1.1. **Definition.** Let V be a finite dimensional real (or complex) vector space. If V has a symmetric bilinear (or hemitian) form $\langle\ ,\ \rangle$, then the linear automorphisms of this inner product space is denoted by $O(V)$. In the real case, $g \in O(V)$ is called reflection if it fixes a hyperplane and in the orthogonal complement multiplies by -1. An the complex case element $g \in O(V)$ is called a complex reflection (of order n) if it fixes a hyperplane and in the orthogonal complement multiplies by some primitive n-th root of unity. If $v \in V$, we write $v^2 = \langle v, v\rangle$ and call it norm of v. Let $v \in V$ with $v^2 \neq 0$. Define $R_v \in O(V)$ by

$$R_v^\xi(w) = w - (1-\xi)\langle w, v\rangle v/v^2.$$

Then R_v^ξ is a complex reflection in v that fixes $v^\perp$ (called mirror of the reflection) and takes v to ξv. We say R_v^ξ is a (complex) reflection in v. When $\xi = -1$, we just write $R_v = R_v^\xi$.

1.2. Let $V \simeq \mathbb{C}^n$. An element $g \subseteq GL(V)$ is called a pseudo-reflection (of order d) if its matrix in some basis is $\mathrm{diag}(\xi, 1, 1, \cdots, 1)$ for some (d-th) root of unity $\xi \neq 1$. If G is a finite subgroup $GL(V)$ generated by pseudo-reflections, then similarly there exists a G invariant positive definite hermitian form on V (just take any form and average over G). So we may assume $G \subseteq O(V) \simeq U(n)$ and the pseudo reflections in G acts by complex reflections.

1.3. **Finite real reflection groups:** Let Δ be among the diagrams A_n, D_n, E_6, E_7, E_8 having n vertices. Let $L(\Delta) \simeq \mathbb{Z}^n$ be the even integral $\mathbb{Z}$-lattice with basis $\{s_j : j \in \Delta\}$ (as usual, here $j \in \Delta$ means j is in the vertex set of Δ) and inner product defined by

- $s_j^2 = \langle s_j, s_j\rangle = 2$ for all $j \in \Delta$,
- $\langle s_j, s_k\rangle = -1$ if $(j,k) \in \mathrm{edge}(\Delta)$ and $\langle s_j, s_k\rangle = 0$ otherwise.

This inner product is positive definite. Let R_j denote the reflection in s_j. Let $\mathfrak{h}_\mathbb{R} = L(\Delta) \otimes \mathbb{R}$. Let $R_j = R_{s_j}$ denote the reflection in s_j. Let $W(\Delta)$ be the subgroup of $O(\mathfrak{h}_\mathbb{R}) \simeq O(n)$ generated by the R_j's. The group $W(\Delta)$ has a nice presentation that can be read off from Δ.

Define $\mathrm{Cox}(\Delta, \infty)$ to be the group with generators $\{r_j : j \in \Delta\}$ and relations $r_j r_k r_j = r_k r_j r_k$ if $(j,k) \in \mathrm{edge}(\Delta)$ and $r_j r_k = r_k r_j$ otherwise. Let $\mathrm{Cox}(\Delta, n)$ be the quotient of $\mathrm{Cox}(\Delta, \infty)$ by the relations $r_j^2 = 1$ for all j. Then $W(\Delta)$ has the presentation

$$W(\Delta) \simeq \mathrm{Cox}(\Delta, 2)$$

where the reflections $R_j \in W(\Delta)$ correspond to the generators r_j in $\mathrm{Cox}(\Delta, 2)$.

1.4. *Remark.*

- The $W(\Delta)$'s are important examples of reflection groups. You can make reflection groups from each finite type Dynkin diagram in similar manner and in fact these are almost all the irreducible finite reflection groups with some exceptions in dimension 2 (the dihedral groups) and 3, 4 (coming from the symmetries of platonic solids).
- $W(\Delta)$'s asise in study of complex simple Lie algebras and their representations. For each finite type Dynkin diagram Δ, there is a complex simple Lie algebra and $W(\Delta)$ is its Weyl group.

1.5. **Simply laced root lattices and their reflection groups:** An integral $\mathbb{Z}$-lattice L is a free abelian group $L \simeq \mathbb{Z}^n$ with a $\mathbb{Z}$-valued nondegenerate bilinear form. Say that L is even all lattice vectors have even norm. A simply laced root lattice L is an even integral positive definite $\mathbb{Z}$-lattice generated by its norm 2 vectors. The norm 2 vectors are called *roots* of L because reflections in these vectors take L to itself. The reflection group of L, denoted $R(L)$ is by definition the subgroup of $\mathrm{Aut}(L)$ generated by reflections in the roots of L.

1.6. **Theorem.** *Suppose L is indecomposible, positive definite integral $\mathbb{Z}$-lattice, $v^2 \in 2\mathbb{Z}$ for all $v \in L$ and L is generated by norm 2 vectors. Then $L \simeq L(\Delta)$'s for $\Delta \in \{A_n, D_n, E_6, E_7, E_8\}$. One has $R(L(\Delta)) = W(\Delta) = \mathrm{Cox}(\Delta, 2)$. Here L is indecomposible means it cannot be a orthogonal direct some of smaller ones.*

1.7. **How to recover the Dynkin diagram Δ from the reflection group** $R(L(\Delta))$**:**

- Choose a random hyperplane in V.
- Let ρ (called the Weyl vector) be the half sum of the roots on one side of this hyperplane.
- Choose the mirrors $M_1, \cdots, M_n$ that are closest to $\mathbb{R}\rho$ in the spherical norm on $S^1(\mathfrak{h}_\mathbb{R})$.

- The mirrors $M_1, \cdots, M_n$ forms correspond to the vertices of the Dynkin diagram.

1.8**. Examples of finite complex reflection groups:** Each finite real reflection group is also a finite complex reflection group (just complexify the vector space) and extend the form as a hermitian form. The finite complex reflection groups were classified in the 1950's by Shephard and Todd. There is one infinite family $G(de, e, r)$ (which includes all the real infinite families: for example $G(1, 1, n+1) \simeq W(A_n)$, $G(2, 2, n) = W(D_n)$, etc) and there are 34 exceptional ones. Some of these arise as reflection groups of "complex lattices". We give some examples below:

1.9**. Eisenstein root lattices:** Let $\omega = e^{2\pi i/3}$ and $\mathcal{E} = \mathbb{Z}[\omega]$ be the ring of Eisenstein integers. Define $L_n = L_{\mathcal{E}}(A_n)$ to be the free $\mathcal{E}$-module with basis $\{s_j : j = 1, \cdots, n\}$ and a hermitian form defined by $s_j^2 = 3$, $\langle s_j, s_{j+1} \rangle = \sqrt{-3}$ for all j and $\langle s_j, s_k \rangle = 0$ if $k > j+1$. Then

- L_n has minimal norm 3.
- Reflection in each norm 3 vector of L takes L to itself. These are the "roots" of L and the order 3 complex reflection in these roots generate the complex reflection group $R(L)$ of L.
- L_n is positive definite if and only if $n \leq 4$, so $R(L_n)$ is a finite complex reflection group for $n \leq 4$. The underlying $\mathbb{Z}$-lattices are $L(A_2), L(D_4), L(E_6), L(E_8)$.

Some other examples come from reflection groups of other interesting lattices, e.g. the complex reflection group of the Gaussian D_4 lattice (rank 4 over $\mathbb{Z}$) or the Eisenstein Coxeter-Todd lattice (rank 12 over $\mathbb{Z}$).

1.10. **Theorem.** *Suppose L is indecomposible, positive definite, integral $\mathbb{Z}$-lattice, $\langle v, w \rangle \in \sqrt{-3}\mathcal{E}$ for all $v, w \in L$, and L is generated by norm 3 vectors. Then $L \simeq L(A_n)$ for $1 \leq n \leq 4$. Further $R(L(A_n)) \simeq \mathrm{Cox}(A_n, 3)$ for $1 \leq n \leq 4$.*

The class of complex reflection groups form an interesting and natural collection of linear groups from the point of view of invariant theory. Let V be a n dimensional complex vector space. Let G be a finite subgroup of $GL(V)$. Let $S = \mathrm{Sym}(V^*) \simeq \mathbb{C}[t_1, \cdots, t_n]$ and $R = S^G$ (the ring of G-invariant polynomial functions).

1.11. **Theorem** (Chevalley-Shephard-Todd)**.** *$R \simeq \mathbb{C}[s_1, \cdots, s_n]$ if and only if G is a finite complex reflection group.*

1.12**. Exercise:** Work out The A_2 example.

1.13**. Diagrams for complex reflection groups:** For each finite complex reflection group $W \subseteq O(V)$, Coxeter Wrote down Dynkin Type diagrams Δ. For example A_4 is the diagram for $R(L_4)$ with 3 written at each vertex. The vertices of Δ form a minimal set of complex reflection generators fo W and the edges indicate relations. Lot of nontrivial properties of W can be read off from Δ: for example "the invariant degrees of W" and the homotopy type of $(V - \{\text{mirrors of } W)/W$ (called the braid space of W). However, the definition of Δ is case by case and ad-hoc.

1.14**. An attempt to characterize the complex coxeter diagrams:** Let $W \subseteq O(V) \simeq O(n)$ be the complex reflection group and Φ be the set of roots of W. Define $\alpha : P(V) \to V$ by $\alpha(w) = \sum_{r \in \Phi} \frac{\langle r, w \rangle}{|\langle r, w \rangle|} r$. Notice that if W was a real reflection group, then the Weyl vector ρ is a fixed point of α (up to scalars). This suggests the following algorithm: *Start with a random $w_0 \in V$ and generate w_m's by $w_m = \alpha(w_{m-1})$. If the sequence converges, let $w = \lim w_m$. Let $(M_1, \cdots, M_n)$ be the mirrors closest to w. Make a diagram whose vertices correspond to $M_1, \cdots, M_n$ and edges indicate the relations between the reflections in these mirrors.*

Suppose the roots of W can be chosen to span a lattice defined over $\mathbb{Z}$ or ring of integers of a imaginary quadratic extension of $\mathbb{Q}$. Then experimentally the above algorithm converges and seems to give the "right diagram". In five exceptional cases, including some where the known diagrams are known to behave badly, this algorithm produces new diagrams.

Define $S(w) = \sum_{r \in \Phi} |\langle r, w \rangle| / |w|$. Then one can show that S does not have a local maxima on the mirrors of W and further $\partial_w S = 0$ if and only if w is a fixed point of α. Since S attains its maxima on $\mathbb{P}(V)$, it follows that α has a fixed point.

Some reference: Humphreys: Reflection groups and Coxeter groups. Broue: Introduction to complex reflection groups and their braid groups. Bourbaki: Lie groups and Lie algebras, chapter 4 - 6.

2. The A_2 singluarity

2.1. Commutative algebra preliminaries. Let G be a group acting on a commutative ring R. Say that ϕ of R is G-invariant if $g\phi = \phi$ for all $g \in G$. Let R^G denote the subgring of all G-invariant elements in R.

2.1. **Lemma.** *Let I be an ideal in R that is (setwise) fixed by G. Then the G action on R induces a G action on R/I such that the projection $(R \to R/I)$ is G-equivariant and we have a natural injection $j : R^G/(R^G \cap I) \hookrightarrow (R/I)^G$ given by $j(\phi + R^G \cap I) = (\phi + I)$ for all $\phi \in R^G$. If G is finite and $|G|$ is invertible in R, then j is an isomorphism.*

Proof. Let $j : R^G \to R/I$ be the composition $(R^G \hookrightarrow R \to R/I)$ Then $\ker(j) = R^G \cap I$. So j induces an injection $R^G/R^G \cap I \to R/I$. On the other hand if $\phi \in R$ is G-invariant then $j(\phi) = \phi + I$ is G-invariant in R/I, that is, $j(R^G) \subseteq (R/I)^G$. It follows that j induces an injection $j : R^G/R^G \cap I \to (R/I)^G$ given by $j(\phi + R^G \cap I) = \phi + I$.

Now suppose G is finite and $|G|$ is invertible in R. Define the function $\mathrm{av} : R \to R^G$ by $\mathrm{av}(\phi) = |G|^{-1}\sum_{g\in G} g\phi$ (Caution: that π need not be a ring homomorphism but only an additive group homomorphism). Given an element $(\phi + I) \in (R/I)^G$, choose a representative for it in R, call it ϕ. Then $g\phi + I = g(\phi + I) = \phi + I$ for all $g \in G$ and hence

$$\mathrm{av}(\phi) + I = \phi + I \ \text{ for all } \ (\phi + I) \in (R/I)^G. \tag{1}$$

The composition $(R^G \hookrightarrow R \xrightarrow{\pi} R^G)$ is identity. Note that $\mathrm{av}(I) = I \cap R^G$, so I is in the kernel of the composition $(R \xrightarrow{\mathrm{av}} R^G \to R^G/R^G \cap I)$. Therefore, we obtain an abelian group homomorphism $R/I \to R^G/I \cap R^G$, and hence an abelian group homomorphism $\kappa : (R/I)^G \to R^G/(I \cap R^G)$ given by $\kappa(\phi + I) = \mathrm{av}(\phi) + I \cap R^G$. Using (1), we obtain $j(\kappa(\phi + I)) = j(\mathrm{av}(\phi) + I \cap R^G) = \mathrm{av}(\phi) + I = \phi + I$. So j is onto. □

One verifies that $\kappa \circ j$ is identity, so κ is the inverse of j, in particular κ preserves multiplication. One can also check this directly using (1).

2.2. **Lemma.** *Assume G is finite and $|G|$ is invertible in R. Let $\phi \in R^G$. Then the $\phi R \cap R^G = \phi R^G$.*

Proof. An element $\alpha \in \phi R \cap R^G$ has the form $\alpha = \phi\psi$ for some ψ in R. Acting by $g \in G$, we get $\alpha = \phi g(\psi)$ since α and ψ are G-invariant. It follows that $\alpha = \phi\,\mathrm{av}(\psi) \in \phi R^G$. □

2.2. Schmes and functors of points. Let R be a Noetherian commutative ring.

2.3. **Affine R-schemes of finite type:** Let $\mathsf{Aff.Sch}_R$ denotes the category of affine R-schemes. The cateogy of R-algebras are equivalent to the category $\mathsf{Aff.Sch}_R^{op}$ and this sets up an equivalence between between the full subcategory of finitely generated R-algebras and the full subcategory of R-schemes of finite type [1]. Note that a finitely generate R-algebra has the form $R[\mathbf{t}]/I$ where $R[\mathbf{t}]$ denotes the polynomial ring in n variables $\mathbf{t} = (t_1, \cdots, t_n)$ and I is an ideal in $R[\mathbf{t}]$.

An R-algebra A corresponds to the scheme $\mathrm{spec}(A)$; this is a topological space with a sheaf of rings on it called the structure sheaf of $\mathrm{spec}(A)$. As a topological space $\mathrm{spec}(A)$ consists of the set of prime ideals of A with the Zariski topology whose closed sets are the sets of the form $V(I) = \{P \in \mathrm{spec}(A) \colon I \subseteq P\}$ for some ideal $I \subseteq A$. Saying that $\mathrm{spec}(A)$ is an R-scheme means that $\mathrm{spec}(A)$ comes with a distinguished map to $\mathrm{spec}(R)$; this corresponds to the ring homomorphism $R \to A$ that makes A into an R-algebra. Given a finite type affine R-scheme X, let $\mathcal{O}_X$ denote the structure sheaf and $\mathcal{O}_X(X) = \mathcal{O}(X)$ denote the ring of functions on X. Then one has $X \simeq \mathrm{spec}(\mathcal{O}(X))$. For a finitely generated R-algebra A, one has $\mathcal{O}(\mathrm{spec}(A)) = A$. Thus the functors $\mathrm{spec}(\ )$ and $\mathcal{O}(\ )$ sets up the equivalence between the categories of finite type affine R-schemes and finitely generated R-algebras.

The Zariski topology on $\mathrm{spec}(A)$ is almost never Hausdorff, since the closure of $P \in \mathrm{spec}(A)$ is $V(Q) = \{Q \in \mathrm{spec}(A) \colon P \subseteq Q\}$. The set of closed points of $\mathrm{spec}(A)$ is thus the set of maximal ideals of A. This set will be denoted by $\max(A)$. Let $\mathbb{A}^n_R = \mathrm{spec}(R[\mathbf{t}])$ denote the affine n-space.

2.4. **The functor of points:** Let X be an affine R-scheme and let A be an R-algebra. Define

$$X(A) = \mathrm{Hom}_{\mathsf{Alg}_R}(\mathcal{O}(X), A) = \mathrm{Hom}(\mathrm{spec}(A), X).$$

[1]Reason for assuming R is Noetherian: if we do not make this assumption, the finitely generated R-algebras may not form a full subcategory and we do not want to deal with this issue.

One says that $X(A)$ is the set of A-valued points of X (or A-points of X).

In general, an R-scheme X gives the representable functor $X(\cdot) : \mathsf{Sch}_R^{op} \to \mathsf{set}$, defined by $X(\cdot) = \mathrm{Hom}(\cdot, X)$. This way, we obtain an embedding of Sch_R into the presheaf category $\mathrm{Fun}(\mathsf{Sch}_R^{op}, \mathsf{set})$. Identify $\mathsf{Sch}_R \subseteq \mathrm{Fun}(\mathsf{Sch}_R^{op}, \mathsf{set})$ via this Yoneda embedding. This gives a natural way to enlarge the category of schemes (to stacks for example); by considering suitable subcategories S such that $\mathsf{Sch}_R^{op} \subseteq S \subseteq \mathrm{Fun}(\mathsf{Sch}_R^{op}, \mathsf{set})$. This consideration is at the heart of Grothendieck's philosophy of thinking of objects in a category in terms of the functors they represent.

When we are working over $\mathbb{C}$ the formalism of affine schemes is related to with classical complex geometry of the 19th century through the functor of points using the nullstellensatz. We want to explain this and in particular spell it out for the affine space. For the rest of this subsection, let k be an algebraically closed field of characteristic zero.

2.5. **Lemma.** *Let X be a finite type affine k-scheme. Then there is a natural bijection $X(k) \simeq \max(\mathcal{O}(X))$. In other words, the k-points of X can be identified with the set of (Zariski) closed points of X.*

Proof. This lemma can be called one form of Hilbert's nullstellensatz. By nullstellensatz, we have a bijection $k^n \simeq \max(k[\mathbf{t}])$ given by $(a_1, \cdots, a_n) \mapsto (t_1 - a_1, \cdots, t_n - a_n)$.

Let X be a finite type affine k-scheme. Write $A = \mathcal{O}(X)$. We can identify A with $k[t_1, \cdots, t_n]/I$ for some ideal I. Let $\pi : k[\mathbf{t}] \to A$ be the natural projection; this corresponds to an inclusion $X \subseteq \mathbb{A}_k^n$. If $\mathfrak{m}$ is a maximal ideal in A, then $\pi^{-1}(\mathfrak{m})$ is a maximal ideal in $k[\mathbf{t}]$ and this sets up a bijection between the closed points of X and the closed points of $\mathbb{A}_k^n$ contained in $V(I)$. In particular, by nullstellensatz, we have $A/\mathfrak{m} \simeq k[\mathbf{t}]/\pi^{-1}(\mathfrak{m}) \simeq k$. Thus, given maximal ideal $\mathfrak{m} \in \max(A)$, we get an element of $\phi \in X(k)$ defined as the natural projection $\phi : A \to A/\mathfrak{m} \simeq k$. Conversely $\phi \in X(k)$ determines the maximal ideal $\ker(\phi)$. These set up mutually inverse bijections between $\max(A)$ and $X(k)$. $\square$

2.6**. Example:** The nullstellensatz and the above lemma gives us the following one to one correspondences:

$$k^n \simeq \mathbb{A}^n(k).$$

An element $\mathbf{a} \in k^n$ corresponds to the evaluation homomorphism $ev_{\mathbf{a}} \in \mathbb{A}^n(k)$, defined by $ev_{\mathbf{a}}(f) = f(\mathbf{a})$; note that $\ker(ev_{\mathbf{a}}) = (t_1 - a_1, \cdots, t_n - a_n)$. Conversely $\phi \in \mathbb{A}^n(k)$ corresponds to $\mathbf{a} = (\phi(t_1), \cdots, \phi(t_n)) \in k^n$. Since $\phi : k[\mathbf{t}] \to k$ is a ring homomorphism, one has $\phi(f(t_1, \cdots, t_n)) = f(\phi(t_1), \cdots, \phi(t_n)) = f(\mathbf{a})$; that is $\phi = ev_{\mathbf{a}}$.

Next we describe how morphisms between affine spaces $\mathbb{A}_k^n$ and $\mathbb{A}_k^m$ correspond to polynomal maps between k-vector spaces $\mathbb{A}^n(k)$ and $\mathbb{A}^m(k)$. Let $F_1, \cdots, F_m \in k[\mathbf{t}]$. A function $k^n \to k^m$ of the form $\mathbf{t} \mapsto (F_1(\mathbf{t}), \cdots, F_m(\mathbf{t}))$ is called a polynomial map.

2.7. **Lemma.** *There is a natural bijection between $\mathrm{Hom}_{\mathsf{Sch}_k}(\mathbb{A}_k^n, \mathbb{A}_k^m)$ and the set of polynomial maps from k^n to k^m.*

Proof. Giving a morphism $\pi : \mathbb{A}_k^n \to \mathbb{A}_k^m$ is equivalent to giving a ring homomorphism $\pi^* : k[\mathbf{s}] \to k[\mathbf{t}]$ where $\mathbf{s} = (s_1, \cdots, s_m)$ and we have identified $\mathcal{O}(\mathbb{A}_k^m)$ with $k[\mathbf{s}]$. Such a morphism π^* determines and is determined by m polynomials $e_1(\mathbf{t}), \cdots, e_m(\mathbf{t}) \in k[\mathbf{t}]$ where $e_j(\mathbf{t}) = \pi^*(s_j)$. Thus π determines a polynomial map $\mathbf{t} \mapsto (e_1(\mathbf{t}), \cdots, e_n(\mathbf{t}))$ and conversely this polynomial map determines π^* via the formula $\pi^*(f(s_1, \cdots, s_m)) = f(e_1(\mathbf{t}), \cdots, e_m(\mathbf{t}))$.

Let us see more explicitly a morphism of k-schemes $\pi : \mathbb{A}_k^n \to \mathbb{A}_k^m$ determines the map of vectors spaces $k^n \to k^m$. Pick $\mathbf{a} \in k^n$. This corresponds to the element $ev_{\mathbf{a}} \in \mathbb{A}^n(k)$. The morphism $\pi(k) : \mathbb{A}^n(k) \to \mathbb{A}^m(k)$ takes $ev_{\mathbf{a}}$ to $ev_{\mathbf{a}} \circ \pi^* \in \mathbb{A}^m(k)$. For $f \in k[\mathbf{s}]$, we have

$$ev_{\mathbf{a}} \circ \pi^*(f) = ev_{\mathbf{a}} f(e_1(\mathbf{t}), \cdots, e_m(\mathbf{t})) = f(e_1(\mathbf{a}), \cdots, e_m(\mathbf{a})).$$

In other words $\pi(k)(ev_{\mathbf{a}}) = ev_{(e_1(\mathbf{a}), \cdots, e_m(\mathbf{a}))} \in \mathbb{A}^m(k)$ and this corresponds to the point $(e_1(\mathbf{a}), \cdots, e_m(\mathbf{a})) \in k^m$. Thus π corresponds to the polynomial map $\mathbf{a} \mapsto (e_1(\mathbf{a}), \cdots, e_m(\mathbf{a}))$. $\square$

2.3. **The permutation group of 3 letters.**

2.8**. The invariants:** Let $\mathcal{O} = \mathbb{C}[t_1, t_2, t_3]/\langle t_1 + t_2 + t_3 \rangle \simeq \mathbb{C}[t_2, t_3]$ and $\mathfrak{h} = \operatorname{spec}(\mathcal{O}) \simeq \mathbb{A}^2_{\mathbb{C}}$. Consider the action of $G = S_3$ on the two dimensional complex vector space $\mathfrak{h}(\mathbb{C}) = \{(t_1, t_2, t_3) \in \mathbb{C}^3 \colon \sum_i t_i = 0\}$ by co-ordinate permutations, here $\mathfrak{h}(\mathbb{C})$ denotes the set of $\mathbb{C}$-points of $\mathfrak{h}$. We want to compute $\mathfrak{h}(\mathbb{C})/G := \operatorname{spec}(\mathcal{O}^G)(\mathbb{C})$. Let $R = \mathbb{C}[t_1, t_2, t_3]$. Let $e_1(t) = (t_1 + t_2 + t_3)$, $e_2(t) = (t_1t_2 + t_1t_3 + t_2t_3)$ and $e_3(t) = t_1t_2t_3$ be the elementary symmetric functions. By the fundamental theorem of symmetric functions, we know that $R^G = \mathbb{C}[e_1(t), e_2(t), e_3(t)] \simeq \mathbb{C}[s_1, s_2, s_3]$, a 3 variable polynomial ring in indeterminates s_1, s_2, s_3, with the isomorphism given by $s_j \mapsto e_j(t)$. Since the G-action on S is induced from the G-action on R, lemma 2.1 implies

$$\mathcal{O}^G = (R/\langle e_1 \rangle)^G = R^G/\langle e_1(t) \rangle \cap R^G \simeq \mathbb{C}[s_1, s_2, s_3]/\langle s_1 \rangle \simeq \mathbb{C}[s_2, s_3].$$

So $\mathfrak{h}/G := \operatorname{spec}(\mathcal{O}^G) \simeq \mathbb{A}^2_{\mathbb{C}}$ with co-ordinates s_2, s_3. The natural inclusion $\mathcal{O}^G \hookrightarrow \mathcal{O}$ is given by $(s_2, s_3) \mapsto (\sigma_2(t), \sigma_3(t))$ and this corresponds to the quotient map $\pi : \mathfrak{h}(\mathbb{C}) \to (\mathfrak{h}/G)(\mathbb{C})$. In co-ordinates, this map is given by $\pi(t) = (\sigma_2(t), \sigma_3(t))$ for $t \in \mathfrak{h}(\mathbb{C})$.

2.9**. The discriminant polynomial:** Let t_1, t_2, t_3 be such that $\sum_i t_i = 0$. Let $\Delta = \prod_{i<j}(t_i - t_j)^2$ be the discriminant of the cubic polynomial $F(x) = \prod_i (x - t_i) = x^3 + e_2(t)x - e_3(t)$. Note that Δ is a symmetric polynomial, so it can be written as a polynomial in e_2 and e_3. The following is one way to find this polynomial by direct computation: Differentiating the identity $\prod_i(x - t_i) = x^3 + e_2x - e_3$ with respect to x, we get $\sum_{i<j}(x - t_i)(x - t_j) = 3x^2 + e_2$. Substituting $x = t_i$ and multiplying the three equations together we get

$$-\Delta = \prod_i (3t_i^2 + e_2) = 27 \prod_i t_i^2 + 9e_2 \sum_{i<j} t_i^2 t_j^2 + 3e_2^2 \sum_i t_i^2 + \sigma_2^3 = 27e_3^2 + 4e_2^3,$$

where the last equality follows by noting $\sum_{i<j} t_i^2 t_j^2 = e_2^2$ and $\sum t_i^2 = -2e_2$ and these follow easily from $\sum_i t_i = 0$.

2.10**. The discriminant subvariety** Write $\mathfrak{d}(s_2, s_3) = 4s_2^3 + 27s_3^2$. By the above calculation one has $-\Delta(t_1, t_2, t_3) = \mathfrak{d}(e_2(t), e_3(t))$. So the inclusion $C[s_2, s_3] \hookrightarrow \mathcal{O}$ takes $-\mathfrak{d}$ to Δ and hence kernel of the composition $(\mathbb{C}[s_2, s_3] \hookrightarrow \mathcal{O} \to \mathcal{O}/\Delta)$ is generated by $\mathfrak{d}$ (by lemma 2.2). So we have a commutative rectangle note that

$$\begin{tikzcd} \mathbb{C}[s_2, s_3] \arrow[r, hook] \arrow[d, two heads] & \mathcal{O} \arrow[d, two heads] \\ \mathbb{C}[s_2, s_3]/\mathfrak{d} \arrow[r, hook] & \mathcal{O}/\Delta \end{tikzcd}$$
corresponding to the commutative square
$$\begin{tikzcd} \mathfrak{h}/G & \mathfrak{h} \arrow[l, two heads] \\ \{\mathfrak{d} = 0\} \arrow[u, hook] & \{\Delta = 0\} \arrow[l, two heads] \arrow[u, hook] \end{tikzcd}$$

So $\{\mathfrak{d} = 0\}$ is the image of $\{\Delta = 0\}$ under the quotient map. Here we use the informal notation : $\{\mathfrak{d} = 0\} = \operatorname{spec}(\mathbb{C}[s_2, s_3]/\mathfrak{d})$.

By using the polynomial map on the complex points, we can directly check on the level of point sets that the set theoretic image of the the complex points of the subscheme $\{\Delta = 0\}$ under the quotient map is the set of complex points of the subscheme $\{\mathfrak{d} = 0\}$ as follows: Consider the subset $M = \{(x_1, x_2, x_3) \in \mathfrak{h} \colon x_1 = x_2\} = \{(u, u, -2u) \colon u \in \mathbb{C}\}$. Then

$$\pi(M) = \{(-3u^2, -2u^3) \colon u \in \mathbb{C}\} = \{(s_2, s_3) \in \mathbb{C}^2 \colon 4s_2^3 + 27s_3^2 = 0\}.$$

Similarly the image of the other mirrors $\{x_i = x_j\}$ turn out to be the same set.

The image of the real part of the Cartan: The complex vector space $\mathfrak{h}(\mathbb{C})$ has a distinguished real subspace $\mathfrak{h}(\mathbb{R}) = \{\mathfrak{t} \in \mathbb{R}^3 \colon t_1 + t_2 + t_3 = 0\}$. We want to compute the image of $\mathfrak{h}(\mathbb{R})$ under the quotient map $\pi : \mathfrak{h}(\mathbb{C}) \to (\mathfrak{h}/S_3)(\mathbb{C})$. Recall that we have identified $(\mathfrak{h}/S_3)(\mathbb{C})$ with $\mathbb{C}^2$ so that π becomes the polynomial map $\pi(\mathfrak{t}) = (e_2(t), e_3(t))$. Note that even though we started working over $\mathbb{C}$, the quotient $(\mathfrak{h}/S_3)$ turns out to be an affine space which is defined over $\mathbb{Z}$. So it makes sense to talk of its real points; which is just $\operatorname{spec}(\mathbb{R}[s_2, s_3])(\mathbb{R}$ sitting inside $\operatorname{spec}(\mathbb{C}[s_2, s_3])(\mathbb{C})$. We use the cumbersome notation $\operatorname{spec}(\mathbb{C}[s_2, s_3])(\mathbb{C})$ in place of $\mathbb{C}^2$ because, we want to remember that s_2, s_3 are the names for the co-ordinate functions on this $\mathbb{C}^2$.

2.11. **Lemma.** *One has $\pi(\mathfrak{h}(\mathbb{R})) = \mathfrak{c} = \{(s_2, s_3) \colon 27s_3^2 + 4s_2^3 \leq 0\}$. If C is a Weyl chamber in $\mathfrak{h}(\mathbb{R})$, then $\pi|_C : C \to \mathfrak{c}$ is an isomorphism.*

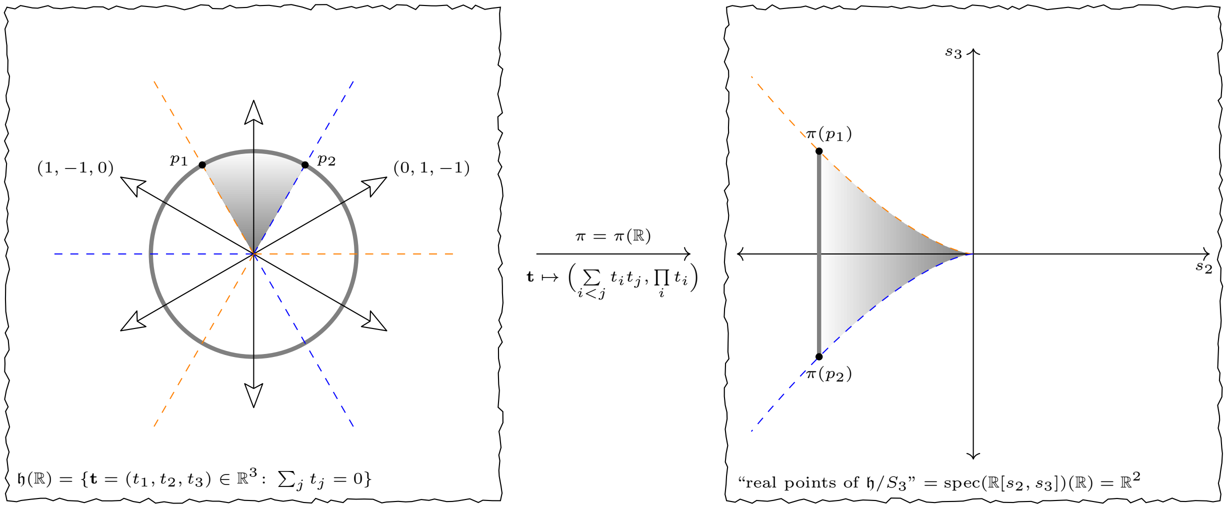

FIGURE 1. Visualizing the projection map π on the set of real points. Suppose a point p goes around the grey circle S on the left once starting at p_1. Then its image $\pi(p)$ traverses the grey vertial straight line segment $\pi(S)$ six times, going down from $\pi(p_1)$ to $\pi(p_2)$ thrice and going back up to $\pi(p_1)$ thrice. The arc of the circle that lies in a Weyl chamber maps bijectively to $\pi(S)$.

Proof. We have the identity $4e_2^3 = -27e_3^2 - \prod_{i<j}(t_i - t_j)^2$. The right hand side is negative if t_1, t_2, t_3 are real numbers. So we must have $e_2 \in (-\infty, 0]$. Fix e_2 at a negative number. Then $27e_3^2 \leq -4e_2^3$ (since $\prod_{i<j}(t_i - t_j)^2 \geq 0$) It follows that $\pi(\mathfrak{h}(\mathbb{R})) \subseteq \mathfrak{c}$.

Since $\pi(gx) = \pi(x)$, each Weyl chamber in $\mathfrak{h}(\mathbb{R})$ maps onto the same subset of $(\mathfrak{h}/S_3)(\mathbb{C})$. Fix a (closed) Weyl chamber C (the shaded chamber in the figure) which contains the vector $(1,0,-1)$. Fix $c > 0$ and consider the circle $S = \mathfrak{h}(\mathbb{R}) \cap \{\mathbf{t}\colon \sum_i t_i^2 = 6c^2\}$. If $\mathbf{t}$ is a point on this circle, then $e_2(\mathbf{t}) = ((\sum_i t_i)^2 - \sum_i t_i^2)/2 = -3c^2$. So $\pi(S)$ is a subset of the vertical line segment $\{(s_2, s_3)\colon s_2 = -3c^2\} \cap \mathfrak{c} = \{-3c^2\} \times [-2c^3, 2c^3]$. The circle S meets the boundary of the Weyl chamber ∂C at two points $p_1 = (2c, -c, -c)$ and $p_2 = (c, c, -2c)$. We compute $\pi(p_1) = (-3c^2, 2c^3)$ and $\pi(p_2) = (-3c^2, -2c^3)$. By intermediate value theorem $\pi(S \cap C)$ is a connected subset of the vertical line segment $\{-3c^2\} \times [-2c^3, 2c^3]$, that contains both the endpoints, so we must have $\pi(S) = \pi(S \cap C) = \{-3c^2\} \times [-2c^3, 2c^3]$ and hence $\pi(C) = \pi(\mathfrak{h}(\mathbb{R})) = \mathfrak{c}$. $\square$

2.12. **Lemma.** *(The braid space)* $(\mathbb{C}^n - \mathcal{M})/S_n$ *strongly deformation retracts onto the subspace* $A = (\{e_1(t) = 0\} - \mathcal{M})/S_n$ *Where* $\mathcal{M}$ *denotes the set of mirrors. So these spaces are homotopy equivalent.*

Proof. Define $F : \mathbb{C}^n \times [0,1] \to \mathbb{C}^n$ by $F(t_1, \cdots, t_n, \epsilon) = (t_1 - \epsilon e_1(t)/n, \cdots, t_n - \epsilon e_n(t)/n)$. Then

- $F(t, 0) = \mathrm{id}_{\mathbb{C}^n}$,
- $F(\cdot, \epsilon)|_{\{e_1(t)=0\}} = \mathrm{id}_{\{e_1(t)=0\}}$ for each fixed ϵ,
- $F(\cdot, 1) \subseteq \{e_1(t) = 0\}$,
- Each $F(\cdot, \epsilon)$ is S_3 equivariant and preserves $\mathcal{M}$.

So F induces the required strong deformation retract. $\square$