w$. Since $\langle u, w\rangle = -1$, it is in the first shell around w and while applying Vinberg's algorithm, we can consider the mirror $u^\perp$ right after starting with the initial set of mirrors $\{s^\perp : s \in \Delta_0\}$. The inner products between u and the simple roots in Δ_0 are all non-positive, which implies that for each $s \in \Delta_0$ the half space $D_{s^\perp}(C_0) = \{v : \langle v, s\rangle \leq 0\}$ and the half space $D_{u^\perp}(w) = \{v : \langle v, w\rangle \leq 0\}$ are opposite half spaces since their they have normal vectors $-s$ and $-u$ respectively. So the Vinberg's algorithm accepts the mirror $u^\perp$ and hence it forms a wall of C. (need to verify the details up to here). The rest of the proof is just reverse engineering the proof of 21.17. □

The theorem gives the following method for constructing Leech cusps. Let N be a Niemeier lattice. Find a primitive null vector w of L and a set of roots Δ in $w^\perp$ that form an affine Dynkin diagram of type N. Find a root u such connects to a unique level one vertex of each component of Δ. Let $-\rho_N$ be the Weyl vector for the spherical simple system $\{v \in \Delta : \langle v, u\rangle = 0\}$. Then $\rho = \rho_N + h(u + w) + (h + 1)w$ is a Leech cusp where h is the Coxeter number of any component of N. Below we give a couple of examples of this.

For consistency with some computations in [CS], we change notation slightly and choose coordinates for $II_{25,1}$ so that it consists of all $(x_0, \cdots, x_{24}; x_\infty)$ such that each x_j is an integer or each $(x_j + 1/2)$ is an integer and $(x_\infty - \sum_{j=0}^{24} x_j)$ is an even integer. We shall see below that the null vector $\rho = (0, 1, 2, \cdots, 24; 70)$ defines a cusp of Leech type and we shal describe a cusp of type A_{24} and a cusp of type D_{24} in the closure of the Weyl chamber containing ρ. Put another way, this describes the environs of two deep holes of type A_{24} and D_{24} in the Leech lattice.

21.22. Leech lattice from a cusp of type A_{24}. Consider the cusp

$$w = (1/2, \cdots, 1/2; 5/2).$$

Note that

$$r_{24} = e_1 - e_2, r_0 = e_2 - e_3, r_1 = e_3 - e_4, \cdots, r_{22} = e_{24} - e_{25}, r_{23} = e_{25} - e_1.$$

[3]In other words, if Δ is a connected component of Δ_0 with balanced numbering $\{n_s : s \in \Delta\}$, then there exists a unique vertex $s_0 \in \Delta$ such that $n_{s_0} = 1$ and $\langle s_0, u\rangle = -1$ and $\langle s, u\rangle = 0$ for all $s \in \Delta - \{s_0\}$. Note that such an u exists, for example, choose any Leech glue root of type w.

are roots in $w^\perp$ that forms an affine A_{24} diagram. To split a hyperbolic cell containing w, find $u \in L$ such that $\langle u, w\rangle = -1$; for example $u = -e_1 - e_2$. Then $w_1 = u + (u^2/2)w = u + w$ is a cusp of L and $\langle w, w_1\rangle = -1$, so $H = \mathsf{span}_{\mathbb{Z}}\{w, w_1\}$ is an hyperbolic cell and $\{w, w_1\}^\perp = N$ is a Niemeier lattice of type A_{24}. Let $s_j = r_j$ for $j = 0, 1, \cdots, 22, 24$ and $s_{23} = r_{23} + w$. Then $\Delta_N = \{s_1, \cdots, s_{24}\}$ forms an A_{24} diagram in $\{w, w_1\}^\perp$. So Δ_N is a simple system for the root system of N. The Weyl vector for Δ_N is

$$-\rho_N = \sum_{j=1}^{12} \tfrac{1}{2} j(25 - j)(s_j + s_{25-j}).$$

It follows that $\rho_N + 25z_1 + 26z$ is a Leech cusp. One computes

$$\rho_N + 25z_1 + 26z = \rho = (0, 1, 2, \cdots, 24; 70).$$

21.23. Leech lattice from a cusp of type D_{24}. Consider the cusp

$$w = e_{25} + e_{26} = (0, \cdots, 0, 1; 1).$$

Note that

$$r_2 = e_2 - e_3, \cdots, r_{23} = e_{23} - e_{24}, r_{24} = e_{23} + e_{24}, r_{25} = e_1 - e_2, r_1 = -e_1 - e_2,$$

are roots in $w^\perp$ that forms an affine D_{24} diagram. To split a hyperbolic cell containing w, find $u \in L$ such that $\langle u, w\rangle = -1$ and that attaches to a level one vertex of the D_{24} diagram; for example $u = \frac{1}{2}(-1, 1, 1, \cdots, 1, 3; 5)$. Then $w_1 = u + w$ is a cusp of L and $\langle w, w_1\rangle = -1$, so $H = \mathsf{span}_{\mathbb{Z}}\{w, w_1\}$ is an hyperbolic cell and $\{w, w_1\}^\perp = N$ is a Niemeier lattice of type D_{24}. Let $s_j = r_j$ for $j = 1, 2, \cdots, 23, 25$ and $s_{24} = r_{24} + w$. Then $\Delta_N = \{s_1, \cdots, s_{24}\}$ forms an D_{24} diagram in $\{w, w_1\}^\perp$. The Weyl vector for Δ_N is

$$-\rho_N = \tfrac{24\cdot 23}{4}(s_{23} + s_{24}) + \sum_{j=1}^{22} \left(24i - \tfrac{i(i+1)}{2}\right) s_i$$

It follows that $\rho_N + 46z_1 + 47z$ is a Leech cusp. One computes

$$\rho_N + 46z_1 + 47z = \rho = (0, 1, 2, \cdots, 24; 70).$$

21.24 Theorem. *Let N be a Niemeier lattice with roots. Let $z \in \mathrm{cl}(C)$ be a cusp of type N. Let $\Phi_{1,z}$ be the set of Leech roots whose mirrors pass through z. Then one can identify $z^{\perp}$ with $N + z\mathbb{Z}$ such that*

$$\Phi_{1,z} = \{s_0^j + z, s_1^j, \cdots, s_{n_j}^j : j = 1, \cdots, r\}$$

where $\{s_1^1, \cdots, s_{n_1}^1\}, \cdots, \{s_1^r, \cdots, s_{n_r}^r\}$ are simple systems for the irreducible components of the root system of N and $s_0^1, \cdots, s_0^r$ are the corresponding lowest roots. Thus $\Phi_{1,z}$ forms the affine Dynkin diagram of N.

In particular, if $v_1, \cdots, v_k$ are the Leech roots corresponding to a component of the affine Dynkin diagram of N and $c_1, \cdots, c_k$ are the balanced numbering of this affine Dynkin diagram, then $\sum_j c_j v_j = z$.

Proof. Recall: z is a cusp of type N means z is a a primitive null vector of L such that $\langle z, \rho\rangle < 0$ and $z^{\perp}/z \simeq N$. Further, $\langle z, x\rangle < 0$ for all $x \in \mathcal{H}^{25}$.

Let C_z be the chamber of $R(L)_z$ containing C (hence $\rho \in \mathrm{cl}(C_z)$). Lemma 14.6 implies that $\mathsf{Wall}(C_z) = \{r^{\perp} : r \in \Phi_{1,z}\}$. Since $\rho \in \mathrm{cl}(C_z)$ and there is no mirror through ρ, for each Leech root r, we have $D_{r^{\perp}}(C_z) = D_{r^{\perp}}(\rho) = \{x \in : \langle x, r\rangle < 0\}$. So

$$C_z = \cap_{r \in \Phi_{1,z}} \{x \in \mathcal{H}^{25} : \langle x, r\rangle < 0\}.$$

Choose z' such that $\langle z, z'\rangle = -1$. The choice of z' gives us a map $J : \mathcal{H}^{25} \to z^{\perp}/z$ (see 19.1) which sets up an isomorphism between the permutation actions of $R(L)_z$ on a horosphere around z and the permutation action of $\mathsf{AR}(N)$ on $(z^{\perp}/z) \otimes \mathbb{R}$. It follows that $J(C_z)$ is chamber of $\mathsf{AR}(N)$ and its walls are

$$\{J(r^{\perp}) : r \in \Phi_{1,z}\} = \{H([r], \langle r, z'\rangle) : r \in \Phi_{1,z}\}.$$

(Recall: $[\,]$ denotes going modulo the span of z). Note that the in the choice of z' and the corresponding choice of J we have a freedom of adding an element of $z^{\perp}$. So by changing z' by an element of $z^{\perp}$ we can choose the map J so that $J(C_z)$ is a chamber of $\mathsf{AR}(N)$ which contains 0 in its closure. If $r \in \Phi_{1,z}$ and $x \in \mathcal{H}^{25}$, then

$$\langle [r], J(x)\rangle = \langle r, x\rangle\langle x, z\rangle^{-1} + \langle r, z'\rangle$$

Since $\langle x, z\rangle < 0$, one has

$$J(C_z) = \cap_{r \in \Phi_{1,z}} \{u \in z^{\perp}/z : \langle r, u\rangle > \langle r, z'\rangle\}.$$

On the other hand, since $J(C_z)$ is a chamber of $\mathsf{AR}(N)$ containing 0 in its closure, we know that there is a simple system for $\{z, z'\}^{\perp}$ with components $s_1^j, \cdots, s_{n_j}^j$ for $j = 1, \cdots, r$ such that

$$J(C_z) = \cap_{j=1}^r (\{u : \langle u, [-s_0^j]\rangle < 1\} \cap \{u : \langle u, [s_1^j]\rangle > 0\} \cap \cdots \cap \{u : \langle u, s_{n_j}^j\rangle > 0\}).$$

where s_0^j is the lowest root in $\{z, z'\}^{\perp}$ for the component $s_1^j, \cdots, s_{n_j}^j$. Observe that if r and s are roots such that $\{u : \langle r, u\rangle > 0\} = \{u : \langle s, u\rangle > 0\}$, then $r = s$.

Compare the two expressions for $J(C_z)$. All the half spaces appearing in the two expressions are essential, that is, each of these half spaces bound a wall of $J(C_z)$. So the half spaces must be in bijection. Let $r \in \Phi_{1,z}$ such that

$$\{u\colon \langle r, u\rangle > \langle r, z'\rangle\} = \{u\colon \langle u, s_i^j\rangle > 0\} \text{ for some } i = 1, \cdots, n_j.$$

Then $[r] = [s_i^j]$ and $\langle s_i^j, z'\rangle = 0$, so $r = s_i^j$. Finally let $r \in \Phi_{1,z}$ be the root such that

$$\{u\colon \langle r, u\rangle > \langle r, z'\rangle\} = \{u\colon \langle s_0^j, u\rangle > -1\}.$$

Then $r = s_0^j + \lambda z$ for some $\lambda \in \mathbb{Z}$ and $-1 = \langle r, z'\rangle = \langle s_0^j + \lambda z, z'\rangle = -\lambda$. So $r = s_0^j + z$. It follows that

$$\Phi_{1,z} = \{s_0^j + z, s_1^j, \cdots, s_{n_j}^j : j = 1, \cdots, r\}.$$

$\square$

21.25 Corollary. *Consider the setup of 21.24. Let h be the Coxeter number of an irreducible component of N. Then $\langle \rho, z\rangle = -h$ and $\rho = \rho_N + hz' + (h+1)z$ where $-\rho_N$ is the Weyl vector for the simple system $\{s_1^j, \cdots, s_{n_j}^j : j = 1, \cdots r\}$.*

Proof. Let $r_0^j = s_0^j + z, r_1^j = s_1^j, \cdots, r_{n_j}^j = s_{n_j}^j$. Let $\{1 = c_0, c_1 \cdots, c_{n_j}\}$ be the balanced numbering on the affine dynkin diagram $\{s_0^j, \cdots, s_{n_j}^j\}$. Then $\sum_i c_i r_i^j = z$. Since each r_i^j is a Leech root, it follows that

$$\langle z, \rho\rangle = -\sum_i c_i = -h.$$

Let $\rho_N = \rho - hz' + \langle \rho, z'\rangle z$. Then $\rho_N \in L \cap \{z, z'\}^\perp = N$ and $\langle \rho_N, s_i^j\rangle = -1$ for $i = 1, \cdots, n_j, j = 1, \cdots, r$. So $-\rho_N$ is the Weyl vector for the simple system $\{s_1^j, \cdots, s_{n_j}^j : j = 1, \cdots r\}$ of N. One knows that $\rho_N^2 = 2h(h+1)$. It follows that $\langle \rho, z'\rangle = -(h+1)$. $\square$

Choose coordinates for $II_{25,1}$ so that it consists of all $(x_0, \cdots, x_{24}; x_\infty)$ such that each x_j is an integer or each $(x_j + 1/2)$ is an integer and $(x_\infty - \sum_{j=0}^{24} x_j)$ is an even integer. The null vector $\rho = (0, 1, 2, \cdots, 24; 70)$ defines a cusp of Leech type. Below we describe a cusp of type A_{24} and a cusp of type D_{24} in the closure of the Weyl chamber containing ρ.This gives two examples of the Holy construction of Leech lattice. Put another way, this describes the environs of two deep holes of type A_{24} and D_{24} in the Leech lattice.

21.26. Leech lattice from a cusp of type A_{24} Consider the cusp $w = (1/2, \cdots, 1/2; 5/2)$. Note that

$$r_{24} = e_1 - e_2, r_0 = e_2 - e_3, r_1 = e_3 - e_4, \cdots, r_{22} = e_{24} - e_{25}, r_{23} = e_{25} - e_1.$$

are roots in $w^\perp$ that forms an affine A_{24} diagram. To split a hyperbolic cell containing w, find $u \in L$ such that $\langle u, w\rangle = -1$; for example $u = -e_1 - e_2$. Then

$w_1 = u+(u^2/2)w = u+w$ is a cusp of L and $\langle w, w_1\rangle = -1$, so $H = \mathsf{span}_{\mathbb{Z}}\{w, w_1\}$ is an hyperbolic cell and $\{w, w_1\}^{\perp} = N$ is a Niemeier lattice of type A_{24}. Let $s_j = r_j$ for $j = 0, 1, \cdots, 22, 24$ and $s_{23} = r_{23} + z$. Then $\Delta_N = \{s_1, \cdots, s_{24}\}$ forms an A_{24} diagram in $\{w, w_1\}^{\perp}$. So Δ_N is a simple system for the root system of N. Note that $(s_0 - z) = -\sum_{j=1}^{24} s_j$ is the lowest root of the simple system Δ_N. It follows that there is a unique chamber C of $R(L)$ such that $z \in \mathrm{cl}(C)$ and and such that $s_0^{\perp}, \cdots, s_{24}^{\perp}$ are precisely the walls of C that pass through z.

The Weyl vector for Δ_N is $-\rho_N = \sum_{j=1}^{12} \frac{1}{2} j(25 - j)(s_j + s_{25-j})$. So the Leech cusp of C is $\rho = \rho_N + 25z_1 + 26z = (0, 1, 2, \cdots, 24; 70)$.

22 The $\sqrt{-3}$-modular Eisenstein lattice of signature $(13,1)$

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