Representation Theory Seminar Sep 14, 2017

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Canonical Galois orders and maximal commutativity

① A hierarchy of algebras.

Galois Orders $\qquad \mathcal{U}(\Lambda, G, \mathcal{M})$

Futorny-Ovsienko 2010, 2014

$\Big\downarrow$ ← Futorny-Molev-Ovsienko 2010

$\boxed{\text{Ragoucy-Sorba 1999} \\ \text{Brundan-Kleshchev 2010}}$

finite W-algebra: $e \in \mathfrak{gl}_N$ nilpotent, $U(\mathfrak{gl}_N, e) \cong W(\pi)$ truncated shifted Yangian

Premet 2002

Gan-Ginzburg 2002

$\Big\downarrow$ $N = pn$, e: Jordan type (n^p)

truncated Yangian: $p \geq 1$, $Y_p(\mathfrak{gl}_n) \xrightarrow{\hbar \to 0} \dfrac{U(\mathfrak{gl}_n[t])}{\langle \mathfrak{gl}_n \cdot t^k \mid k \geq p \rangle}$ truncated current algebra

Drinfeld 1986

$\Big\downarrow$ $p = 1$

enveloping algebra: $U(\mathfrak{gl}_n)$

② Galois orders

Let: Λ integrally closed domain (e.g. a UFD)
G finite subgroup of $\mathrm{Aut}(\Lambda)$
$\mathcal{M}$ submonoid of $\mathrm{Aut}(\Lambda)$

such that

(i) $(\mathcal{M}\mathcal{M}^{-1}) \cap G = \{\mathrm{Id}_\Lambda\}$

(ii) $\forall g \in G \ \forall \mu \in \mathcal{M}: \ {}^g\mu = g \circ \mu \circ g^{-1} \in \mathcal{M}$

(iii) Λ is Noetherian as a Λ^G-module.

Put $\quad L = \mathrm{Frac}\,\Lambda, \quad \mathcal{L} = L * \mathcal{M}$

$\Gamma = \Lambda^G \quad K = L^G, \quad \mathcal{K} = \mathcal{L}^G$

$$\begin{array}{ccccc} \Lambda & \subset & L & \subset & \mathcal{L} \\ \cup & & \cup & & \cup \\ \Gamma & \subset & K & \subset & \mathcal{K} \\ \text{I.D.s} & & \text{fields} & & \text{noncomm. rings} \end{array}$$

Lemma

(i) Λ and Γ are Noetherian

(ii) $K = \mathrm{Frac}\,\Gamma$, L/K Galois, $G = \mathrm{Gal}(L/K)$

Proof (i) Λ is finitely generated Γ-module
Λ is Noetherian ring
$\Longrightarrow \Gamma$ is Noetherian.
$\uparrow$ Eakin-Nagata's Thm

Def (i) A Γ-subring $\mathcal{U} \subset \mathcal{K}$ is a Galois Γ-ring if $\mathcal{U}K = \mathcal{K} = K\mathcal{U}$.

(ii) A Galois Γ-ring $\mathcal{U} \subset \mathcal{K}$ is a Galois Γ-order if for any f.d. left K-subspace $W \subset \mathcal{K}$, the left Γ-module $\mathcal{U} \cap W$ is finitely generated. And same on the right.

For $a = \sum_{\mu \in \mathcal{M}} a_\mu \mu$, $\operatorname{Supp}(a) = \{\mu \mid a_\mu \neq 0\}$.

Thm (Futorny-Ovsienko '10, Futorny-H* '14)

(i) Let $X \subset \mathcal{K}$ and $\mathcal{U} = \langle \Gamma \cup X \rangle$. Then $\mathcal{U}$ is a Galois Γ-ring iff $\bigcup_{x \in X} \operatorname{Supp}(x)$ generates $\mathcal{M}$.

(ii) A Galois Γ-ring is a Galois Γ-order iff Γ is maximal commutative in $\mathcal{U}$.

(i.e. $\mathcal{U} \subset \mathcal{C} \subset \mathcal{U}$, $\mathcal{C}$ comm $\Rightarrow \mathcal{U} = \mathcal{C}$.)

(i) is practical.

(ii) is nice but not practical.

③ Module categories

$$\mathcal{F} \subset \mathcal{O} \subset \mathcal{W} \subset \mathcal{GZ}$$

finite-dim'l modules — Category $\mathcal{O}$ — weight modules — Gelfand-Zeitlin modules

Fix: $\mathcal{A}$ an algebra
$\Gamma \subset \mathcal{A}$ a commutative subalgebra.

Def. An $\mathcal{A}$-module M is a Gelfand-Zeitlin module (w.r.t. Γ) if Γ acts locally finite on M:

$$\forall v \in M: \dim \Gamma v < \infty$$

Equivalently:

$$M = \bigoplus_{\mathfrak{m} \in \operatorname{Specm}(\Gamma)} M^{\mathfrak{m}}, \quad M^{\mathfrak{m}} = \{v \in M \mid \exists n \in \mathbb{N}: \mathfrak{m}^n v = 0\}$$

and $\dim M^{\mathfrak{m}} < \infty \ \forall \mathfrak{m}$

Fiber of $\mathfrak{m} \in \operatorname{Specm} \Gamma$:

$$\Phi_{\mathcal{A}}(\mathfrak{m}) = \{\text{simple } M \in \mathcal{GZ}_{\mathcal{A}} \text{ with } M_{\mathfrak{m}} \neq 0\} / \text{isomorphism}$$

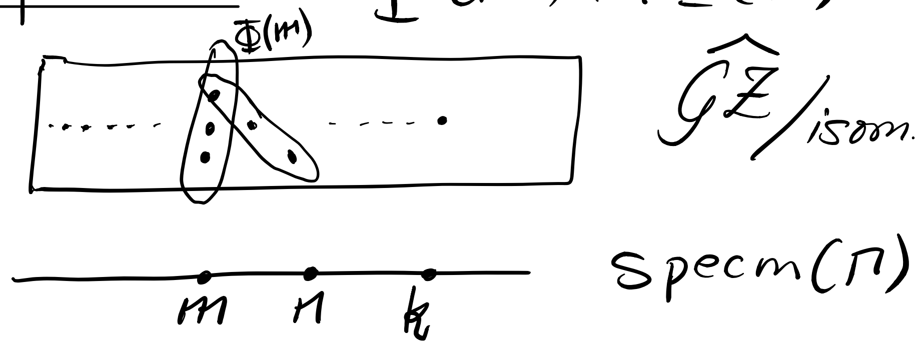
Thm (Futorny–Ovsienko '10, '14)

(i) Let $\mathcal{U}$ be a Galois Γ-ring. Then $\exists$ massive subset $\Omega \subset \mathrm{Specm}(\Gamma)$: $\forall m \in \Omega$: $\Phi_{\mathcal{U}}(m) \neq \emptyset$

(ii) Let $\mathcal{U}$ be a Galois Γ-order. Then $\forall m \in \mathrm{Specm}(\Gamma)$: $\Phi_{\mathcal{U}}(m)$ is nonempty and finite:

$$|\Phi_{\mathcal{U}}(m)| \leq |G_{\mathcal{M}(\bar{m})} : G_{\bar{m}}| \qquad \bar{m} \in \mathrm{Spec}(\Lambda), \quad m = \Gamma \cap \bar{m}$$

Open problems 1) $\Phi(m) \cap \Phi(n) = ?$



2) When is $\mathcal{U}$ free as a left Γ-module?
(right)

(4) New criterion.

Thm (H* 2017)
$\mathcal{U}$ = Galois Γ-ring $\subset \mathcal{K}$
Then $\mathcal{U}$ is a Galois Γ-order iff
$\exists X \subset \mathcal{U}$:

$$\forall x \in X : \quad x(\Gamma) \subset \Gamma$$

were $x = \sum_\mu x_\mu \mu$ is evaluated at $\gamma \in \Gamma$ by

$$x(\gamma) = \sum_\mu x_\mu \cdot \mu(\gamma) \in K$$

Corollary The following Galois rings are actually Galois orders:

1) $U_q(\mathfrak{gl}_n)$

2) (Quantum) OGZ algebras
$U_{(q)}(\lambda)$ $\quad \lambda$ strict partition.

$$U_{(q)}\big((n, n-1, \ldots, 1)\big) \cong U_{(q)}(\mathfrak{gl}_n)$$

3) Type D n.c. Kleinian singularities

Moreover this gives new simpler proofs that $Y_p(\mathfrak{gl}_n)$, $W(\pi)$ are Galois orders.

(5) Examples.

(I) $\Lambda = \mathbb{C}[x_{ki} \mid 1 \le i \le k \le n]$

$$G = S_n \times S_{n-1} \times \cdots \times S_1$$

$$\mathcal{M} = \langle \delta^{ki} \mid 1 \le i \le k \le n \rangle \cong \mathbb{Z}^{n(n-1)/2}$$

$\sigma = (\sigma_n, \dots, \sigma_1) \in G$,

$$\sigma(x_{ki}) = x_{k\sigma_k(i)}$$

$$\delta^{lj}(x_{ki}) = \begin{cases} x_{ki} + 1 & l = k \,\&\, j = i \\ x_{ki} & \text{else} \end{cases}$$

$$X_i^{\pm} = \sum_{i=1}^{k} A_{ki}^{\pm} \cdot \delta^{ki}$$

$$A_{ki}^{\pm} = \frac{\prod_{j=1}^{k\pm 1} (x_{k\pm 1, j} - x_{ki})}{\prod_{j \ne i} (x_{kj} - x_{ki})}$$

Fact: $\mathcal{U} = \langle \Gamma \cup \{X_i^{\pm}\} \rangle \cong U(gl_n)$

$$f = X_i^{\pm}(\delta) = \sum_{i=1}^{k} A_{ki}^{\pm} \, \delta^{ki}(\delta)$$

Know: $f \in K = L^G$

WTS $f \in \Gamma$.

Note: $f \cdot \underbrace{\prod_{k} \prod_{i \neq j} (x_{ki} - x_{kj})}_{\Delta}$

is antisymmetric

Also $f \in \Lambda$ because all denominators are gone.

$\Rightarrow \Delta \mid f \cdot \Delta$ in Λ

$\Rightarrow f \in \Gamma$!

This shows $U(gl_n)$ is a Galois order.

(II) $G \leq GL(V)$ finite refl. grp.

$\Lambda = S(V^*)$

$\mathcal{M} = V^*$ acting by translations

$\mathcal{L} = (\operatorname{Frac} S(V^*)) \rtimes V^*$

$\mathcal{K} = \mathcal{L}^G$

$\chi : G \to \Bbbk^\times$ linear character

$N \subset S(V^*) \rtimes V^*$ f.g. left $S(V^*)^G$-submod.

$T_\chi : S(V^*) \rtimes V^* \longrightarrow \mathcal{K}$

$$T_\chi(X) = \frac{1}{d_\chi} \sum_{g \in G} \chi(g)\, g(X)$$

<u>Thm</u> $T_\chi(N)$ is a Galois order in $\mathcal{K}$.

(III) $\Lambda = \mathbb{Z}$ $G = 1$ $\mathcal{M} = \mathbb{N}$

$K = L = \mathbb{Q}$,

$\mathcal{K} = \mathcal{L} = \mathbb{Q} * \mathbb{N} \cong \mathbb{Q}[t]$

$\mathcal{U} = \langle p(t) = a_0 + a_1 t + a_2 t^2 + \dots \rangle$

$a_1 \neq 0 \Rightarrow \mathcal{U}$ Galois ring.

When is $\mathcal{U}$ a Galois order?

Need $p(\mathbb{Z}) \subset \mathbb{Z}$

<u>Exercise</u> Show that

$$\{p(t) \in \mathbb{Q}[t] \mid p(\mathbb{Z}) \subset \mathbb{Z}\} =$$

$$= \bigoplus_{n \geq 0} \mathbb{Z}\, t^{(n)}, \quad \text{where}$$

$$t^{(n)} = \frac{t(t-1)\cdots(t-n+1)}{n!}$$