The product rule for derivations on finite dimensional split semi-simple Lie algebras over a field of characteristic zero

Richard L. Kramer

Department of Mathematics
Iowa State University
Ames, Iowa 50011, USA

Abstract

In this article we consider maps $\pi : R \to R$ on a non-associative ring R which satisfy the product rule $\pi(ab) = (\pi a)b + a\pi b$ for arbitrary $a, b \in R$, calling such a map a *production* on R. After some general preliminaries, we restrict ourselves to the case where R is the underlying Lie ring of a finite dimensional split semi-simple Lie algebra over a field $\mathbb{F}$ of characteristic zero. In this case we show that if π is a production on R, then π necessarily satisfies the sum rule $\pi(a + b) = \pi a + \pi b$, that is, we show that the product rule implies the sum rule, making π a derivation on the underlying Lie ring of R. We further show that there exist unique derivations on the field $\mathbb{F}$, one for each simple factor of R, such that appropriate product rules are satisfied for the Killing form of two elements of R, and for the scalar product of an element of $\mathbb{F}$ with an element of R.

Key words: derivation, product rule, Lie ring, Lie algebra, non-associative ring, non-associative algebra

1 Preliminaries

Definition 1 *Let R be a non-associative ring. A map $\pi : R \to R$ is called a* production *on R provided that $\pi(xy) = (\pi x)y + x\pi y$ for all $x, y \in R$. A production which also satisfies $\pi(x + y) = \pi x + \pi y$ for all $x, y \in R$ is called a* derivation *on R.*

Email address: `ricardo@iastate.edu` (Richard L. Kramer).

Preprint submitted to Journal of Algebra *2 December 2006*

Proposition 2 *Let R be a non-associative ring and let $\pi : R \to R$ be a production on R. Then $\pi 0 = 0$.*

PROOF. We calculate $\pi(0) = \pi(0 \cdot 0) = (\pi 0) \cdot 0 + 0 \cdot \pi 0 = 0$. $\square$

Proposition 3 *Let R be a non-associative ring with identity and let $\pi : R \to R$ be a production on R. Then $\pi 1 = 0$ and $2\pi(-1) = 0$.*

PROOF. To see that $\pi 1 = 0$, we calculate $\pi 1 = \pi(1 \cdot 1) = (\pi 1)1 + 1\pi 1$. To see that $2\pi(-1) = 0$ whenever $x^2 = 1$, we calculate $0 = \pi 1 = \pi((-1) \cdot (-1)) = (\pi(-1)) \cdot (-1) + (-1) \cdot \pi(-1) = -2\pi(-1)$. $\square$

Example 4 *Let R be the ring $\mathbb{Z}/4$ of integers modulo 4. Then it is easy to see that the productions on R are precisely the maps $\pi : R \to R$ such that $\pi 0 = \pi 1 = 0$ and $\pi 2, \pi 3 \in \{0, 2\}$. Thus there are exactly four productions on the ring $R = \mathbb{Z}/4$.*

Remark 5 *Note that Example 4 shows that the second conclusion of Proposition 3 cannot in general be improved to read $\pi(-1) = 0$, even in the case of commutative rings with identity.*

Lemma 6 *Let S, A, B, T and C be abelian groups. Suppose that we are given a bilinear map (denoted by juxtaposition) $A \times B \to C$. Suppose that we have bilinear maps (also denoted by juxtaposition) $S \times A \to A$ and $S \times C \to C$ such that the following diagram with the obvious maps commute:*

$$\begin{array}{ccc} S \times A \times B & \longrightarrow & S \times C \\ \downarrow & & \downarrow \\ A \times B & \longrightarrow & C \end{array} \qquad (sa)b = s(ab)$$

Suppose further that we also have bilinear maps (also denoted by juxtaposition) $B \times T \to B$ and $C \times T \to C$ such that the following diagram with the obvious maps commute:

$$\begin{array}{ccc} A \times B \times T & \longrightarrow & C \times T \\ \downarrow & & \downarrow \\ A \times B & \longrightarrow & C \end{array} \qquad a(bt) = (ab)t$$

Suppose that we have maps $\pi_A : A \to A$, $\pi_B : B \to B$, and $\pi_C : C \to C$ which jointly satisfy the equation the following production equation *for any $a \in A$ and $b \in B$.*

$$\pi_C(ab) = \pi_A(a)b + a\pi_B(b) \tag{1}$$

Then, given any $s_1, \ldots, s_n \in S$, $a_1, \ldots, a_n \in A$, and any $b_1, \ldots, b_m \in B$ and $t_1, \ldots, t_m \in T$ the following equation holds for π_C, where we write $a = \sum_{i=1}^n s_i a_i$ and $b = \sum_{j=1}^m b_j t_j$.

$$\pi_C(ab) + \sum_{i=1}^n \sum_{j=1}^m s_i \pi_C(a_i b_j) t_j = \sum_{i=1}^n s_i \pi_C(a_i b) + \sum_{j=1}^m \pi_C(a b_j) t_j \qquad (2)$$

Remark 7 *Note that expressions such as sabt (with $s \in S$, $a \in A$, $b \in B$ and $t \in T$) are unambiguous, because of the associativity represented by the commutative diagrams above. However, sct (with $c \in C$) in general is ambiguous, since it is in general possible that $(sc)t \neq s(ct)$. Of course, if c can be written as $c = \sum a_i b_i$ with $a_i \in A$ and $b_i \in B$, then $(sc)t = (s(\sum a_i b_i))t = \sum(s(a_i b_i))t = \sum((s a_i) b_i)t = \sum(s a_i)(b_i t) = \sum s(a_i(b_i t)) = \sum s((a_i b_i)t) = s((\sum a_i b_i)t) = s(ct)$. In particular, $s\pi_C(ab)t = s(\pi_A(a)b + a\pi_B(b))t$ is unambiguous.*

Remark 8 *It should be emphasized that* (1) *is the* only *assumption made about the maps π_A, π_B and π_C. In particular no assumption is made that any of the maps are additive. Since the map $A \times B \to C$ is bilinear, it is easy to prove that $\pi_C(0_C) = 0_C$. We need only note that $\pi_C(0_C) = \pi_C(0_A 0_B) = \pi_A(0_A)0_B + 0_A \pi_B(0_B) = 0_C + 0_C = 0_C$. However, without knowledge of the linear map $A \times B \to C$, this is all that can be said, and nothing analogous need hold for π_A and π_B. In fact, if the bilinear map $A \times B \to C$ is the trivial map $(a, b) \mapsto 0$, then* (1) *reduces to simply saying that $\pi_c(0_C) = 0_C$, so that π_A and π_B are totally arbitrary, as is π_C, so long as it maps 0_C to 0_C.*

PROOF.

$$\begin{aligned}
\pi_C(ab) &= \pi_A(a)b + a\pi_B(b) \\
&= \pi_A(a) \sum_{j=1}^m b_j t_j + \left(\sum_{i=1}^n s_i a_i \right) \pi_B(b) \\
&= \sum_{j=1}^m \pi_A(a) b_j t_j + \sum_{i=1}^n s_i a_i \pi_B(b) \\
&= \sum_{j=1}^m \left(\pi_C(a b_j) - a\pi_B(b_j) \right) t_j + \sum_{i=1}^n s_i \left(\pi_C(a_i b) - \pi_A(a_i) b \right) \\
&= \sum_{j=1}^m \left(\pi_C(a b_j) - \left(\sum_{i=1}^n s_i a_i \right) \pi_B(b_j) \right) t_j \\
&\qquad + \sum_{i=1}^n s_i \left(\pi_C(a_i b) - \pi_A(a_i) \sum_{j=1}^m b_j t_j \right) \\
&= \sum_{j=1}^m \pi_C(a b_j) t_j + \sum_{i=1}^n s_i \pi_C(a_i b) - \sum_{i=1}^n \sum_{j=1}^m s_i \left(\pi_A(a_i) b_j + a_i \pi_B(y_j) \right) t_j
\end{aligned}$$

$$= \sum_{j=1}^{m} \pi_C(xy_j)t_j + \sum_{i=1}^{n} s_i\pi_C(a_ib) - \sum_{i=1}^{n}\sum_{j=1}^{m} s_i\pi_C(a_ib_j)t_j$$

□

Corollary 9 *Let R be a non-associative ring and let $\pi : R \to R$ be a production on R. Let $x_i \in R$ for $1 \leq i \leq n$ and $y_j \in R$ for $1 \leq j \leq m$. Then we have*

$$\pi(xy) + \sum_{i=1}^{n}\sum_{j=1}^{m} \pi(x_iy_j) = \sum_{i=1}^{n} \pi(x_iy) + \sum_{j=1}^{m} \pi(xy_i) \tag{3}$$

where $x = \sum_{i=1}^{n} x_i$ and $y = \sum_{j=1}^{m} y_j$.

PROOF. Apply Lemma 6 with $A = B = C = R$ and $S = T = \mathbb{Z}$, together with the obvious bilinear maps. Let $\pi_A = \pi_B = \pi_C = \pi$, $s_i = t_j = 1$, $a = x$ and $b = y$. All of the hypotheses are satisfied, and (2) reduces to (3). □

Corollary 10 *Let R be a non-associative ring and let $\pi : R \to R$ be a production on R. Let $x_i, y_i \in R$ for $1 \leq i \leq n$. Suppose further that $x_iy_j = 0$ whenever $i \neq j$, so that $xy = \sum_{i=1}^{n} x_iy_i$, where $x = \sum_{i=1}^{n} x_i$ and $y = \sum_{i=1}^{n} y_i$. Then we have*

$$\pi(xy) = \sum_{i=1}^{n} \pi(x_iy_i).$$

PROOF. First note that $\pi(x_iy_j) = \pi 0 = 0$ for $i \neq j$, by Proposition 2. Note also that xy_i=$x_iy = x_iy_i$. With this in mind, Corollary 9 now says that

$$\pi(xy) + \sum_{i=1}^{n} \pi(x_iy_i) = \sum_{i=1}^{n} \pi(x_iy_i) + \sum_{i=1}^{n} \pi(x_iy_i)$$

from which the corollary follows. □

Corollary 11 *Let R be a non-associative ring and let $\pi : R \to R$ be a production on R. Let $u, v \in R$ satisfy $u^2 = v^2 = 0$. Then we have*

$$\pi(uv + vu) = \pi(uv) + \pi(vu).$$

PROOF. Setting $x_1 = y_1 = u$ and $x_2 = y_2 = v$ in Corollary 9 with $n = m = 2$, we see that

$$\pi(uv + vu) + (\pi(uv) + \pi(vu)) = (\pi(uv) + \pi(vu)) + (\pi(vu) + \pi(uv))$$

from which the corollary follows. □

Corollary 12 *Let R be a non-associative ring and let $\pi : R \to R$ be a production on R. Let $u, v \in R$ satisfy $u^2 = v^2 = 0$ and $vu = -uv$. Then we have*

$$\pi(-uv) = -\pi(uv).$$

PROOF. By Proposition 2 and Corollary 11, we see that $\pi(uv) + \pi(-uv) = \pi(uv) + \pi(vu) = \pi(uv + vu) = \pi 0 = 0$. $\square$

Corollary 13 *Let R be a non-associative algebra over the commutative ring with identity Λ, and let $\pi : R \to R$ be a production on the underlying ring of R. Let $a, b \in R$ with $c = ab$. Then for any $\lambda, \mu \in \Lambda$ we have*

$$\pi(\lambda\mu c) + \lambda\mu\pi(c) = \mu\pi(\lambda c) + \lambda\pi(\mu c). \tag{4}$$

PROOF. Apply Lemma 6 with $A = B = C = R$ and $S = T = \Lambda$, together with the obvious bilinear maps. Let $\pi_A = \pi_B = \pi_C = \pi$ and $n = m = 1$ with $s_1 = \lambda$, $t_1 = \mu$. All of the hypotheses are satisfied, and (2) reduces to (4). $\square$

Corollary 14 *Let R be a non-associative algebra over the commutative ring with identity Λ, and let $\pi : R \to R$ be a production on the underlying ring of R. Let $a_0, a_1, b_0, b_1 \in R$ satisfy $a_0b_1 = a_1b_0 = 0$ and $a_0b_0 = a_1b_1 = c$. Then for any $\lambda, \mu \in \Lambda$ we have*

$$\pi(\lambda c + \mu c) = \pi(\lambda c) + \pi(\mu c) \tag{5}$$

and

$$\pi(\lambda\mu c) + \lambda\mu\pi(c) = \mu\pi(\lambda c) + \lambda\pi(\mu c). \tag{6}$$

PROOF. First note that if $a = \lambda_0 a_0 + \lambda_1 a_1$ and $b = \mu_0 b_0 + \mu_1 b_1$, then the hypotheses imply that $ab = (\lambda_0 a_0 + \lambda_1 a_1)(\mu_0 b_0 + \mu_1 b_1) = (\lambda_0\mu_0 + \lambda_1\mu_1)c$. Apply Lemma 6 with $A = B = C = R$ and $S = T = \Lambda$, together with the obvious bilinear maps. Let $\pi_A = \pi_B = \pi_C = \pi$ and $n = m = 2$ with $s_i = \lambda_i$ and $t_j = \mu_j$. All of the hypotheses are satisfied, and the (2) reduces to (7).

$$\begin{aligned}\pi\Big((\lambda_0\mu_0 + \lambda_1\mu_1)\,c\Big) + (\lambda_0\mu_0 + \lambda_1\mu_1)\,\pi\,(c) = \\ \mu_0\pi\,(\lambda_0 c) + \lambda_0\pi\,(\mu_0 c) + \mu_1\pi\,(\lambda_1 c) + \lambda_1\pi\,(\mu_1 c)\end{aligned} \tag{7}$$

Letting $\lambda_1 = \mu_1 = 0$, $\lambda_0 = \lambda$ and $\mu_0 = \mu$ in (7), we see immediately that (6) holds. Similarly, by letting $\lambda_1 = \mu_0 = 1$, $\lambda_0 = \lambda$ and $\mu_1 = \mu$ we see that

$$\pi\Big((\lambda + \mu)\,c\Big) + (\lambda + \mu)\,\pi(c) = \pi\,(\lambda c) + \lambda\pi(c) + \mu\pi(c) + \pi\,(\mu c)\,.$$

From this, (5) follows immediately. $\square$

Remark 15 *Note that if R is an anti-symmetric non-associative algebra over Λ, that is, if $x^2 = 0$ for every $x \in R$, then any $c = ab$ will satisfy the conclusions* (5) *and* (6) *of Corollary 14. To see this, we need only note that if we define $a_0 = b_1 = a$ and $b_0 = -a_1 = b$, then $a_0 b_1 = a^2 = 0$, $a_1 b_0 = -b^2 = 0$, $a_0 b_0 = ab = c$, and $a_1 b_1 = -ba = ab = c$. Note in particular that this applies to any Lie algebra.*

Let R be a non-associative ring which is direct sum of ideals $R = R_1 \oplus \cdots \oplus R_n$. If $\pi_i : R_i \to R_i$ is a production on R_i for each $i = 1, \ldots, n$, then the map $\pi : R \to R$ defined by $\pi(x_1 + \cdots + x_n) = \pi_1(x_1) + \cdots + \pi_n(x_n)$ for $x_i \in R_i$ is easily seen to be a production on R. Theorem 16 provides a partial converse to this. We say that a non-associative ring is *annihilator free* if for any $a \in R_i$ which satisfies $ax = xa = 0$ for all $x \in R_i$, we have $a = 0$. In case R is a direct sum $R = R_1 \oplus \cdots \oplus R_n$, note that R is annihilator free if and only if each R_i is annihilator free.

Theorem 16 *Let R be a non-associative ring which is direct sum of ideals $R = R_1 \oplus \cdots \oplus R_n$, and let $\pi : R \to R$ be a production on R. Suppose further that R is annihilator free. (Equivalently, that each R_i is annihilator free.) Then there exist unique productions $\pi_i : R_i \to R_i$ such that $\pi(x_1 + \cdots + x_n) = \pi_1(x_1) + \cdots + \pi_n(x_n)$ for $x_i \in R_i$.*

PROOF. The uniqueness is trivial, since $\pi_i(0) = 0$ for every i, so that $\pi_i(x) = \pi(x)$ for any $x \in R_i$.

For existence, we first need to show that $\pi(x) \in R_i$ whenever $x \in R_i$. Let $x \in R_i$. Write $\pi(x) = a_0 + \cdots + a_n$ with $a_j \in R_j$. Given any $j \neq i$ and any $y \in R_j$, we have $a_j y = \pi(x)y \in R_j$ and also $a_j y = \pi(x)y = \pi(xy) - x\pi(y) = \pi(0) - x\pi(y) = -x\pi(y) \in R_i$. Thus, $a_j y = 0$ for any $y \in R_j$. Similarly, $y a_j = 0$ for any $y \in R_j$. Since R_j is annihilator free, we must have $a_j = 0$. Since $a_j = 0$ for every $j \neq i$, we have $\pi(x) = a_i \in R_i$, as desired.

Now, we may define $\pi_i : R_i \to R_i$, for any $i = 1, \ldots, n$, by $\pi_i(x) = \pi(x)$ for any $x \in R_i$. Clearly, each π_i is a production on R_i. It remains only to show that $\pi(x_1 + \cdots + x_n) = \pi_1(x_1) + \cdots + \pi_n(x_n)$ whenever $x_i \in R_i$ for each i. Let $x = x_1 + \cdots + x_n$ and $y = y_1 + \cdots + y_n$ be arbitrary, with $x_i, y_i \in R_i$. Making use of Corollary 10, we may calculate as follows:

$$\begin{aligned}
(\pi(x) - \pi_1(x_1) - \cdots - \pi_n(x_n))y &= \pi(x)y - \pi(x_1)y - \ldots - \pi(x_n)y \\
&= (\pi(xy) - x\pi(y)) - (\pi(x_1 y) - x_1\pi(y)) - \ldots - (\pi(x_n y) - x_n\pi(y)) \\
&= \pi(xy) - \pi(x_1 y) - \cdots - \pi(x_n y) - (x - x_1 - \cdots - x_n)\,\pi(y) \\
&= \pi(xy) - \pi(x_1 y_1) - \cdots - \pi(x_n y_n) \\
&= 0
\end{aligned}$$

Thus, $(\pi(x) - \pi_1(x_1) - \cdots - \pi_n(x_n))y = 0$ for any $y \in R$. Similarly, $y(\pi(x) - \pi_1(x_1) - \cdots - \pi_n(x_n)) = 0$ for any $y \in R$. Since R is annihilator free, this gives $\pi(x_1 + \cdots + x_n) = \pi_1(x_1) + \cdots + \pi_n(x_n)$, as desired. The proof is complete. □

Proposition 17 *Let R be a non-associative ring with direct sum decomposition $R = \bigoplus_{\alpha \in I} R_\alpha$ as an abelian group under addition, and let $\pi : R \to R$ be a production on R. Suppose that π is additive on each of the summands R_α, in the sense that for any $\alpha \in I$ and for any $x, y \in R_\alpha$, we have $\pi(x+y) = \pi x + \pi y$. Suppose further that for each $\alpha, \beta \in I$ there exists some $\gamma \in I$ such that $xy \in R_\gamma$ for any $x \in R_\alpha$ and $y \in R_\beta$. Define $\delta : R \to R$ by $\delta\left(\sum_{\alpha \in I} x_\alpha\right) = \sum_{\alpha \in I} \pi x_\alpha$, where $x_\alpha \in R_\alpha$ for each $\alpha \in I$ and $x_\alpha = 0$ for all but finitely many $\alpha \in I$. Then δ is a derivation on R.*

PROOF. We define $m : I \times I \to I$ so that $xy \in R_{m(\alpha,\beta)}$ whenever $x \in R_\alpha$ and $y \in R_\beta$. For any $x \in R$, we write $x = \sum_{\alpha \in I} x_\alpha$, where $x_\alpha \in R_\alpha$ for each $\alpha \in I$ and $x_\alpha = 0$ for all but finitely many α. Then we see that δ is a production as follows.

$$\begin{aligned}
\delta(xy) &= \delta\Big(\Big(\sum_\alpha x_\alpha\Big)\Big(\sum_\beta y_\beta\Big)\Big) \\
&= \delta\Big(\sum_\gamma\Big(\sum_{m(\alpha,\beta)=\gamma} x_\alpha y_\beta\Big)\Big) \\
&= \sum_\gamma \pi\Big(\sum_{m(\alpha,\beta)=\gamma} x_\alpha y_\beta\Big) \\
&= \sum_\gamma\Big(\sum_{m(\alpha,\beta)=\gamma} \pi(x_\alpha y_\beta)\Big) \\
&= \sum_\alpha \sum_\beta \pi(x_\alpha y_\beta) \\
&= \sum_\alpha \sum_\beta ((\pi x_\alpha) y_\beta + x_\alpha \pi y_\beta) \\
&= \Big(\sum_\alpha \pi x_\alpha\Big) y + x \sum_\beta \pi y_\beta \\
&= (\delta x) y + x \delta y
\end{aligned}$$

To see that δ is a derivation on R, it remains only to show additivity of δ on R, which we see as follows.

$$\begin{aligned}
\delta(x+y) &= \delta\Big(\sum_\alpha x_\alpha + \sum_\alpha y_\alpha\Big) \\
&= \delta\Big(\sum_\alpha (x_\alpha + y_\alpha)\Big) \\
&= \sum_\alpha \pi(x_\alpha + y_\alpha)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_\alpha (\pi x_\alpha + \pi y_\alpha) \\
&= \sum_\alpha \pi x_\alpha + \sum_\alpha \pi y_\alpha \\
&= \delta\Big(\sum_\alpha x_\alpha\Big) + \delta\Big(\sum_\alpha y_\alpha\Big) \\
&= \delta x + \delta y
\end{aligned}
$$

□

Theorem 18 *Let L be a Lie ring, and $\pi : L \to L$ a production on L. Then we have*

$$\pi[[x, y], z] + \pi[[y, z], x] + \pi[[z, x], y] = 0$$

for every $x, y, z \in L$.

PROOF.

$$
\begin{aligned}
\pi[[x, y], z] + \pi[[y, z], x] + \pi[[z, x], y] &= [\pi[x, y], z] + [[x, y], \pi z] \\
&\quad + [\pi[y, z], x] + [[y, z], \pi x] \\
&\quad + [\pi[z, x], y] + [[z, x]\pi y] \\
&= [[\pi x, y], z] + [[x, \pi y], z] + [[x, y], \pi z] \\
&\quad + [[\pi y, z], x] + [[y, \pi z], x] + [[y, z], \pi x] \\
&\quad + [[\pi z, x], y] + [[z, \pi x], y] + [[z, x], \pi y] \\
&= [[\pi x, y], z] + [[z, \pi x], y] + [[y, z], \pi x] \\
&\quad + [[x, \pi y], z] + [[z, x], \pi y] + [[\pi y, z], x] \\
&\quad + [[x, y], \pi z] + [[\pi z, x], y] + [[y, \pi z], x] \\
&= 0 + 0 + 0
\end{aligned}
$$

□

2 Finite dimensional split semi-simple Lie algebras over a field of characteristic zero

In this section, we assume that L is a finite dimensional split semi-simple Lie algebra over a field $\mathbb{F}$ of characteristic zero, with $\langle -, - \rangle$ as its Killing form. (Recall that any Lie algebra over an algebraically closed field of characteristic zero is split.) Let

$$L = H \oplus \bigoplus_{\alpha \in \Delta} L_\alpha \tag{8}$$

be a fixed Cartan decomposition for L, where $\Delta = \Delta_+ \cup \Delta_-$ is the set of (non-zero) roots of L, and Δ_+ is the set of positive roots under a given ordering, with Δ_- the corresponding set of negative roots. We write h_α for the coroot of α. Let Δ_+^0 be the set of simple positive roots, that is, the set of all $\alpha \in \Delta_+$ such that there do not exist $\beta, \gamma \in \Delta_+$ with $\alpha = \beta + \gamma$. Then $\{ h_\alpha \mid \alpha \in \Delta_+ \}$ is a basis for H. Note that $[h, x_\alpha] = \alpha(h) x_\alpha$ for any $h \in H$ and $x_\alpha \in L_\alpha$, and that $[x_\alpha, x_{-\alpha}] = \langle x_\alpha, x_{-\alpha} \rangle h_\alpha$ for any $x_\alpha \in L_\alpha$ and $x_{-\alpha} \in L_{-\alpha}$. Recall that $\langle h_\alpha, h_\beta \rangle \in \mathbb{Q}$ is rational for all $\alpha, \beta \in \Delta$.

Throughout this section we will assume that the map $\pi : L \to L$ is a production on the underlying Lie ring of L.

Lemma 19 *Let L be a finite dimensional split semi-simple Lie algebra over a field $\mathbb{F}$ of characteristic zero, with Cartan decomposition* (8)*. Suppose that $\pi h = 0$ for every $h \in H$ and that for every $\alpha \in \Delta$ and every $x_\alpha \in L_\alpha$, we have $\pi x_\alpha = 0$. Then π is the trivial production $\pi x = 0$ for all $x \in L$.*

PROOF. First, we will show that

$$\pi(x_\alpha + x_{-\alpha}) = 0 \tag{9}$$

whenever $x_\alpha \in L_\alpha$ and $x_{-\alpha} \in L_{-\alpha}$. If one or both of x_α and $x_{-\alpha}$ are 0, then we have nothing to prove, so assume that both are non-zero. Then $\langle x_\alpha, x_{-\alpha} \rangle \neq 0$. Thus, we may define $y_\alpha \in L_\alpha$ by $x_\alpha = \langle x_\alpha, x_{-\alpha} \rangle \langle h_\alpha, h_\alpha \rangle y_\alpha$. Then, we have $\langle y_\alpha, x_{-\alpha} \rangle \langle h_\alpha, h_\alpha \rangle = 1$. Now, we use Theorem 18 to calculate as follows.

$$\begin{aligned}
0 &= \pi[[y_\alpha, x_{-\alpha}], x_\alpha - x_{-\alpha}] + \pi[[x_{-\alpha}, x_\alpha - x_{-\alpha}], y_\alpha] + \pi[[x_\alpha - x_{-\alpha}, y_\alpha], x_{-\alpha}] \\
&= \pi[\langle y_\alpha, x_{-\alpha} \rangle h_\alpha, x_\alpha - x_{-\alpha}] + \pi[-\langle x_\alpha, x_{-\alpha} \rangle h_\alpha, y_\alpha] + \pi[\langle y_\alpha, x_{-\alpha} \rangle h_\alpha, x_{-\alpha}] \\
&= \pi\Big(\langle y_\alpha, x_{-\alpha} \rangle \langle h_\alpha, h_\alpha \rangle x_\alpha + \langle y_\alpha, x_{-\alpha} \rangle \langle h_\alpha, h_\alpha \rangle x_{-\alpha} \Big) \\
&\qquad + \pi\Big(-\langle x_\alpha, x_{-\alpha} \rangle \langle h_\alpha, h_\alpha \rangle y_\alpha \Big) + \pi\Big(-\langle y_\alpha, x_{-\alpha} \rangle \langle h_\alpha, h_\alpha \rangle x_{-\alpha} \Big) \\
&= \pi(x_\alpha + x_{-\alpha}) + \pi(-x_\alpha) + \pi(-x_{-\alpha}) \\
&= \pi(x_\alpha + x_{-\alpha})
\end{aligned}$$

This proves (9), as desired.

Next, we show that

$$\pi(x) \in H \text{ whenever } x = \sum_{\alpha \in \Delta} x_\alpha, \tag{10}$$

where $x_\alpha \in L_\alpha$. It is enough to show that $[h, \pi(x)] = 0$ for any $h \in H$. Since $[H, L] \subseteq \bigoplus_{\alpha \in \Delta} L_\alpha$, this is equivalent to showing that $[h, \pi(x)] \in H$ for any $h \in H$. We do this by induction on the number k of non-zero terms x_α occurring in the sum $x = \sum_{\alpha \in \Delta} x_\alpha$. If $k < 2$, then $\pi(x) = 0$ by hypothesis. Similarly, if $k = 2$ and there is some $\alpha \in \Delta$ with $x_\alpha, x_{-\alpha} \neq 0$, then $x = x_\alpha + x_{-\alpha}$, so that $\pi(x) = 0$ by (9). Either way, we have $[h, \pi(x)] = [h, 0] = 0$ trivially. So, we may assume that there exists $\alpha', \alpha'' \in \Delta$ with $x_{\alpha'}, x_{\alpha''} \neq 0$ with such that

$\alpha' \neq \pm\alpha''$. Note that any $h \in H$ can be written in the form $h = h' + h''$ for some $h', h'' \in H$ satisfying $\alpha'(h') = \alpha''(h'') = 0$.

$$\begin{aligned}
[h, \pi(x)] &= [h' + h'', \pi(x)] \\
&= [h', \pi(x)] + [h'', \pi(x)] \\
&= \Big(\pi[h', x] - [\pi(h'), x]\Big) + \Big(\pi[h'', x] - [\pi(h''), x]\Big) \\
&= \pi[h', x] + \pi[h'', x] \\
&= \pi\Big[h', \sum_{\alpha\in\Delta} x_\alpha\Big] + \pi\Big[h'', \sum_{\alpha\in\Delta} x_\alpha\Big] \\
&= \pi\Big(\sum_{\alpha\in\Delta} \alpha(h')x_\alpha\Big) + \pi\Big(\sum_{\alpha\in\Delta} \alpha(h'')x_\alpha\Big) \\
&\in H
\end{aligned}$$

The last statement follows directly from the induction hypothesis, since we have $\alpha'(h') = \alpha''(h'') = 0$ and $x_{\alpha'}, x_{\alpha''} \neq 0$, so that both sums $\sum_{\alpha\in\Delta} \alpha(h')x_\alpha$ and $\sum_{\alpha\in\Delta} \alpha(h'')x_\alpha$ have strictly fewer non-zero terms than the sum $\sum_{\alpha\in\Delta} x_\alpha$, and therefore the images of both sums under π are in H. Thus, (10) is proved.

Next, we show that

$$\pi(x) \in H \text{ for any } x \in L. \tag{11}$$

As before, it is enough to show that $[h, \pi(x)] = 0$ for any $x \in L$ and $h \in H$, which in turn is equivalant to showing that $[h, \pi(x)] \in H$ for any $x \in L$ and $h \in H$. By writing x as $x = h' + \sum_{\alpha\in\Delta} x_\alpha$, we see that

$$\begin{aligned}
[h, \pi(x)] &= \pi[h, x] - [\pi(h), x] \\
&= \pi\Big[h, h' + \sum_{\alpha\in\Delta} x_\alpha\Big] - [0, x] \\
&= \pi\Big(\sum_{\alpha\in\Delta} \alpha(h)x_\alpha\Big) \\
&\in H,
\end{aligned}$$

by (10). Thus, (11) is proved.

Finally, we complete the proof of Lemma 19 by showing that

$$\pi(x) = 0 \text{ for all } x \in L. \tag{12}$$

Since $\pi(x) \in H$ by (11), it is enough to show that $[\pi(x), y_\beta] = 0$ for every $\beta \in \Delta$ and $y_\beta \in L_\beta$. If we let $h = \pi(x)$, we see immediately that $[\pi(x), y_\beta] = [h, y_\beta] = \beta(h)y_\beta \in L_\beta$. But

$$[\pi(x), y_\beta] = \pi[x, y_\beta] - [x, \pi(y_\beta)]$$

$$= \pi[x, y_\beta] - [x, 0]$$
$$\in H,$$

by (11), so that $[\pi(x), y_\beta] \in L_\beta \cap H = \{0\}$. Thus, (12) holds, and the proof is complete. □

Theorem 20 *Let L be a finite dimensional split semi-simple Lie algebra over a field $\mathbb{F}$ of characteristic zero. Suppose that π is a production on the underlying Lie ring of L. Then π is additive, that is,*

$$\pi(x + y) = \pi(x) + \pi(y) \tag{13}$$

for all $x, y \in L$.

PROOF. First, we show that the Cartan decomposition (8) satisfies the hypotheses of Proposition 17. That is, we show that π is additive on each of the summands of (8).

Given $\alpha \in \Delta$ and $x_\alpha \in L_\alpha$ with $x_\alpha \neq 0$, note that $[h_\alpha, x_\alpha] = \langle h_\alpha, h_\alpha \rangle x_\alpha$, so that π is additive on $\mathbb{F}\langle h_\alpha, h_\alpha \rangle x_\alpha = \mathbb{F}x_\alpha = L_\alpha$, by Remark 15.

Note also that if $x_\alpha \in L_\alpha$ and $x_{-\alpha} \in L_{-\alpha}$ with $x_\alpha, x_{-\alpha} \neq 0$, then $\langle x_\alpha, x_{-\alpha} \rangle \neq 0$ and $[x_\alpha, x_{-\alpha}] = \langle x_\alpha, x_{-\alpha} \rangle h_\alpha$, so that π is additive on $\mathbb{F}\langle x_\alpha, x_{-\alpha} \rangle h_\alpha = \mathbb{F}h_\alpha$, by Remark 15. Recall that $H = \bigoplus_{\alpha \in \Delta_+^0} \mathbb{F}h_\alpha$. For convenience, let us choose $x_\alpha \in L_\alpha$ and $x_{-\alpha} \in L_{-\alpha}$ for each $\alpha \in \Delta_+^0$ such that $\langle x_\alpha, x_{-\alpha} \rangle = 1$. Then $[x_\alpha, x_{-\alpha}] = h_\alpha$ for all $\alpha \in \Delta_+^0$. Note also that $[x_\alpha, x_{-\beta}] = 0$ for any $\alpha, \beta \in \Delta_+^0$ with $\alpha \neq \beta$.

We now show that π is additive on H. Let $h, h' \in H$. Then we can write $h = \sum_{\alpha \in \Delta_+^0} r_\alpha h_\alpha$ and $h' = \sum_{\alpha \in \Delta_+^0} r'_\alpha h_\alpha$ with $r_\alpha, r'_\alpha \in \mathbb{F}$ for $\alpha \in \Delta_+^0$. The following calculation uses Corollary 10 together with the fact that π is additive on $\mathbb{F}h_\alpha$ for all $\alpha \in \Delta_+^0$.

$$\begin{aligned}
\pi(h + h') &= \pi\Big(\sum_{\alpha \in \Delta_+^0} r_\alpha h_\alpha + \sum_{\alpha \in \Delta_+^0} r'_\alpha h_\alpha\Big) \\
&= \pi\Big(\sum_{\alpha \in \Delta_+^0} (r_\alpha + r'_\alpha)\, h_\alpha\Big) \\
&= \pi\Big[\sum_{\alpha \in \Delta_+^0} (r_\alpha + r'_\alpha)\, x_\alpha, \sum_{\alpha \in \Delta_+^0} x_{-\alpha}\Big] \\
&= \sum_{\alpha \in \Delta_+^0} \pi\Big[(r_\alpha + r'_\alpha)\, x_\alpha, x_{-\alpha}\Big] \\
&= \sum_{\alpha \in \Delta_+^0} \pi\Big((r_\alpha + r'_\alpha)\, h_\alpha\Big)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\alpha\in\Delta_+^0} \pi\left(r_\alpha h_\alpha + r'_\alpha h_\alpha\right) \\
&= \sum_{\alpha\in\Delta_+^0} \Big(\pi\left(r_\alpha h_\alpha\right) + \pi\left(r'_\alpha h_\alpha\right)\Big) \\
&= \sum_{\alpha\in\Delta_+^0} \pi\left(r_\alpha h_\alpha\right) + \sum_{\alpha\in\Delta_+^0} \pi\left(r'_\alpha h_\alpha\right) \\
&= \sum_{\alpha\in\Delta_+^0} \pi\left[r_\alpha x_\alpha, x_{-\alpha}\right] + \sum_{\alpha\in\Delta_+^0} \pi\left[r'_\alpha x_\alpha, x_{-\alpha}\right] \\
&= \pi\Big[\sum_{\alpha\in\Delta_+^0} r_\alpha x_\alpha, \sum_{\alpha\in\Delta_+^0} x_{-\alpha}\Big] + \pi\Big[\sum_{\alpha\in\Delta_+^0} r'_\alpha x_\alpha, \sum_{\alpha\in\Delta_+^0} x_{-\alpha}\Big] \\
&= \pi\Big(\sum_{\alpha\in\Delta_+^0} r_\alpha h_\alpha\Big) + \pi\Big(\sum_{\alpha\in\Delta_+^0} r'_\alpha h_\alpha\Big) \\
&= \pi(h) + \pi(h')
\end{aligned}
$$

Thus, we see that π is additive on H. Since π has already been seen to be additive on L_α for each $\alpha \in \Delta$, we see that the Cartan decomposition (8) satisfies the hypotheses for Proposition 17. Let $\delta : L \to L$ be the map whose existence is claimed in Proposition 17. Note that δ is a derivation (and therefore a production) on the underlying Lie ring of L, which agrees with π on H and on L_α for all $\alpha \in \Delta$. It is trivial to see that the set of all productions on the underlying Lie ring of L form a vector space over the field $\mathbb{F}$, under the obvious pointwise definitions. In particular, the map $\pi' = \pi - \delta$ defined by $\pi'(x) = \pi(x) - \delta(x)$ is a production on the underlying Lie ring of L. Furthermore, $\pi'(h) = 0$ for all $h \in H$ and $\pi'(x_\alpha) = 0$ for all $\alpha \in \Delta$ and all $x_\alpha \in L_\alpha$. Thus, Lemma 19 implies that π' is the trivial production, so that $\pi = \delta$. Thus, π is a derivation on the underlying Lie ring L, and hence additive. □

Theorem 21 *Let L be a finite dimensional split simple Lie algebra over a field $\mathbb{F}$ of characteristic zero, with Killing form $\langle -, - \rangle$, and let π be a production on the underlying Lie ring of L. Then π is a derivation on the underlying Lie ring of L, and there exists a unique derivation $\delta : \mathbb{F} \to \mathbb{F}$ on the field of scalars $\mathbb{F}$ such that*

$$\pi(\lambda x) = \delta(\lambda)x + \lambda\pi(x) \tag{14}$$

and

$$\delta \langle x, y \rangle = \langle \pi(x), y \rangle + \langle x, \pi(y) \rangle , \tag{15}$$

for any $\lambda \in \mathbb{F}$ and any $x, y \in L$.

PROOF. The fact that π is a additive on L, and thus a derivation on the underlying Lie ring of L, is the content of Theorem 20. It follows that π is actually $\mathbb{Q}$-linear. In particular, $\pi(\langle h_\alpha, h_\beta\rangle x) = \langle h_\alpha, h_\beta\rangle \pi(x)$ for all $x \in L$ and all $\alpha, \beta \in \Delta$, since $\langle h_\alpha, h_\beta\rangle \in \mathbb{Q}$ is rational. We will use this fact freely without mention.

Note also that the uniqueness of δ is trivial, by either (14) or (15), so it enough to show the existence of a δ with the desired properties.

Our first task is to define δ. For any $\alpha \in \Delta$, we define a map $\delta_\alpha : \mathbb{F} \to \mathbb{F}$ by

$$\delta_\alpha(\lambda) = \frac{\langle \pi(\lambda h_\alpha), h_\alpha\rangle}{\langle h_\alpha, h_\alpha\rangle}, \tag{16}$$

for any $\lambda \in \mathbb{F}$. We shall see shortly that $\delta_\alpha = \delta_\beta$ for all $\alpha, \beta \in \Delta$.

First, we show that δ_α is a derivation on $\mathbb{F}$ for any $\alpha \in \Delta$. The fact that δ_α is additive is an immediate consequence of the fact that π is additive. To see that δ_α is a production on $\mathbb{F}$, choose any $x_\alpha \in L_\alpha$ and $x_{-\alpha} \in L_{-\alpha}$ such that $\langle x_\alpha, x_{-\alpha}\rangle = 1$. Then $h_\alpha = [x_\alpha, x_{-\alpha}]$, so we may apply Corollary 13 with $c = h_\alpha$ in (4) to see that

$$\pi(\lambda\mu h_\alpha) + \lambda\mu\pi(h_\alpha) = \mu\pi(\lambda h_\alpha) + \lambda\pi(\mu h_\alpha),$$

and thus

$$\delta_\alpha(\lambda\mu) + \lambda\mu\delta_\alpha(1) = \mu\delta_\alpha(\lambda) + \lambda\delta_\alpha(\mu),$$

for any $\lambda, \mu \in \mathbb{F}$. Thus, to see that δ_α is a production, we need only show that

$$\delta_\alpha(1) = 0, \tag{17}$$

for all $\alpha \in \Delta$. To see this, once again we choose any $x_\alpha \in L_\alpha$ and $x_{-\alpha} \in L_{-\alpha}$ such that $\langle x_\alpha, x_{-\alpha}\rangle = 1$, and calculate as follows:

$$\begin{aligned}
\delta_\alpha(1) &= \frac{\langle \pi(h_\alpha), h_\alpha\rangle}{\langle h_\alpha, h_\alpha\rangle} \\
&= \frac{\langle \pi(h_\alpha), [x_\alpha, x_{-\alpha}]\rangle}{\langle h_\alpha, h_\alpha\rangle} \\
&= \frac{\langle [\pi(h_\alpha), x_\alpha], x_{-\alpha}\rangle}{\langle h_\alpha, h_\alpha\rangle} \\
&= \frac{\langle \pi[h_\alpha, x_\alpha] - [h_\alpha, \pi(x_\alpha)], x_{-\alpha}\rangle}{\langle h_\alpha, h_\alpha\rangle} \\
&= \frac{\langle \pi[h_\alpha, x_\alpha], x_{-\alpha}\rangle}{\langle h_\alpha, h_\alpha\rangle} - \frac{\langle [h_\alpha, \pi(x_\alpha)], x_{-\alpha}\rangle}{\langle h_\alpha, h_\alpha\rangle} \\
&= \frac{\langle \pi\left(\langle h_\alpha, h_\alpha\rangle x_\alpha\right), x_{-\alpha}\rangle}{\langle h_\alpha, h_\alpha\rangle} + \frac{\langle \pi(x_\alpha), [h_\alpha, x_{-\alpha}]\rangle}{\langle h_\alpha, h_\alpha\rangle} \\
&= \langle \pi(x_\alpha), x_{-\alpha}\rangle - \langle \pi(x_\alpha), x_{-\alpha}\rangle \\
&= 0
\end{aligned}$$

Thus, δ_α is a derivation on $\mathbb{F}$ for any $\alpha \in \Delta$, as claimed.

Next, we show that

$$\langle \pi \left(\langle x_\alpha, x_{-\alpha} \rangle h_\alpha \right), h \rangle = \langle h_\alpha, h \rangle \Big(\langle \pi(x_\alpha), x_{-\alpha} \rangle + \langle x_\alpha, \pi(x_{-\alpha}) \rangle \Big), \tag{18}$$

for any $\alpha \in \Delta$, $h \in H$, $x_\alpha \in L_\alpha$ and $x_{-\alpha} \in L_{-\alpha}$. We see this as follows:

$$\begin{aligned} \langle \pi \left(\langle x_\alpha, x_{-\alpha} \rangle h_\alpha \right), h \rangle &= \langle \pi \left[x_\alpha, x_{-\alpha} \right], h \rangle \\ &= \langle [\pi(x_\alpha), x_{-\alpha}], h \rangle + \langle [x_\alpha, \pi(x_{-\alpha})], h \rangle \\ &= \langle \pi(x_\alpha), [x_{-\alpha}, h] \rangle + \langle [h, x_\alpha], \pi(x_{-\alpha}) \rangle \\ &= \langle h_\alpha, h \rangle \Big(\langle \pi(x_\alpha), x_{-\alpha} \rangle + \langle x_\alpha, \pi(x_{-\alpha}) \rangle \Big) \end{aligned}$$

If $\langle h_\alpha, h \rangle \neq 0$, we can rewrite this as

$$\frac{\langle \pi \left(\langle x_\alpha, x_{-\alpha} \rangle h_\alpha \right), h \rangle}{\langle h_\alpha, h \rangle} = \langle \pi(x_\alpha), x_{-\alpha} \rangle + \langle x_\alpha, \pi(x_{-\alpha}) \rangle . \tag{19}$$

If we set $h = h_\alpha$ in (19), we see from (16) that

$$\delta_\alpha(\langle x_\alpha, x_{-\alpha} \rangle) = \langle \pi(x_\alpha), x_{-\alpha} \rangle + \langle x_\alpha, \pi(x_{-\alpha}) \rangle . \tag{20}$$

Given any $\lambda \in \mathbb{F}$ and any $\alpha \in \Delta$, we can find $x_\alpha \in L_\alpha$ and $x_{-\alpha} \in L_{-\alpha}$ such that $\lambda = \langle x_\alpha, x_{-\alpha} \rangle$. Thus, we may combine (20) with (18) to yield

$$\langle \pi(\lambda h_\alpha), h \rangle = \delta_\alpha(\lambda) \langle h_\alpha, h \rangle . \tag{21}$$

Next, we show that

$$\langle \pi(h), x_\alpha \rangle + \langle h, \pi(x_\alpha) \rangle = 0, \tag{22}$$

for any $\alpha \in \Delta$, $h \in H$ and $x_\alpha \in L_\alpha$. To see this, we calculate as follows:

$$\begin{aligned} \langle \pi(h), x_\alpha \rangle + \langle h, \pi(x_\alpha) \rangle &= \left\langle \pi(h), \left[h_\alpha, \frac{x_\alpha}{\langle h_\alpha, h_\alpha \rangle} \right] \right\rangle + \left\langle h, \pi \left[h_\alpha, \frac{x_\alpha}{\langle h_\alpha, h_\alpha \rangle} \right] \right\rangle \\ &= \left\langle [\pi(h), h_\alpha], \frac{x_\alpha}{\langle h_\alpha, h_\alpha \rangle} \right\rangle + \left\langle h, \left[\pi(h_\alpha), \frac{x_\alpha}{\langle h_\alpha, h_\alpha \rangle} \right] \right\rangle \\ &\quad + \left\langle h, \left[h_\alpha, \pi \left(\frac{x_\alpha}{\langle h_\alpha, h_\alpha \rangle} \right) \right] \right\rangle \\ &= \left\langle [\pi(h), h_\alpha], \frac{x_\alpha}{\langle h_\alpha, h_\alpha \rangle} \right\rangle + \left\langle [h, \pi(h_\alpha)], \frac{x_\alpha}{\langle h_\alpha, h_\alpha \rangle} \right\rangle \\ &\quad + \left\langle [h, h_\alpha], \pi \left(\frac{x_\alpha}{\langle h_\alpha, h_\alpha \rangle} \right) \right\rangle \\ &= \left\langle \pi[h, h_\alpha], \frac{x_\alpha}{\langle h_\alpha, h_\alpha \rangle} \right\rangle + \left\langle [h, h_\alpha], \frac{x_\alpha}{\langle h_\alpha, h_\alpha \rangle} \right\rangle \end{aligned}$$

$$
\begin{aligned}
&= \left\langle \pi(0), \frac{x_\alpha}{\langle h_\alpha, h_\alpha\rangle} \right\rangle + \left\langle 0, \frac{x_\alpha}{\langle h_\alpha, h_\alpha\rangle} \right\rangle \\
&= 0 + 0
\end{aligned}
$$

Using (22), we may show that

$$
\langle \pi(\lambda h), x_\alpha\rangle = \lambda \langle \pi(h), x_\alpha\rangle, \tag{23}
$$

for any $\alpha \in \Delta$, $\lambda \in \mathbb{F}$, $h \in H$ and $x_\alpha \in L_\alpha$ as follows:

$$
\begin{aligned}
\langle \pi(\lambda h), x_\alpha\rangle &= -\langle \lambda h, \pi(x_\alpha)\rangle \\
&= -\lambda \langle h, \pi(x_\alpha)\rangle \\
&= \lambda \langle \pi(h), x_\alpha\rangle
\end{aligned}
$$

From (23), we see immediately that $\langle \pi(\lambda h) - \lambda\pi(h), x_\alpha\rangle = 0$ for any $x_\alpha \in L_\alpha$, so that

$$
\pi(\lambda h) - \lambda\pi(h) \in H, \tag{24}
$$

for any $h \in H$ and $\lambda \in \mathbb{F}$.

Next, we show that

$$
\pi(\lambda h_\alpha) = \delta_\alpha(\lambda)h_\alpha + \lambda\pi(h_\alpha), \tag{25}
$$

for all $\alpha \in \Delta$ and $\lambda \in \mathbb{F}$. By (24), we see that $\pi(\lambda h_\alpha) - \lambda\pi(h_\alpha) - \delta_\alpha(\lambda)h_\alpha \in H$. Thus, to show (25), it suffices to show that $\langle \pi(\lambda h_\alpha) - \lambda\pi(h_\alpha) - \delta_\alpha(\lambda)h_\alpha, h\rangle = 0$ for any $h \in H$. We see this, using (21) and (17) as follows:

$$
\begin{aligned}
&\langle \pi(\lambda h_\alpha) - \lambda\pi(h_\alpha) - \delta_\alpha(\lambda)h_\alpha, h\rangle \\
&\qquad = \langle \pi(\lambda h_\alpha), h\rangle - \lambda \langle \pi(h_\alpha), h\rangle - \delta_\alpha(\lambda) \langle h_\alpha, h\rangle \\
&\qquad = \delta_\alpha(\lambda) \langle h_\alpha, h\rangle - \lambda\delta_\alpha(1) \langle h_\alpha, h\rangle - \delta_\alpha(\lambda) \langle h_\alpha, h\rangle \\
&\qquad = 0
\end{aligned}
$$

Next, we show that

$$
\langle \pi(h_\alpha + h_\beta), [x_\alpha, x_\beta]\rangle = \langle h_\alpha + h_\beta, h_\alpha + h_\beta\rangle \langle \pi(x_\alpha), x_\beta\rangle, \tag{26}
$$

for any $\alpha, \beta \in \Delta$, $x_\alpha \in L_\alpha$ and $x_\beta \in L_\beta$, by calculating as follows:

$$
\begin{aligned}
\langle \pi(h_\alpha + h_\beta), [x_\alpha, x_\beta]\rangle &= \langle [\pi(h_\alpha + h_\beta), x_\alpha], x_\beta\rangle \\
&= \langle \pi[h_\alpha + h_\beta, x_\alpha] - [h_\alpha + h_\beta, \pi(x_\alpha)], x_\beta\rangle \\
&= \langle \pi[h_\alpha + h_\beta, x_\alpha], x_\beta\rangle - \langle [h_\alpha + h_\beta, \pi(x_\alpha)], x_\beta\rangle \\
&= \langle \pi(\langle h_\alpha + h_\beta, h_\alpha\rangle x_\alpha), x_\beta\rangle + \langle \pi(x_\alpha), [h_\alpha + h_\beta, x_\beta]\rangle \\
&= \langle h_\alpha + h_\beta, h_\alpha\rangle \langle \pi(x_\alpha), x_\beta\rangle + \langle h_\alpha + h_\beta, h_\beta\rangle \langle \pi(x_\alpha), x_\beta\rangle \\
&= \langle h_\alpha + h_\beta, h_\alpha + h_\beta\rangle \langle \pi(x_\alpha), x_\beta\rangle
\end{aligned}
$$

From (26), we see immediately that

$$\langle \pi(x_\alpha), x_\beta \rangle = \frac{\langle \pi(h_\alpha + h_\beta), [x_\alpha, x_\beta] \rangle}{\langle h_\alpha + h_\beta, h_\alpha + h_\beta \rangle} \text{ for } \alpha + \beta \neq 0. \tag{27}$$

Since $[x_\alpha, x_\beta] + [x_\beta, x_\alpha] = 0$, we can use (27) to conclude that

$$\langle \pi(x_\alpha), x_\beta \rangle + \langle x_\alpha, \pi(x_\beta) \rangle = 0 \text{ for } \alpha + \beta \neq 0, \tag{28}$$

for any $\alpha, \beta \in \Delta$, $x_\alpha \in L_\alpha$ and $x_\beta \in L_\beta$.

Our next goal is to prove that

$$\pi(\lambda x_\alpha) = \delta_\alpha(\lambda) x_\alpha + \lambda \pi(x_\alpha), \tag{29}$$

for any $\alpha \in \Delta$, $\lambda \in \mathbb{F}$ and $x_\alpha \in L_\alpha$. We start by using (22) to show that

$$\langle \pi(\lambda x_\alpha) - \delta_\alpha(\lambda) x_\alpha - \lambda \pi(x_\alpha), h \rangle = 0 \tag{30}$$

for any $h \in H$, by calculating as follows:

$$\begin{aligned} \langle \delta_\alpha(\lambda) x_\alpha, h \rangle + \langle \lambda \pi(x_\alpha), h \rangle &= 0 + \lambda \langle \pi(x_\alpha), h \rangle \\ &= -\lambda \langle x_\alpha, \pi(h) \rangle \\ &= - \langle \lambda x_\alpha, \pi(h) \rangle \\ &= \langle \pi(\lambda x_\alpha), h \rangle \end{aligned}$$

Next, we use (28) to show that

$$\langle \pi(\lambda x_\alpha) - \delta_\alpha(\lambda) x_\alpha - \lambda \pi(x_\alpha), x_\beta \rangle = 0 \text{ for } \alpha + \beta \neq 0, \tag{31}$$

foe any $x_\alpha \in L_\beta$, by calculating as follows:

$$\begin{aligned} \langle \delta_\alpha(\lambda) x_\alpha, x_\beta \rangle + \langle \lambda \pi(x_\alpha), x_\beta \rangle &= 0 + \lambda \langle \pi(x_\alpha), x_\beta \rangle \\ &= -\lambda \langle x_\alpha, \pi(x_\beta) \rangle \\ &= - \langle \lambda x_\alpha, \pi(x_\beta) \rangle \\ &= \langle \pi(\lambda x_\alpha), x_\beta \rangle \end{aligned}$$

Finally, we use (20) to show that

$$\langle \pi(\lambda x_\alpha) - \delta_\alpha(\lambda) x_\alpha - \lambda \pi(x_\alpha), x_{-\alpha} \rangle = 0, \tag{32}$$

for any $x_{-\alpha}$, by calculating as follows:

$$\begin{aligned} \langle \pi(\lambda x_\alpha), x_{-\alpha} \rangle &= \delta_\alpha \langle \lambda x_\alpha, x_{-\alpha} \rangle - \langle \lambda x_\alpha, \pi(x_{-\alpha}) \rangle \\ &= \delta_\alpha \left(\lambda \langle x_\alpha, x_{-\alpha} \rangle \right) - \lambda \langle x_\alpha, \pi(x_{-\alpha}) \rangle \\ &= \delta_\alpha (\lambda) \langle x_\alpha, x_{-\alpha} \rangle + \lambda \delta_\alpha \langle x_\alpha, x_{-\alpha} \rangle - \lambda \langle x_\alpha, \pi(x_{-\alpha}) \rangle \\ &= \delta_\alpha (\lambda) \langle x_\alpha, x_{-\alpha} \rangle + \lambda \Big(\delta_\alpha \langle x_\alpha, x_{-\alpha} \rangle - \langle x_\alpha, \pi(x_{-\alpha}) \rangle \Big) \end{aligned}$$

$$
\begin{aligned}
&= \delta_\alpha(\lambda) \langle x_\alpha, x_{-\alpha} \rangle + \lambda \langle \pi(x_\alpha), x_{-\alpha} \rangle \\
&= \langle \delta_\alpha(\lambda) x_\alpha, x_{-\alpha} \rangle + \langle \lambda \pi(x_\alpha), x_{-\alpha} \rangle
\end{aligned}
$$

Since the Killing form is non-degenerate, we may use (30), (31) and (32) to conclude that (29) holds, as desired.

We are now in a position to define the derivation δ on $\mathbb{F}$. First, note that since $h_{-\alpha} = -h_\alpha$, we immediately see from (16) that

$$\delta_\alpha(\lambda) = \delta_{-\alpha}(\lambda), \tag{33}$$

for any $\alpha \in \Delta$. Next, we show that

$$\delta_{\alpha+\beta}(\lambda\mu) = \delta_\alpha(\lambda)\mu + \lambda\delta_\beta(\mu) \text{ for } \alpha, \beta, \alpha + \beta \in \Delta, \tag{34}$$

where $\lambda\mu \in \mathbb{F}$, $x_\alpha \in L_\alpha$ and $x_\beta \in L_\beta$. To see this, note that $[x_\alpha, x_\beta] \in L_{\alpha+\beta}$, since we are assuming that $\alpha + \beta \in \Delta$. Using (29), we see that

$$
\begin{aligned}
\delta_{\alpha+\beta}(\lambda\mu)[x_\alpha, x_\beta] &= \pi\left(\lambda\mu[x_\alpha, x_\beta]\right) - \lambda\mu\pi[x_\alpha, x_\beta] \\
&= \pi[\lambda x_\alpha, \mu x_\beta] - \lambda\mu\pi[x_\alpha, x_\beta] \\
&= [\pi(\lambda x_\alpha), \mu x_\beta] + [\lambda x_\alpha, \pi(\mu x_\beta)] - \lambda\mu\pi[x_\alpha, x_\beta] \\
&= [\delta_\alpha(\lambda) x_\alpha + \lambda\pi(x_\alpha), \mu x_\beta] + [\lambda x_\alpha, \delta_\beta(\mu) x_\beta + \mu\pi(x_\beta)] - \lambda\mu\pi[x_\alpha, x_\beta] \\
&= \Big(\delta_\alpha(\lambda)\mu + \lambda\delta_\beta(\mu)\Big)[x_\alpha, x_\beta] + \lambda\mu\Big([\pi(x_\alpha), x_\beta] + [x_\alpha, \pi(x_\beta)] - \pi[x_\alpha, x_\beta]\Big) \\
&= \Big(\delta_\alpha(\lambda)\mu + \lambda\delta_\beta(\mu)\Big)[x_\alpha, x_\beta],
\end{aligned}
$$

which easily implies (34), by any choice of $x_\alpha, x_\beta \neq 0$ so that $[x_\alpha, x_\beta] \neq 0$. By letting $\mu = 1$ in (34), we see that $\delta_{\alpha+\beta}(\lambda) = \delta_\alpha(\lambda) + \lambda\delta_\beta(0) = \delta_\alpha(\lambda)$. Similarly, $\delta_{\alpha+\beta}(\mu) = \delta_\beta(\mu)$. Thus, we have

$$\delta_\alpha = \delta_\beta = \delta_{\alpha+\beta} \text{ for } \alpha, \beta, \alpha + \beta \in \Delta. \tag{35}$$

Since L is assumed to be simple, from (33) and (35) we may conclude that

$$\delta_\alpha = \delta_\beta \text{ for } \alpha, \beta \in \Delta. \tag{36}$$

(Note that if L were merely assumed to be semi-simple, then (36) would only follow in case α and β were roots of the same simple factor.)

We now define the derivation $\delta : \mathbb{F} \to \mathbb{F}$ on $\mathbb{F}$ by

$$\delta = \delta_\alpha \text{ for any } \alpha \in \Delta. \tag{37}$$

By (36), δ is well defined. Also, δ is a derivation on $\mathbb{F}$ since δ_α is.

It remains to prove (14) and (15).

Now, we show that

$$\pi(\lambda h) = \delta(\lambda)h + \lambda\pi(h), \tag{38}$$

for any $\lambda \in \mathbb{F}$ and $h \in H$. To see this, we write $h = \sum_{\alpha\in\Delta_+^0} \mu_\alpha h_\alpha$, with $\mu_\alpha \in \mathbb{F}$ for all $\alpha \in \Delta_+^0$, and use (25) and (37) as follows:.

$$\begin{aligned}
\pi(\lambda h) &= \pi\Big(\lambda \sum_{\alpha\in\Delta_+^0} \mu_\alpha h_\alpha\Big) \\
&= \pi\Big(\sum_{\alpha\in\Delta_+^0} \lambda\mu_\alpha h_\alpha\Big) \\
&= \sum_{\alpha\in\Delta_+^0} \pi\left(\lambda\mu_\alpha h_\alpha\right) \\
&= \sum_{\alpha\in\Delta_+^0} \Big(\delta(\lambda\mu_\alpha)h_\alpha + \lambda\mu_\alpha\pi(h_\alpha)\Big) \\
&= \sum_{\alpha\in\Delta_+^0} \Big(\Big(\delta(\lambda)\mu_\alpha + \lambda\delta(\mu_\alpha)\Big)h_\alpha + \lambda\mu_\alpha\pi(h_\alpha)\Big) \\
&= \delta(\lambda) \sum_{\alpha\in\Delta_+^0} \mu_\alpha h_\alpha + \lambda \sum_{\alpha\in\Delta_+^0} \Big(\delta(\mu_\alpha)h_\alpha + \mu_\alpha\pi(h_\alpha)\Big) \\
&= \delta(\lambda)h + \lambda \sum_{\alpha\in\Delta_+^0} \pi(\mu_\alpha h_\alpha) \\
&= \delta(\lambda)h + \lambda\pi\Big(\sum_{\alpha\in\Delta_+^0} \mu_\alpha h_\alpha\Big) \\
&= \delta(\lambda)h + \lambda\pi(h)
\end{aligned}$$

We are finally in a position to prove (14). Writing $x = h+\sum_{\alpha\in\delta} x_\alpha$,with $h \in H$ and $x_\alpha \in L_\alpha$ for $\alpha \in \Delta$, we use (29), (38) and (37) to compute as follows:

$$\begin{aligned}
\pi(\lambda x) &= \pi\Big(\lambda\Big(h + \sum_{\alpha\in\delta} x_\alpha\Big)\Big) \\
&= \pi\Big(\lambda h + \sum_{\alpha\in\delta} \lambda x_\alpha\Big) \\
&= \pi(\lambda h) + \sum_{\alpha\in\delta} \pi(\lambda x_\alpha) \\
&= \delta(\lambda)h + \lambda\pi(h) + \sum_{\alpha\in\delta}\Big(\delta(\lambda)x_\alpha + \lambda\pi(x_\alpha)\Big) \\
&= \delta(\lambda)\Big(h + \sum_{\alpha\in\Delta} x_\alpha\Big) + \lambda\Big(\pi(h) + \sum_{\alpha\in\Delta} \pi(x_\alpha)\Big) \\
&= \delta(\lambda)x + \lambda\pi(x)
\end{aligned}$$

Thus, (14) is proved.

Next, we show that

$$\delta \langle \lambda h_\alpha, \mu h_\beta \rangle = \langle \pi(\lambda h_\alpha), \mu h_\beta \rangle + \langle \lambda h_\alpha, \pi(\mu h_\beta) \rangle , \tag{39}$$

for any $\alpha, \beta \in \Delta$ and $h_\alpha, h_\beta \in H$, by using (21) and computing as follows:

$$\begin{aligned}
\delta \langle \lambda h_\alpha, \mu h_\beta \rangle &= \delta\Big(\lambda\mu \langle h_\alpha, h_\beta \rangle\Big) \\
&= \delta(\lambda\mu) \langle h_\alpha, h_\beta \rangle \\
&= \Big(\delta(\lambda)\mu + \lambda\delta(\mu)\Big) \langle h_\alpha, h_\beta \rangle \\
&= \delta(\lambda) \langle h_\alpha, \mu h_\beta \rangle + \delta(\mu) \langle h_\beta, \lambda h_\alpha \rangle \\
&= \langle \pi(\lambda h_\alpha), \mu h_\beta \rangle + \langle \pi(\mu h_\beta), \lambda h_\alpha \rangle \\
&= \langle \pi(\lambda h_\alpha), \mu h_\beta \rangle + \langle \lambda h_\alpha, \pi(\mu h_\beta) \rangle
\end{aligned}$$

It is easy to see that

$$\delta \langle h, x_\alpha \rangle = \langle \pi(h), x_\alpha \rangle + \langle h, \pi(x_\alpha) \rangle , \tag{40}$$

for all $\alpha \in \delta$, $h \in H$ and $x_\alpha \in L_\alpha$. We need only note that $\delta \langle h, x_\alpha \rangle = \delta(0) = 0$, and apply (22).

It is equally easy to see that

$$\delta \langle x_\alpha, x_\beta \rangle = \langle \pi(x_\alpha), x_\beta \rangle + \langle x_\alpha, \pi(x_\beta) \rangle , \tag{41}$$

for all $\alpha, \beta \in \Delta$, $x_\alpha \in L_\alpha$ and $x_\beta \in L_\beta$. If $\alpha + \beta \neq 0$, we need only note that $\delta \langle x_\alpha, x_\beta \rangle = \delta(0) = 0$ and apply (28). If $\alpha + \beta = 0$, we need only apply (20).

Since every $x \in L$ can be written as $x = \sum_{\alpha \in \Delta^0_+} \lambda_\alpha h_\alpha + \sum_{\alpha \in \Delta} x_\alpha$ where $\lambda_\alpha \in \mathbb{F}$ for all $\alpha \in \Delta^0_+$ and $x_\alpha \in L_\alpha$ for all $x_\alpha \in L_\alpha$, it is easy to see that we can use (39), (40) and (41), together with the fact that both π and δ are additive, and that the Killing form is bilinear, to prove (15).

This completes the proof of Theorem 21 $\square$

Theorem 22 *Let L be a finite dimensional split semi-simple Lie algebra over a field $\mathbb{F}$ of characteristic zero, with Killing form $\langle -, - \rangle$, and let π be a production on the underlying Lie ring of L. Let*

$$L = \bigoplus_{i=1}^{n} L_i$$

be the decomposition of L into its simple factors. Then π is a derivation on the underlying Lie ring of L, and there Then there exists a unique sequence of *derivations* $\delta_1, \ldots, \delta_n : \mathbb{F} \to \mathbb{F}$ *on the field of scalars* $\mathbb{F}$ *such that*

$$\pi(\lambda x) = \sum_{i=1}^{n} \delta_i(\lambda) x_i + \lambda \pi(x)$$

and

$$\sum_{i=1}^{n} \delta_i \langle x_i, y_i \rangle = \langle \pi(x), y \rangle + \langle x, \pi(y) \rangle ,$$

for any $\lambda \in \mathbb{F}$ *and any* $x, y \in L$*,where we write* $x = \sum_{i=1}^{n} x_i$ *and* $y = \sum_{i=1}^{n} y_i$ *with* $x_i, y_i \in L_i$ *for* $i = 1, \ldots, n$*.*

PROOF. This is an easy consequence of Theorem 16 and Theorem 21. □

References

[1] Jacobson, Nathan, *Lie Algebras*, Interscience Publishers, New York, 1962.

[2] Kramer, Richard L. *The undefinability of intersection from perpendicularity in the three-dimensional Euclidean geometry of lines*, Geometriae Dedicata, **46** (1993), pp. 207–210.

[3] Kramer, Richard L. *Definability in the foundations of Euclidean geometry and the product rule for derivations*, Model Theory in Bogotá (Summer 2005), presented August 16, 2005, `http://www.public.iastate.edu/~ricardo/geotalk.pdf` (unpublished).

[4] Schwabhäuser, Wolfram and Szczerba, Lesław W. *Relations on lines as primitive notions for Euclidean goemetry*, Fundamenta Mathematicae, **82** (1974/75), pp. 347–355.