§. Graded rings and modules.

k = algebraically closed field of characteristic 0.

A = graded k-algebra ; $A = \bigoplus_{n=0}^{\infty} A_n$ with $A_0 = k$ and

$A_m A_n \subseteq A_{m+n}$.

$M = \bigoplus_{\substack{n \in \mathbb{Z} \\ n >> -\infty}} M_n$ - a graded A-module. ie. $A_m M_n \subseteq M_{m+n}$.

If $m \in M$, write $m = \sum_d' (m)_d$ with $(m)_d \in M_d$.

(deg d piece of m)

- m is a homog element of deg d, if $m = (m)_d$.

- Note - $(am)_d = \sum_k' (a)_k (m)_{d-k}$

- - If a is homog, then $(am)_d = a \cdot (m)_{d-\deg(a)}$

- A submodule $N \subseteq M$ is homog - if it is generated by homog elements.

Lemma TFAE : (i) N is homog. submod.
(ii) If $n \in N$, then all graded pieces of n belong to N.
(iii). N is a graded module $N = \bigoplus_n (N \cap M_n)$

proof (i) ⇒ (ii). Let $n_1, n_2, \ldots$ be homog generators for N.

Write $n = n_1 x_1 + n_2 x_2 + \cdots$

Let d be the top deg term in n. Then

$(n)_d = n_1 (x_1)_{d-\deg(n_1)} + n_2 (x_2)_{d-\deg(n_2)} + \cdots \in N$.

Now induct on degree □.

§ 2. Graded dimension.

Let $M = \bigoplus_{r=0}^{\infty} M_r$ be a graded k module with f.d graded pieces. Define.

$$\mathrm{grd}_k(M) = \mathrm{grd}_k(M)(t) = \sum_{r=0}^{\infty}{}' \dim(M_r)\, t^r.$$

(graded dim of M)

- Let $M = k[y]$ where $\deg(y) = d$. Then

$$\mathrm{grd}_k(M) = 1 + t^d + t^{2d} + \cdots = \frac{1}{(1-t^d)}.$$

- If M, N are graded k modules, then,

$$\mathrm{grd}_k(M \otimes N) = \mathrm{grd}_k(M) \cdot \mathrm{grd}_k(N)$$

- Let M be isom to a poly. ring $k[y_1, \ldots, y_n]$ where $y_j \in M_{d_j}$.

Then $k[y_1, \ldots y_n] \underset{gr}{\simeq} k[y_1] \otimes \cdots \otimes k[y_n]$. So.

$$\mathrm{grd}_k(k[y_1, \ldots y_n]) = \prod_{i=1}^{n} \frac{1}{(1-t^{d_i})}$$

§3 Invariants of a finite group.

Setup.

- finite gp $G \circlearrowright V$ (f. dim. k-vsp.)
- $S = \mathrm{sym}(V^*) \simeq k[x_1 \ldots x_n]$ $\quad n = \dim_k(V)$
- $R = S^G$ - (the ring of invariants)

(Since G acts linearly on V, the G action on S is degree preserving).

- R_+ = maximal ideal in R consisting of the + degree elements.
- Let $\rho = \frac{1}{|G|} \sum_{g \in G} g : S \longrightarrow R$. (the Reynolds operator).

Theorem (Hilbert). R is a finitely generated. k-algebra.
[In fact if $f_1 \ldots f_r$ are any set of homogeneous invariants that generate the ideal R_+S, then $R = k[f_1 \ldots f_r]$.

Lemma: Let $\mathfrak{a}$ be an ideal in a Noetherian ring A. Then any set of generators of $\mathfrak{a}$ contains a finite subset that generates $\mathfrak{a}$.

proof. let J be a set of generators of $\mathfrak{a}$. Suppose no finite subset of J generates $\mathfrak{a}$. Choose $a_1 \in J$. Then $\langle a_1 \rangle \neq \mathfrak{a}$, so $J \not\subseteq \langle a_1 \rangle$, so $\exists\, a_2 \in J - \langle a_1 \rangle$. Again $\langle a_1, a_2 \rangle \neq \mathfrak{a}$, so $J \not\subseteq \langle a_1, a_2 \rangle$, so $\exists\, a_3 \in J - \langle a_1, a_2 \rangle$. Get $\langle a_1 \rangle \subsetneq \langle a_1, a_2 \rangle \subsetneq \langle a_1, a_2, a_3 \rangle \subsetneq \cdots$ □

Remark: • Since S is noetherian and the ideal R_+S is generated by the homg elements of R_+, we can choose a finite set of homg invariants $f_1 \ldots f_r \in R_+^{\text{hmg.}}$, that generate the ideal R_+S.

- Consider. $\tilde{R} = k[xy, xy^2, \ldots] \subset k[x,y] = \tilde{S}$. Then $\tilde{R}_+\tilde{S}$ is f.g. but $\tilde{R}$ is not a f.g. k-algebra. So the proof of Hilbert thm must use the fact that R is the ring of invariants of finite gp G.

proof. Choose a finite set of homog. elements $f_1 \cdots f_r \in R_+$ that generate R_+S. Let $x \in R_+^{homog}$. It suffices to show that $x \in k[f_1 \cdots f_r]$.

Write $x = \sum_i h_i f_i$. Since x and f_i's are homog, wlg we may assume h_i's are homog. Now

$$x = \rho(x) = \sum \rho(h_i) f_i$$

Since $\deg(\rho(h_i)) < \deg(x)$, by induction on degree $\rho(h_i) \in k[f_1 \cdots f_r]$. So x does too. □.

Now assume G is generated by complex reflections.

The next lemma uses properties of reflection groups.

§4. Invariants of complex reflection groups.

(*) Lemma. Let $h_1 \cdot h_r \in R$ such that $h_1 \notin h_2 R + \cdots + h_r R$. Let $p_1, \cdots, p_r \in S^{homg}$ such that $p_1 h_1 + \cdots + p_r h_r = 0$. Then $p_1 \in R_+ S$.

proof. Let $g \in G$ be a reflection with hyperplane $\{f = 0\}$. Then $({}^g p_j - p_j)$ vanishes on $\{f = 0\}$. So, by Nullstellensatz, $f \mid ({}^g p_j - p_j)$. Since G is generated by reflection it follows that $f \mid ({}^g p_j - p_j)$ $\forall g \in G$. So $\sum p_i h_i = 0$ implies

$$\sum \frac{({}^g p_j - p_j)}{f} \cdot h_j = 0.$$ By induction on degree it follows that $({}^g p_1 - p_1) \in R_+ S$. Since G is generated by reflections $({}^g p_1 - p_1) \in R_+ S$ for all $g \in G$. Averaging over G we find that $p_1 \in \rho(p_1) + R_+(S)$.

Note that $\deg(p_1) \neq 0$, since $\deg(p_1) = 0$ would imply that

$$h_1 = p_1^{-1}(p_2 h_2 + \cdots + p_r h_r)$$

so $h_1 = \rho(h_1) = p^{-1}(\rho(p_2) h_2 + \cdots + \rho(p_r) h_r) \in h_2 R + \cdots + h_r R$,

contradicting our assumption.

So. $\deg(p_1) > 0$, so $\rho(p_1) \in R_+$. So $p_1 \in \rho(p_1) + R_+ S = R_+ S$. $\square$

<u>Theorem</u> [Chevalley - Shephard - Todd - Serre]. TFAE.

(1) G is generated by complex reflections.
(2) R is a polynomial algebra.
(3) S is a free R module.
(4) $R \otimes (S/R_+S) \simeq S$ as a graded R-algebra.

<u>proof of (1) $\Rightarrow$ (2)</u> : [after Chevalley].

Choose a minimal set of homogeneous elements $f_1 \ldots f_r \in R_+$ that generate the ideal R_+S. By Hilbert's thm, $R = k[f_1 \ldots f_r]$.

We need to show that $f_1 \ldots f_r$ are algebraically indep.

Let $d_i = \deg(f_i)$.

Suppose $h(y_1 \ldots y_r)$ be a nonzero polynomial such that $h(f_1 \ldots f_r) = 0$. Wlg may assume h is weighted homogeneous of deg d, with $\deg(y_1) = d_1, \ldots, \deg(y_r) = d_r$.

[Write $h = h_{(0)} + h_{(1)} + \ldots$, where $h_{(d)}$ contains the monomials $y_1^{e_1} \ldots y_r^{e_r}$ such that $\sum d_i e_i = d$. Then

$$h(\underline{f}(x)) = h_{(0)}(\underline{f}(x)) + h_{(1)}(\underline{f}(x)) + \cdots$$

and $h_{(j)}(\underline{f}(x))$ is homg of deg d in $x_1 \ldots x_n$.

So if $h(\underline{f}(x)) = 0$ then $h_{(j)}(\underline{f}(x)) = 0$ for all j.]

Fix $k = 1, \dots, r$.

Now $h(f_1 \dots f_r) = 0$ implies

$$(*) : \sum_{j=1}^{r}{}' h_j \left(\frac{\partial f_j}{\partial x_k}\right) = 0, \quad \text{where} \quad h_j = (\partial_j h)(f_1 \dots f_r) \in R_{(d - d_j)}.$$

Choose a minimal set J of h_j's that generate the ideal.

$(h_1 R + \dots + h_r R)$. After reindexing, we may assume $J = \{h_1, \dots, h_m\}$.

Write $h_j = \sum_{i=1}^{m}{}' g_{ji} h_i$, $j = r+1, \dots, m$. Wlg. assume $g_{ji} \in R_{(d_i - d_j)}$

$$(*) \Rightarrow \sum_{i=1}^{m}{}' h_i \left(\frac{\partial f_i}{\partial x_k}\right) + \sum_{j=m+1}^{r}{}' \left(\sum_{i=1}^{m}{}' g_{ji} h_i\right) \cdot \frac{\partial f_j}{\partial x_k} = 0$$

So $\displaystyle\sum_{i=1}^{m}{}' h_i p_i = 0$ where $p_i = \left(\frac{\partial f_i}{\partial x_k} + \sum_{j=m+1}^{r}{}' g_{ji} \frac{\partial f_j}{\partial x_k}\right) \in S_{(d_i - 1)}$.

So by lemma, $p_1 \in R_+ S$, ie

$$\frac{\partial f_1}{\partial x_k} + \sum_{j=m+1}^{r}{}' g_{j1} \frac{\partial f_j}{\partial x_k} = \sum_{i=1}^{r} f_i q_i$$

Euler's formula $\deg(f) \cdot f = \sum_i{}' x_i \frac{\partial f}{\partial x_i}$, implies,

$$d_1 f_1 + \sum_{j=m+1}^{r} g_{j1} d_j f_j = \sum{}' f_i r_i \quad \text{with } \deg(r_i) > 0.$$

Since $\deg(r_i) > 0$, Equating $\deg(d_1)$ terms on both sides,

we find $d_1 f_1$ is a S linear combo of $f_2 \dots f_r$, contradicting

the minimality of $f_1 \dots f_r$. $\square$.

§. 5. The invariant degrees.

Define: $\chi_g(t) = \sum_d{}' \mathrm{Tr}(g|_{S_{(d)}}) \cdot t^d$

$$\frac{1}{|G|}\sum_{g\in G} \chi_g(t) = \sum_n{}' \mathrm{Tr}(p|_{S_{(d)}})\, t^d = \sum{}' \dim\left[(S_{(d)})^G\right] t^d$$

$$= \sum{}' \dim\left[R_{(d)}\right] t^d$$

$$= \mathrm{grdim}_k(R)(t).$$

Let $g\in G$.

Diagonalise $g \circlearrowright V$

Let $\lambda_1, \dots, \lambda_n =:$ eigenvalues of $g|_V$ with basis of eigenvectors $v_1, \dots, v_n$.

Then $g(\underline{v}^i) = \underline{\lambda}^i \underline{v}^i$.

So $\mathrm{Tr}\left(g|_{\mathrm{Sym}_{(d)}(V)}\right) = \sum_{i_1+\dots+i_n=d} \underline{\lambda}^i$

$$\sum_d{}' \mathrm{Tr}(g|_{\mathrm{Sym}_d(V)}) t^d = \sum_{i_1\dots i_n}{}' \lambda_1^{i_1} \cdots \lambda_n^{i_n} t^{i_1+\dots+i_n}$$

$$= \left(\sum_{i_1}{}' (\lambda_1 t)^{i_1}\right)\left(\sum_{i_2}{}' (\lambda_2 t)^{i_2}\right)\cdots$$

$$= \frac{1}{\prod_t (1-\lambda_j t)} = \frac{1}{\det(1-tg)}.$$

So

$$\mathrm{grdim}_k(R)(t) = \frac{1}{|G|}\sum_{g\in G}\frac{1}{\det(1-tg)}.$$

Thm. [Molien]

Since $R \simeq$ a polynomial ring with generators in degree $d_1, \dots, d_n$, we have

$$\frac{1}{|G|}\sum_{g\in G}\frac{1}{\det(1-tg)} = \prod_{i=1}^n \frac{1}{(1-t^{d_i})}$$

Multiplying both sides by $(1-t)^n$, we get,

$$\prod_{i=1}^{n}\left(\frac{1}{1+t+\cdots t^{d_i-1}}\right) = \frac{1}{|G|}\cdot\sum_{g}\frac{(1-t)^n}{(1-tg)}$$

So $t=1$. All the terms on RHS vanish except when $g = \mathrm{Id}$.

So $\prod d_i = |G|$.

G_Δ	Δ	$d_1 \ldots\ldots d_n$
$SL_n(\cdot)$	A_n	$2, 3, \ldots\ldots, n+1$
$SO_{2n+1}(\cdot)$	B_n	$2, 4, 6, \ldots\ldots 2n$
$Sp_{2n}(\cdot)$	C_n	(same as B_n)
$SO_{2n}(\cdot)$	D_n	$2, 4, 6, \ldots, 2n-2, n$
$E_6(\cdot)$	E_6	$2, 5, 6, 8, 9, 12$
$E_7(\cdot)$	E_7	$2, 6, 8, 10, 12, 14, 18$
$E_8(\cdot)$	E_8	$2\ 8, 12\ 14\ 18, 20, 24, 30.$
	$\vdots$	

The d_i's contain a lot of information. For example:

- $\sum(d_i-1) = \frac{1}{2}$ (# of nontrivial reflections in G). $= \frac{1}{2}\,|\#$ of roots$|$. (in the case of Weyl gps)

In type E_8: $\sum(d_i-1) = 120$.

- The order of the Chevalley group $G_\Delta(\mathbb{F}_q)/$center. $= q^{\sum(d_i-1)}\prod_i(q^{d_i}-1)$

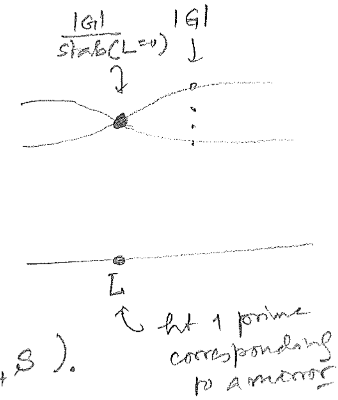
  if "$G_\Delta(\mathbb{F}_q)$ is simply connected" ie the Schur multiplier $H^2(G,\mathbb{C}^*)$ vanishes.

- $Z(U(\mathfrak{g}_\Delta)) \underset{\uparrow}{\simeq} \mathrm{Sym}(\mathfrak{h})^{W_\Delta} \simeq \mathbb{R}$.

- $H^*(\text{compact Lie gp of type } \Delta, \mathbb{R})$

  $\simeq$ Exterior algebra on generators of degree $(2d_i-1)$.

$$
\begin{array}{c} S/R_+S \\ | \\ k \end{array}
\qquad
\begin{tikzcd} R_+S \arrow[r, hook] \arrow[d, dash] & S \arrow[d, dash, "G."] \\ R_+ \arrow[r, hook] & R \end{tikzcd}
\qquad
\begin{array}{c} \mathrm{Frac}(S) \\ | \\ \mathrm{Frac}(R) \end{array}
\qquad
\begin{tikzcd} \mathrm{spec}(S) = \mathbb{A}^n \arrow[d] \\ \mathrm{spec}(R) = \mathbb{A}^n /\!\!/ G. \end{tikzcd}
$$



$$|G| = [\mathrm{Frac}(S) : \mathrm{Frac}(R)] = \mathrm{rk}_R(S) = \dim_k(S/R_+S).$$

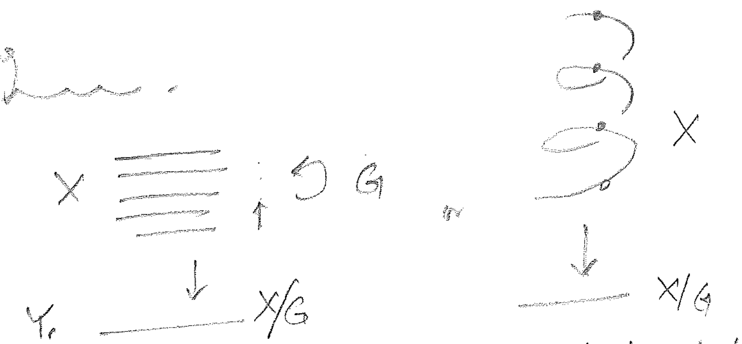
$$
\begin{array}{c} S/q \\ | \\ R/p \end{array}
\qquad
\begin{array}{c} q \\ | \\ p \end{array}
\qquad
\begin{tikzcd} B \arrow[r, dash] \arrow[d, dash] & L \arrow[d, dash, "G"] \\ R \arrow[r, dash] & K \end{tikzcd}
$$

$$G^d(q) = \{ g \in G : gq = q \}.$$
$$\cup$$
$$G^i(q) = \{ g \in G^d(q) : g|_{S/q} = \mathrm{id}_{S/q} \},$$

$$0 \to G^i(q) \longrightarrow G^d(q) \longrightarrow \mathrm{Aut}\left(\frac{B/q}{R/p}\right) \to 0.$$

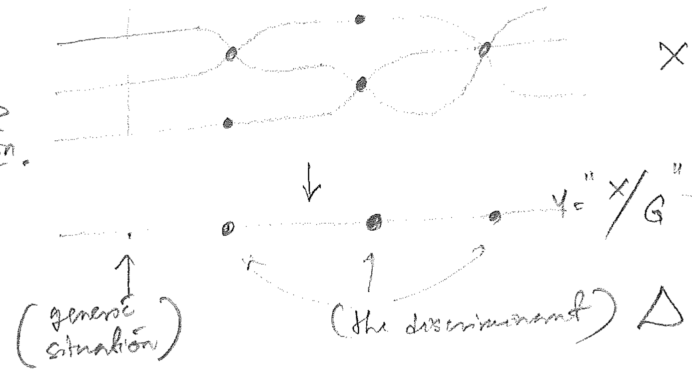
§1. Ramified/Branched covering.

- Best kind of ~~group~~ family/group action.



Often some G acting simply transitively on each fiber

- Next best: kind of family/gp action.

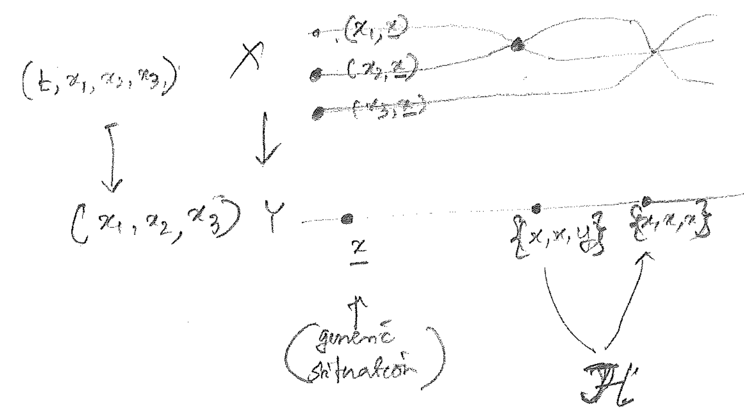


- Example ① (Geometry)

Consider $f(t, x) = (t - x_1)(t - x_2)(t - x_3) = t^3 - u_1(x)t^2 + u_2(x)t - u_3(x) = 0$.

elementary symmetric poly → $u_1(x) = x_1 + x_2 + x_3$, $u_2(x) = x_1x_2 + x_1x_3 + x_2x_3$, $u_3(x) = x_1x_2x_3$.

$$\mathbb{A}^4 \supseteq X = \{f = 0\}$$
$$\downarrow$$
$$(\mathbb{A}^3/S_3)$$



$$\Delta = \prod_{i<j}(x_i - x_j)^2$$

(Vandermonde determinant).

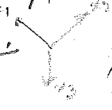
often in interesting example there is a G acting on X such that $Y = X/G$

best kind = covering spaces.

next best = generically covering but ~~...~~ may have finite stabilizer.

[A_2 case] Geometry / Gp theory.

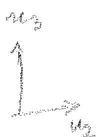
Example (2).



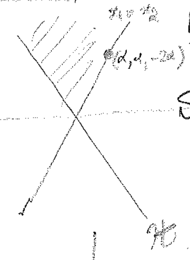
$\mathfrak{h}$

$\downarrow$

assume $(k = \bar{k})$ (σ_2, σ_3)



$\mathfrak{h}/G$

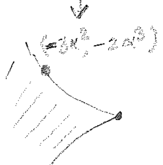


poly functions

$S = k[x_2, x_3] = \dfrac{k[x_1, x_2, x_3]}{\langle x_1 + x_2 + x_3 \rangle}$

$\mathfrak{h} = \{ x \in \mathbb{A}^3 : x_1 + x_2 + x_3 = 0 \}$.

$\uparrow$



$R = k[u_2, u_3] = S^G$

$= \dfrac{k[u_1, u_2, u_3]}{\langle u_1 \rangle}$

$(\mathcal{H}/G) = -(27u_3^2 + 4u_2^3) = \Delta$

(picture on real points)

$\Delta = \prod_{i<j} (x_i - x_j)^2 = -(27u_3^2 + 4u_2^3)$.

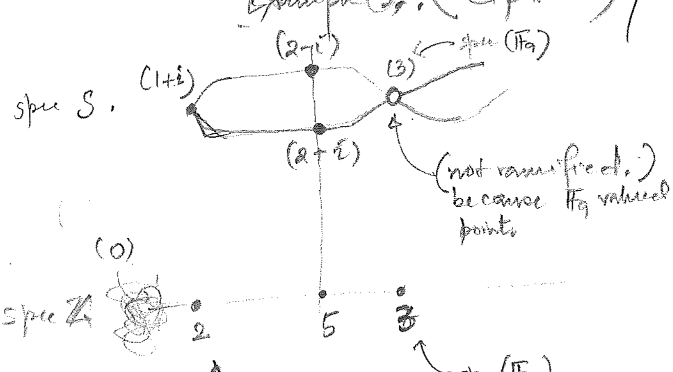
[weighted homog with $\deg(u_2) = 2$, $\deg(u_3) = 3$.

weighted homogeneous poly with

- It is easy to verify on the level of point sets that $(\mathcal{H}/S_3) \subseteq \{\Delta = 0\}$.

Example (3). (Gp theory / arithmetic.)

[The arithmetic situation is more subtle because while the points in (2) are all k valued, in arithmetic the residue field can grow.



$S = \mathbb{Z}[i]$

$R = \mathbb{Z}$

$G = \mathrm{Gal}(\mathbb{Q}[i]/\mathbb{Q})$.

Residue field extension.

$S/3S \simeq \mathbb{F}_9 \circlearrowleft G$

$\downarrow$

$\mathbb{Z}/3 = \mathbb{F}_3$.

$(1+i)^2 = 2$.

$S = \dfrac{\mathbb{Z}[x]}{\langle x^2 + 1 \rangle}$, $\Delta = \det \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$

$p \mid \Delta \iff p$ ramifies in S.

Picture at 5

$\mathbb{Z}/2 \circlearrowleft$ $S \longrightarrow S/(2+i) = \mathbb{F}_5$

$S \longrightarrow S/(2-i) = \mathbb{F}_5$

$R \longrightarrow R/5 = \mathbb{F}_5$

Picture at 3.

$S \longrightarrow S/3 = \mathbb{F}_9$

$\circlearrowleft \mathbb{Z}/2$

$R \longrightarrow R/3 = \mathbb{F}_3$

§2. Discriminants of complex reflection groups.

[we work over $\mathbb{C}$.
$k = \bar{k}$, char $k \nmid 2$.
probably enough for much of it.]

Setup • $G \overset{\text{finite}}{\subseteq} U(n)$. $\curvearrowright \mathbb{C}^n$; $\hat{V} = (\mathbb{C}^n)^*$, $S = \mathrm{Sym}(\hat{V})$. $R = S^G$.
(of order n) $\cong k[x_1 \ldots x_n]$.

$g \in G$ is a complex reflection if s fixes a hyperplane s_H in $\mathbb{C}^n$ and if $s_H = v^\perp$ then $s(v) = \zeta v$ for some root of unity ζ of order $n > 1$. So $s \sim \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & \zeta \end{pmatrix}$. notation: $s = R_v^\zeta$.

$R_v^\zeta(x) = x - (1-\zeta)\frac{\langle x|v\rangle}{\langle v|v\rangle} v$.

Theorem (Hilbert) R is a f.g. k-algebra. In fact if $f_1 \ldots f_r$ are any set of hmg. invariants generating $(R_+ S)$ then $R = k[f_1 \ldots f_r]$.

(Last time) Theorem (Chevalley ...) TFAE.

- G is generated by complex reflections.
- R is isom to a polynomial ring (in r variables).
- S is a free module$/R$ of (rank $|G|$).
- $R \otimes (S/R_+S) \xrightarrow{\sim} S$ as graded R modules.

Example/Exercise: in A_2-case basis for S/R: $1, x_2, x_3, x_2^2, x_2 x_3, x_2^2 x_3$

Def. Let $f_1 \ldots f_r$ be hmg. set of alg. indep. invariants. So
$R = k[f_1 \ldots f_r] \cong k[t_1 \ldots t_r]$
$d_i = \deg(f_i)$ — these are called invariant degrees of G.

Theorem (Molien)

$$\mathrm{grdim}_k(R)(t) = \sum' \dim_k(R_{(d)})\, t^d = \frac{1}{|G|}\sum_d \mathrm{Tr}\left(\rho\big|_{S_{(d)}}\right) t^d; \quad \left[\text{where } \rho = \frac{1}{|G|}\sum_g' g.\right]$$

$$= \frac{1}{|G|}\sum_{g \in G}' \frac{1}{\det(1 - tg)} = \prod \left(\frac{1}{1 - t^{d_i}}\right)$$

from the description $R \cong k[f_1 \ldots f_r]$.

Consequences: 1. d_i's are invariants of R (hence of G).

2. $|G| = \prod d_i$

3. $|\#$ of reflections in $G| = \sum (d_i - 1)$

§ The Jacobian and discriminant

Defⁿ: $\mathrm{Jac}(\underline{f}) = \left(\frac{\partial f_i}{\partial x_j} \right)$; $J = \det(\mathrm{Jac}(\underline{f}))$.

① Theorem (Steinberg). $J \in k^* \prod_{H \in \text{mirrors of } G} \left(\ell_H^{\,e_H - 1} \right)$

In particular J does not depend on the choice of $\{f_i\}$'s,

and $\{J = 0\} = \bigcup$ of mirrors of G.

② Thm. $gJ = \det(g|_V)^{-1} \cdot J$, $\forall g \in G$.

In fact if $\chi : G \longrightarrow \mathbb{C}^*$ is $\chi(g) = \det(g|_V)^{-1}$, then

$$S^G_\chi = R.J. = S^G. J.$$

Def Choose Δ s.t. $\Delta R = JS \cap R$.

$$\begin{array}{cc} J & S \\ | & | \\ \Delta & R \end{array}$$

Δ = discriminant.

Here:

Exercise: Verify the results for $G = S_3$. $J = \prod_{i<j} (x_i - x_j)$

$\Delta = J^2$.

③ Theorem (Terao) $\mathrm{Der}_k(S)^G = \{ \theta \in \mathrm{Der}_k(R) : \theta(\Delta) \in \Delta R \} =: D_R(\Delta)$

proof sketch. (1) Enough to show: $\ell_H^{e_H - 1} \mid J$. because degrees match. by consequence 2 of Molien thm.

Fix H. Choose a basis $\ell_H = t_1, t_2, \ldots, t_n$ for V. write

$f_i = f_i(t_1, \ldots, t_n)$. (change basis from $x_1 \ldots x_n$ to $t_1 \ldots t_n$)

Then $J = \det\left(\left(\frac{\partial f_i}{\partial t_j}\right)\right)$. So enough to show

$$\ell_H \Big| \frac{\partial f}{\partial t_1} \quad \forall\, f \in R.$$

$$s_H\big|_S(f) = f \Rightarrow f(\zeta t_1, \ldots, t_n) = f(t_1, \ldots, t_n)$$

$$\Rightarrow \frac{\partial f}{\partial t_1}(\zeta t_1 \ldots t_n) = \zeta^{e_H - 1} f(t_1 \ldots t_n).$$

$$\Rightarrow t_1^{e_H - 1} \Big| f. \quad \square$$

proof of (2). Exercise

The map in (3). there exists $\mathrm{Der}_k(R) \longrightarrow \mathrm{Der}_k(\mathrm{Frac}(S))$. since $\mathrm{Frac}(S)/\mathrm{Frac}(R)$

$\theta \longmapsto \theta_S$

a finite field extension.

Let $\widetilde{\mathrm{Der}_k}(R) = \{\theta \in \mathrm{Der}_k(R) : \theta_S \in \mathrm{Der}_k(S)\}$,

If $\theta \in \widetilde{\mathrm{Der}_k}(R)$, then $\theta_S \in \mathrm{Der}_k(S)^G$

Easy show: $\widetilde{\mathrm{Der}_k}(R) \cong \mathrm{Der}_k(S)^G$. inverse given by $\eta \mapsto \eta\big|_R$.

Thm: $\widetilde{\mathrm{Der}_k}(R) = \mathcal{D}_R(\mathcal{A})$.

(proof of Easy inclusion: $\subseteq$): let $\theta \in \widetilde{\mathrm{Der}_k}(R)$. let $\chi(g) = \det(g)_V^{-1}$

$$g\big|_S(\theta_S(J)) = \left(g\big|_{\mathrm{Der}(S)}\theta_S\right)(gJ) = \theta_S(gJ) = \theta_S(\chi(g)J) = \chi(g)\,\theta_S(J).$$

So $\theta_S(J) \in S^G_\chi = S^G \cdot J = RJ$.

Write $\Delta = J\cdot p$. Then

$$\theta(\Delta) = \theta_S(Jp) = p\,\theta_S(J) + J\,\theta_S(p) \in JS \cap R = \Delta R.$$

$\square$

Back to S_3 example.

$x_2 \in S$ satisfies $x_2^3 + u_2 x_2 - u_3 = 0$ (since $\sigma_1 = 0$)

$$\Rightarrow \quad 3x_2^2\left(\frac{\partial x_2}{\partial u_2}\right) + \sigma_2\left(\frac{\partial x_2}{\partial u_2}\right) + x_2 = 0$$

$$\Rightarrow \quad \frac{\partial x_2}{\partial u_2} = -\frac{x_2}{u_2 + 3x_2^2} = -\frac{x_2}{(x_2 - x_3)(x_2 - x_1)} = \frac{x_2(x_1 - x_3)}{J}$$

and

$$3x_2^2\frac{\partial x_2}{\partial u_3} + u_2\frac{\partial x_2}{\partial u_3} - 1 = 0$$

$$\frac{\partial x_2}{\partial u_3} = \frac{1}{u_2 + 3x_2^2} = \frac{1}{(x_2 - x_3)(x_2 - x_1)}$$

$$= \frac{-(x_1 - x_3)}{J}.$$

$$\left(\left(\frac{\partial x_i}{\partial u_j}\right)\right) = \frac{1}{J}\begin{array}{cc} x_1 & x_2 & \\ \left[\begin{array}{cc} x_2(x_1 - x_3) & x_3(x_1 - x_2) \\ -(x_1 - x_3) & -(x_1 - x_2) \end{array}\right] & & \begin{array}{c}\sigma_1 \\ \sigma_2\end{array} \end{array}$$

$P = (-3a^2, -2a^3)$

$$\left(\frac{\partial u_3}{\partial u_2}\right)_P = a. = -\frac{2u_2^2}{9u_3}.$$

$$\left(9u_3\frac{\partial}{\partial u_2} - 2u_2^2\frac{\partial}{\partial u_3}\right) \overset{?}{\in} \mathrm{Der}_k(S)$$