Global Weyl modules for non-standard maximal parabolics of twisted affine Lie algebras

Matthew Lee

Department of Mathematics
University of California, Riverside

Representation Theory Seminar September 7, 2017

- For a simple Lie algebra $\mathfrak{g} \supset \mathfrak{h}$
- $\Delta = \{\alpha_i : i \in I\}$
- $\Phi^+ = \{\sum_{i \in I} a_i \alpha_i : a_i \geq 0 \, \forall i\}$
- $\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha = \mathfrak{h} \oplus \mathfrak{n}^+$
- P^+ dominant integral weights

Universal highest modules for families of representations

	$\mathfrak{b}$ borel	$\mathfrak{p}$ parabolic

s.s.	• Verma module
	  • Module for the borel
		• Parabolic Verma Module
		  • Module for the parabolic

affine	• Global Weyl module • Bimodule	• ???

Universal highest modules for families of representations

	$\mathfrak{b}$ borel	$\mathfrak{p}$ parabolic

s.s.	• Verma module
	  • Module for the borel
		• Parabolic Verma Module
		  • Module for the parabolic

affine	• Global Weyl module • Bimodule	• ???

Universal highest modules for families of representations

	$\mathfrak{b}$ borel	$\mathfrak{p}$ parabolic

s.s.	• Verma module
	  • Module for the borel
		• Parabolic Verma Module
		  • Module for the parabolic

affine	• Global Weyl module • Bimodule	• ???

Universal highest modules for families of representations

	$\mathfrak{b}$ borel	$\mathfrak{p}$ parabolic

s.s.	• Verma module
	  • Module for the borel
		• Parabolic Verma Module
		  • Module for the parabolic

affine	• Global Weyl module • Bimodule	• ???

Universal highest modules for families of representations

	$\mathfrak{b}$ borel	$\mathfrak{p}$ parabolic

s.s.	• Verma module
	  ◦ Module for the borel
		• Parabolic Verma Module
		  ◦ Module for the parabolic

affine	• Global Weyl module ◦ Bimodule	• ???

Universal highest modules for families of representations

	$\mathfrak{b}$ borel	$\mathfrak{p}$ parabolic

s.s.	• Verma module
	  ◦ Module for the borel
		• Parabolic Verma Module
		  ◦ Module for the parabolic

affine	• Global Weyl module ◦ Bimodule	• ???

Universal highest modules for families of representations

	$\mathfrak{b}$ borel	$\mathfrak{p}$ parabolic

s.s.	• Verma module
	  • Module for the borel
		• Parabolic Verma Module
		  • Module for the parabolic

affine	• Global Weyl module • Bimodule	• ???

- s.s. Lie algebra ($\mathcal{O}$)
  - $\mathfrak{n}^+$ locally finite
  - Verma modules: $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$
  - Projective
- Parabolic Category $\mathcal{O}$ ($\mathcal{O}^\mathfrak{p}$)
  - Locally $\mathfrak{u}_\mathfrak{l}$-finite
  - Parabolic Verma modules: $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} L_\mathfrak{p}(\lambda)$
  - Projective

- s.s. Lie algebra ($\mathcal{O}$)
  - $\mathfrak{n}^+$ locally finite
  - Verma modules: $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$
  - Projective
- Parabolic Category $\mathcal{O}$ ($\mathcal{O}^{\mathfrak{p}}$)
  - Locally $\mathfrak{u}_{\mathfrak{l}}$-finite
  - Parabolic Verma modules: $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} L_{\mathfrak{p}}(\lambda)$
  - Projective

- s.s. Lie algebra ($\mathcal{O}$)
  - $\mathfrak{n}^+$ locally finite
  - Verma modules: $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$
  - Projective
- Parabolic Category $\mathcal{O}$ ($\mathcal{O}^{\mathfrak{p}}$)
  - Locally $\mathfrak{u}_{\mathfrak{l}}$-finite
  - Parabolic Verma modules: $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} L_{\mathfrak{p}}(\lambda)$
  - Projective

- s.s. Lie algebra ($\mathcal{O}$)
  - $\mathfrak{n}^+$ locally finite
  - Verma modules: $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$
  - Projective
- Parabolic Category $\mathcal{O}$ ($\mathcal{O}^{\mathfrak{p}}$)
  - Locally $\mathfrak{u}_{\mathfrak{l}}$-finite
  - Parabolic Verma modules: $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} L_{\mathfrak{p}}(\lambda)$
  - Projective

- s.s. Lie algebra ($\mathcal{O}$)
  - $\mathfrak{n}^+$ locally finite
  - Verma modules: $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$
  - Projective
- Parabolic Category $\mathcal{O}$ ($\mathcal{O}^{\mathfrak{p}}$)
  - Locally $\mathfrak{u}_{\mathfrak{l}}$-finite
  - Parabolic Verma modules: $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} L_{\mathfrak{p}}(\lambda)$
  - Projective

- s.s. Lie algebra ($\mathcal{O}$)
  - $\mathfrak{n}^+$ locally finite
  - Verma modules: $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$
  - Projective
- Parabolic Category $\mathcal{O}$ ($\mathcal{O}^\mathfrak{p}$)
  - Locally $\mathfrak{u}_\mathfrak{l}$-finite
  - Parabolic Verma modules: $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} L_\mathfrak{p}(\lambda)$
  - Projective

- s.s. Lie algebra ($\mathcal{O}$)
  - $\mathfrak{n}^+$ locally finite
  - Verma modules: $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$
  - Projective
- Parabolic Category $\mathcal{O}$ ($\mathcal{O}^\mathfrak{p}$)
  - Locally $\mathfrak{u}_\mathfrak{l}$-finite
  - Parabolic Verma modules: $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} L_\mathfrak{p}(\lambda)$
  - Projective

- s.s. Lie algebra ($\mathcal{O}$)
  - $\mathfrak{n}^+$ locally finite
  - Verma modules: $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$
  - Projective
- Parabolic Category $\mathcal{O}$ ($\mathcal{O}^{\mathfrak{p}}$)
  - Locally $\mathfrak{u}_{\mathfrak{l}}$-finite
  - Parabolic Verma modules: $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} L_{\mathfrak{p}}(\lambda)$
  - Projective

- s.s. Lie algebra ($\mathcal{O}$)
  - $\mathfrak{n}^+$ locally finite
  - Verma modules: $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$
  - Projective
- Parabolic Category $\mathcal{O}$ ($\mathcal{O}^\mathfrak{p}$)
  - Locally $\mathfrak{u}_\mathfrak{l}$-finite
  - Parabolic Verma modules: $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} L_\mathfrak{p}(\lambda)$
  - Projective

- s.s. Lie algebra ($\mathcal{O}$)
  - $\mathfrak{n}^+$ locally finite
  - Verma modules: $U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$
  - Projective
- Parabolic Category $\mathcal{O}$ ($\mathcal{O}^{\mathfrak{p}}$)
  - Locally $\mathfrak{u}_{\mathfrak{l}}$-finite
  - Parabolic Verma modules: $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} L_{\mathfrak{p}}(\lambda)$
  - Projective

Affine Lie algebras

- Integrability replaces locally finite

Proposition (Chari–Pressley, 2001)

- *Let V be an integrable $U(\mathfrak{g}[t,t^{-1}])$-module generated by a non-zero element $v \in V_\lambda^+$. Then V is a quotient of $W(\lambda)$.*
- *Let V be finite-dimensional $U(\mathfrak{g})$-module generated by a vector $v \in V_\lambda^+$ and such that $\dim V_\lambda = 1$. Then V is a quotient of $W(\pi)$*

- $W(\lambda)$ is a $(U(\mathfrak{g}[t,t^{-1}]), \mathbf{A}_\lambda)$-bimodule

Affine Lie algebras

- Integrability replaces locally finite

Proposition (Chari–Pressley, 2001)

- *Let V be an integrable $U(\mathfrak{g}[t, t^{-1}])$-module generated by a non-zero element $v \in V_\lambda^+$. Then V is a quotient of $W(\lambda)$.*
- *Let V be finite-dimensional $U(\mathfrak{g})$-module generated by a vector $v \in V_\lambda^+$ and such that $\dim V_\lambda = 1$. Then V is a quotient of $W(\pi)$*

- $W(\lambda)$ is a $(U(\mathfrak{g}[t, t^{-1}]), \mathbf{A}_\lambda)$-bimodule

Affine Lie algebras

- Integrability replaces locally finite

Proposition (Chari–Pressley, 2001)

- *Let V be an integrable $U(\mathfrak{g}[t,t^{-1}])$-module generated by a non-zero element $v \in V_\lambda^+$. Then V is a quotient of $W(\lambda)$.*
- *Let V be finite-dimensional $U(\mathfrak{g})$-module generated by a vector $v \in V_\lambda^+$ and such that $\dim V_\lambda = 1$. Then V is a quotient of $W(\pi)$*

- $W(\lambda)$ is a $(U(\mathfrak{g}[t,t^{-1}]), \mathbf{A}_\lambda)$-bimodule

Affine Lie algebras

- Integrability replaces locally finite

Proposition (Chari–Pressley, 2001)

- *Let V be an integrable $U(\mathfrak{g}[t, t^{-1}])$-module generated by a non-zero element $v \in V_\lambda^+$. Then V is a quotient of $W(\lambda)$.*
- *Let V be finite-dimensional $U(\mathfrak{g})$-module generated by a vector $v \in V_\lambda^+$ and such that $\dim V_\lambda = 1$. Then V is a quotient of $W(\pi)$*

- $W(\lambda)$ is a $(U(\mathfrak{g}[t, t^{-1}]), \mathbf{A}_\lambda)$-bimodule

- Global Weyl module, $W(\lambda)$, is cyclic generated by w_λ with the relations
  - $h.w_\lambda = \lambda(h)w_\lambda, \quad \mathfrak{n}^+[t,t^{-1}].w_\lambda = 0, \quad (x_i^- \otimes 1)^{\lambda(h_i)+1}.w_\lambda = 0$
- Projective module, not projective as an $\mathbf{A}_\lambda$-module
- Local Weyl modules, $W(\pi)$, for twisted loop algebras were studied by Chari, Fourier, Senesi (2007)
- Global Weyl modules, $W(\lambda)$, for twisted loop algebras were studied by Fourier, Manning, Senesi (2011)

- Global Weyl module, $W(\lambda)$, is cyclic generated by w_λ with the relations
  - $h.w_\lambda = \lambda(h)w_\lambda, \quad \mathfrak{n}^+[t,t^{-1}].w_\lambda = 0, \quad (x_i^- \otimes 1)^{\lambda(h_i)+1}.w_\lambda = 0$
- Projective module, not projective as an $\mathbf{A}_\lambda$-module
- Local Weyl modules, $W(\pi)$, for twisted loop algebras were studied by Chari, Fourier, Senesi (2007)
- Global Weyl modules, $W(\lambda)$, for twisted loop algebras were studied by Fourier, Manning, Senesi (2011)

- Global Weyl module, $W(\lambda)$, is cyclic generated by w_λ with the relations
  - $h.w_\lambda = \lambda(h)w_\lambda, \quad \mathfrak{n}^+[t, t^{-1}].w_\lambda = 0, \quad (x_i^- \otimes 1)^{\lambda(h_i)+1}.w_\lambda = 0$
- Projective module, not projective as an $\mathbf{A}_\lambda$-module
- Local Weyl modules, $W(\pi)$, for twisted loop algebras were studied by Chari, Fourier, Senesi (2007)
- Global Weyl modules, $W(\lambda)$, for twisted loop algebras were studied by Fourier, Manning, Senesi (2011)

- Global Weyl module, $W(\lambda)$, is cyclic generated by w_λ with the relations
  - $h.w_\lambda = \lambda(h)w_\lambda, \quad \mathfrak{n}^+[t, t^{-1}].w_\lambda = 0, \quad (x_i^- \otimes 1)^{\lambda(h_i)+1}.w_\lambda = 0$
- Projective module, not projective as an $\mathbf{A}_\lambda$-module
- Local Weyl modules, $W(\pi)$, for twisted loop algebras were studied by Chari, Fourier, Senesi (2007)
- Global Weyl modules, $W(\lambda)$, for twisted loop algebras were studied by Fourier, Manning, Senesi (2011)

Background
Notation

- $\mathfrak{g}$ simply laced of type A_{2n-1}, D_n or E_6
- δ basic imaginary root
- If $\alpha \in \Phi^+$, then we can write $\alpha = \sum_{i \in I} r_i \alpha_i$ where $r_i \in \mathbb{Z}_+$
  - Define $\mathbf{a}_i(\alpha) = r_i$
- Current algebra $\mathfrak{g}[t] := \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t]$
- σ Dynkin diagram automorphism order k
  - ξ a k^{th} root of unity

Background
Notation

- $\mathfrak{g}$ simply laced of type A_{2n-1}, D_n or E_6
- δ basic imaginary root
- If $\alpha \in \Phi^+$, then we can write $\alpha = \sum_{i \in I} r_i \alpha_i$ where $r_i \in \mathbb{Z}_+$
    - Define $\mathbf{a}_i(\alpha) = r_i$
- Current algebra $\mathfrak{g}[t] := \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t]$
- σ Dynkin diagram automorphism order k
    - ξ a kth root of unity

Background
Notation

- $\mathfrak{g}$ simply laced of type A_{2n-1}, D_n or E_6
- δ basic imaginary root
- If $\alpha \in \Phi^+$, then we can write $\alpha = \sum_{i \in I} r_i \alpha_i$ where $r_i \in \mathbb{Z}_+$
  - Define $\mathbf{a}_i(\alpha) = r_i$
- Current algebra $\mathfrak{g}[t] := \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t]$
- σ Dynkin diagram automorphism order k
  - ξ a kth root of unity

Background
Notation

- $\mathfrak{g}$ simply laced of type A_{2n-1}, D_n or E_6
- δ basic imaginary root
- If $\alpha \in \Phi^+$, then we can write $\alpha = \sum_{i \in I} r_i \alpha_i$ where $r_i \in \mathbb{Z}_+$
    - Define $\mathbf{a}_i(\alpha) = r_i$
- Current algebra $\mathfrak{g}[t] := \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t]$
- σ Dynkin diagram automorphism order k
    - ξ a kth root of unity

Background
Notation

- $\mathfrak{g}$ simply laced of type A_{2n-1}, D_n or E_6
- δ basic imaginary root
- If $\alpha \in \Phi^+$, then we can write $\alpha = \sum_{i \in I} r_i \alpha_i$ where $r_i \in \mathbb{Z}_+$
  - Define $\mathbf{a}_i(\alpha) = r_i$
- Current algebra $\mathfrak{g}[t] := \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t]$
- σ Dynkin diagram automorphism order k
  - ξ a kth root of unity

Background
Notation

- $\mathfrak{g}$ simply laced of type A_{2n-1}, D_n or E_6
- δ basic imaginary root
- If $\alpha \in \Phi^+$, then we can write $\alpha = \sum_{i \in I} r_i \alpha_i$ where $r_i \in \mathbb{Z}_+$
  - Define $\mathbf{a}_i(\alpha) = r_i$
- Current algebra $\mathfrak{g}[t] := \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t]$
- σ Dynkin diagram automorphism order k
  - ξ a kth root of unity

Background
Notation

- $\mathfrak{g}$ simply laced of type A_{2n-1}, D_n or E_6
- δ basic imaginary root
- If $\alpha \in \Phi^+$, then we can write $\alpha = \sum_{i \in I} r_i \alpha_i$ where $r_i \in \mathbb{Z}_+$
  - Define $\mathbf{a}_i(\alpha) = r_i$
- Current algebra $\mathfrak{g}[t] := \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t]$
- σ Dynkin diagram automorphism order k
  - ξ a kth root of unity

Background
Notation

- $\mathfrak{g}$ simply laced of type A_{2n-1}, D_n or E_6
- δ basic imaginary root
- If $\alpha \in \Phi^+$, then we can write $\alpha = \sum_{i \in I} r_i \alpha_i$ where $r_i \in \mathbb{Z}_+$
  - Define $\mathbf{a}_i(\alpha) = r_i$
- Current algebra $\mathfrak{g}[t] := \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[t]$
- σ Dynkin diagram automorphism order k
  - ξ a k^{th} root of unity

Maximal parabolic

- $\mathfrak{g} = \bigoplus_{s=0}^{k-1} \mathfrak{g}_s$, $\mathfrak{g}_0$ is simple, and $\mathfrak{g}_s$ are irreducible $\mathfrak{g}_0$-modules
- For D_4, $\mathfrak{g}_1 \simeq \mathfrak{g}_2$ and for $m \neq 0$, $\mathfrak{g}_m \simeq V(\theta_s)$
- Define $\sigma : \mathbb{C}[t] \to \mathbb{C}[t]$ by $\sigma(f(t)) = f(\xi^{-1} t)$
- $\mathfrak{g}[t]^\sigma = \bigoplus_{s=1}^{k} \mathfrak{g}_s \otimes t^s \mathbb{C}[t^k]$
- $\{x_i^\pm : 1 \le i \le n_0\} \in \Delta_0$ and $x_0^\pm \in V(\theta_s)$
    - Similarly, define P_0^+

Maximal parabolic

- $\mathfrak{g} = \bigoplus_{s=0}^{k-1} \mathfrak{g}_s$, $\mathfrak{g}_0$ is simple, and $\mathfrak{g}_s$ are irreducible $\mathfrak{g}_0$-modules
- For D_4, $\mathfrak{g}_1 \simeq \mathfrak{g}_2$ and for $m \neq 0$, $\mathfrak{g}_m \simeq V(\theta_s)$
- Define $\sigma : \mathbb{C}[t] \to \mathbb{C}[t]$ by $\sigma(f(t)) = f(\xi^{-1}t)$
- $\mathfrak{g}[t]^\sigma = \bigoplus_{s=1}^{k} \mathfrak{g}_s \otimes t^s\mathbb{C}[t^k]$
- $\{x_i^\pm : 1 \leq i \leq n_0\} \in \Delta_0$ and $x_0^\pm \in V(\theta_s)$
  - Similarly, define P_0^+

Maximal parabolic

- $\mathfrak{g} = \bigoplus_{s=0}^{k-1} \mathfrak{g}_s$, $\mathfrak{g}_0$ is simple, and $\mathfrak{g}_s$ are irreducible $\mathfrak{g}_0$-modules
- For D_4, $\mathfrak{g}_1 \simeq \mathfrak{g}_2$ and for $m \neq 0$, $\mathfrak{g}_m \simeq V(\theta_s)$
- Define $\sigma : \mathbb{C}[t] \to \mathbb{C}[t]$ by $\sigma(f(t)) = f(\xi^{-1} t)$
- $\mathfrak{g}[t]^\sigma = \bigoplus_{s=1}^{k} \mathfrak{g}_s \otimes t^s \mathbb{C}[t^k]$
- $\{x_i^\pm : 1 \leq i \leq n_0\} \in \Delta_0$ and $x_0^\pm \in V(\theta_s)$
    - Similarly, define P_0^+

Maximal parabolic

- $\mathfrak{g} = \bigoplus_{s=0}^{k-1} \mathfrak{g}_s$, $\mathfrak{g}_0$ is simple, and $\mathfrak{g}_s$ are irreducible $\mathfrak{g}_0$-modules
- For D_4, $\mathfrak{g}_1 \simeq \mathfrak{g}_2$ and for $m \neq 0$, $\mathfrak{g}_m \simeq V(\theta_s)$
- Define $\sigma : \mathbb{C}[t] \to \mathbb{C}[t]$ by $\sigma(f(t)) = f(\xi^{-1} t)$
- $\mathfrak{g}[t]^\sigma = \bigoplus_{s=1}^{k} \mathfrak{g}_s \otimes t^s \mathbb{C}[t^k]$
- $\{x_i^\pm : 1 \leq i \leq n_0\} \in \Delta_0$ and $x_0^\pm \in V(\theta_s)$
  - Similarly, define P_0^+

Maximal parabolic

- $\mathfrak{g} = \bigoplus_{s=0}^{k-1} \mathfrak{g}_s$, $\mathfrak{g}_0$ is simple, and $\mathfrak{g}_s$ are irreducible $\mathfrak{g}_0$-modules
- For D_4, $\mathfrak{g}_1 \simeq \mathfrak{g}_2$ and for $m \neq 0$, $\mathfrak{g}_m \simeq V(\theta_s)$
- Define $\sigma : \mathbb{C}[t] \to \mathbb{C}[t]$ by $\sigma(f(t)) = f(\xi^{-1}t)$
- $\mathfrak{g}[t]^\sigma = \bigoplus_{s=1}^{k} \mathfrak{g}_s \otimes t^s\mathbb{C}[t^k]$
- $\{x_i^\pm : 1 \leq i \leq n_0\} \in \Delta_0$ and $x_0^\pm \in V(\theta_s)$
  - Similarly, define P_0^+

Maximal parabolic

- $\mathfrak{g} = \bigoplus_{s=0}^{k-1} \mathfrak{g}_s$, $\mathfrak{g}_0$ is simple, and $\mathfrak{g}_s$ are irreducible $\mathfrak{g}_0$-modules
- For D_4, $\mathfrak{g}_1 \simeq \mathfrak{g}_2$ and for $m \neq 0$, $\mathfrak{g}_m \simeq V(\theta_s)$
- Define $\sigma : \mathbb{C}[t] \to \mathbb{C}[t]$ by $\sigma(f(t)) = f(\xi^{-1} t)$
- $\mathfrak{g}[t]^\sigma = \bigoplus_{s=1}^{k} \mathfrak{g}_s \otimes t^s \mathbb{C}[t^k]$
- $\{x_i^\pm : 1 \leq i \leq n_0\} \in \Delta_0$ and $x_0^\pm \in V(\theta_s)$
  - Similarly, define P_0^+

Maximal parabolic

- $\mathfrak{g} = \bigoplus_{s=0}^{k-1} \mathfrak{g}_s$, $\mathfrak{g}_0$ is simple, and $\mathfrak{g}_s$ are irreducible $\mathfrak{g}_0$-modules
- For D_4, $\mathfrak{g}_1 \simeq \mathfrak{g}_2$ and for $m \neq 0$, $\mathfrak{g}_m \simeq V(\theta_s)$
- Define $\sigma : \mathbb{C}[t] \to \mathbb{C}[t]$ by $\sigma(f(t)) = f(\xi^{-1} t)$
- $\mathfrak{g}[t]^\sigma = \bigoplus_{s=1}^{k} \mathfrak{g}_s \otimes t^s \mathbb{C}[t^k]$
- $\{x_i^\pm : 1 \le i \le n_0\} \in \Delta_0$ and $x_0^\pm \in V(\theta_s)$
  - Similarly, define P_0^+

Maximal parabolic

- $\mathfrak{g} = \bigoplus_{s=0}^{k-1} \mathfrak{g}_s$, $\mathfrak{g}_0$ is simple, and $\mathfrak{g}_s$ are irreducible $\mathfrak{g}_0$-modules
- For D_4, $\mathfrak{g}_1 \simeq \mathfrak{g}_2$ and for $m \neq 0$, $\mathfrak{g}_m \simeq V(\theta_s)$
- Define $\sigma : \mathbb{C}[t] \to \mathbb{C}[t]$ by $\sigma(f(t)) = f(\xi^{-1} t)$
- $\mathfrak{g}[t]^\sigma = \bigoplus_{s=1}^{k} \mathfrak{g}_s \otimes t^s \mathbb{C}[t^k]$
- $\{x_i^\pm : 1 \leq i \leq n_0\} \in \Delta_0$ and $x_0^\pm \in V(\theta_s)$
  - Similarly, define P_0^+

Maximal parabolic

- $\mathfrak{g} = \bigoplus_{s=0}^{k-1} \mathfrak{g}_s$, $\mathfrak{g}_0$ is simple, and $\mathfrak{g}_s$ are irreducible $\mathfrak{g}_0$-modules
- For D_4, $\mathfrak{g}_1 \simeq \mathfrak{g}_2$ and for $m \neq 0$, $\mathfrak{g}_m \simeq V(\theta_s)$
- Define $\sigma : \mathbb{C}[t] \to \mathbb{C}[t]$ by $\sigma(f(t)) = f(\xi^{-1}t)$
- $\mathfrak{g}[t]^\sigma = \bigoplus_{s=1}^{k} \mathfrak{g}_s \otimes t^s\mathbb{C}[t^k]$
- $\{x_i^\pm : 1 \leq i \leq n_0\} \in \Delta_0$ and $x_0^\pm \in V(\theta_s)$
  - Similarly, define P_0^+

Maximal parabolic

- $\mathfrak{g} = \bigoplus_{s=0}^{k-1} \mathfrak{g}_s$, $\mathfrak{g}_0$ is simple, and $\mathfrak{g}_s$ are irreducible $\mathfrak{g}_0$-modules
- For D_4, $\mathfrak{g}_1 \simeq \mathfrak{g}_2$ and for $m \neq 0$, $\mathfrak{g}_m \simeq V(\theta_s)$
- Define $\sigma : \mathbb{C}[t] \to \mathbb{C}[t]$ by $\sigma(f(t)) = f(\xi^{-1}t)$
- $\mathfrak{g}[t]^\sigma = \bigoplus_{s=1}^{k} \mathfrak{g}_s \otimes t^s\mathbb{C}[t^k]$
- $\{x_i^\pm : 1 \leq i \leq n_0\} \in \Delta_0$ and $x_0^\pm \in V(\theta_s)$
  - Similarly, define P_0^+

Realization

Kac–Moody	Realization
$\widehat{\mathfrak{g}} =< e_i, f_i : 0 \leq i \leq n >/$ rel'ns	$\mathfrak{g}[t, t^{-1}]^\sigma \oplus \mathbb{C}c \oplus \mathbb{C}d$
$\tilde{\mathfrak{g}} = [\widehat{\mathfrak{g}}, \widehat{\mathfrak{g}}]/\mathbb{C}c$	Let η be a $\mathbf{a}_j(\delta) \cdot k^{th}$ root of unity
Maximal parabolic: $\widetilde{\mathfrak{p}}_j$	$\tau : \mathfrak{g}^\sigma \simeq \mathfrak{g}^\sigma$ by $x_i^\pm \mapsto \eta^{\pm\delta_{i,j}} x_i^\pm$
$\mathrm{gr}(e_j) = 1$	$\tau : \mathbb{C}[t] \simeq \mathbb{C}[t]$ by $f(t) \mapsto f(\eta^{-1} t)$

$\widetilde{\mathfrak{p}}_j \simeq$ subalgebra of $\mathfrak{g}[t, t^{-1}]^\sigma \simeq \mathfrak{g}[t]^{\sigma\tau}$

Pick j such that $\mathbf{a}_j(\delta) > 1$

Realization

Kac–Moody	Realization
$\widehat{\mathfrak{g}} =< e_i, f_i : 0 \leq i \leq n >/$ rel'ns	$\mathfrak{g}[t, t^{-1}]^\sigma \oplus \mathbb{C}c \oplus \mathbb{C}d$
$\tilde{\mathfrak{g}} = [\widehat{\mathfrak{g}}, \widehat{\mathfrak{g}}]/\mathbb{C}c$	Let η be a $\mathbf{a}_j(\delta) \cdot k^{th}$ root of unity
Maximal parabolic: $\widetilde{\mathfrak{p}_j}$	$\tau : \mathfrak{g}^\sigma \simeq \mathfrak{g}^\sigma$ by $x_i^\pm \mapsto \eta^{\pm\delta_{i,j}} x_i^\pm$
$\mathrm{gr}(e_j) = 1$	$\tau : \mathbb{C}[t] \simeq \mathbb{C}[t]$ by $f(t) \mapsto f(\eta^{-1} t)$

$\widetilde{\mathfrak{p}_j} \simeq$ subalgebra of $\mathfrak{g}[t, t^{-1}]^\sigma \simeq \mathfrak{g}[t]^{\sigma\tau}$

Pick j such that $\mathbf{a}_j(\delta) > 1$

Realization

Kac–Moody	Realization
$\widehat{\mathfrak{g}} =< e_i, f_i : 0 \le i \le n >/$ rel'ns	$\mathfrak{g}[t, t^{-1}]^\sigma \oplus \mathbb{C}c \oplus \mathbb{C}d$
$\tilde{\mathfrak{g}} = [\widehat{\mathfrak{g}}, \widehat{\mathfrak{g}}]/\mathbb{C}c$	Let η be a $\mathbf{a}_j(\delta) \cdot k^{th}$ root of unity
Maximal parabolic: $\widetilde{\mathfrak{p}}_j$	$\tau : \mathfrak{g}^\sigma \simeq \mathfrak{g}^\sigma$ by $x_i^\pm \mapsto \eta^{\pm\delta_{i,j}} x_i^\pm$
$\mathrm{gr}(e_j) = 1$	$\tau : \mathbb{C}[t] \simeq \mathbb{C}[t]$ by $f(t) \mapsto f(\eta^{-1} t)$

$\widetilde{\mathfrak{p}}_j \simeq$ subalgebra of $\mathfrak{g}[t, t^{-1}]^\sigma \simeq \mathfrak{g}[t]^{\sigma\tau}$

Pick j such that $\mathbf{a}_j(\delta) > 1$

Realization

Kac–Moody	Realization
$\widehat{\mathfrak{g}} =< e_i, f_i : 0 \leq i \leq n >/$ rel'ns	$\mathfrak{g}[t, t^{-1}]^\sigma \oplus \mathbb{C}c \oplus \mathbb{C}d$
$\tilde{\mathfrak{g}} = [\widehat{\mathfrak{g}}, \widehat{\mathfrak{g}}]/\mathbb{C}c$	Let η be a $\mathbf{a}_j(\delta) \cdot k^{th}$ root of unity
Maximal parabolic: $\widetilde{\mathfrak{p}}_j$	$\tau : \mathfrak{g}^\sigma \simeq \mathfrak{g}^\sigma$ by $x_i^\pm \mapsto \eta^{\pm\delta_{i,j}} x_i^\pm$
$\quad \mathrm{gr}(e_j) = 1$	$\tau : \mathbb{C}[t] \simeq \mathbb{C}[t]$ by $f(t) \mapsto f(\eta^{-1}t)$

$\widetilde{\mathfrak{p}}_j \simeq$ subalgebra of $\mathfrak{g}[t, t^{-1}]^\sigma \simeq \mathfrak{g}[t]^{\sigma\tau}$

Pick j such that $\mathbf{a}_j(\delta) > 1$

Realization

Kac–Moody	Realization
$\widehat{\mathfrak{g}} =< e_i, f_i : 0 \leq i \leq n >/$ rel'ns	$\mathfrak{g}[t, t^{-1}]^\sigma \oplus \mathbb{C}c \oplus \mathbb{C}d$
$\tilde{\mathfrak{g}} = [\widehat{\mathfrak{g}}, \widehat{\mathfrak{g}}]/\mathbb{C}c$	Let η be a $\mathbf{a}_j(\delta) \cdot k^{th}$ root of unity
Maximal parabolic: $\widetilde{\mathfrak{p}}_j$	$\tau : \mathfrak{g}^\sigma \simeq \mathfrak{g}^\sigma$ by $x_i^\pm \mapsto \eta^{\pm\delta_{i,j}} x_i^\pm$
$\mathrm{gr}(e_j) = 1$	$\tau : \mathbb{C}[t] \simeq \mathbb{C}[t]$ by $f(t) \mapsto f(\eta^{-1}t)$

$\widetilde{\mathfrak{p}}_j \simeq$ subalgebra of $\mathfrak{g}[t, t^{-1}]^\sigma \simeq \mathfrak{g}[t]^{\sigma\tau}$

Pick j such that $\mathbf{a}_j(\delta) > 1$

Realization

Kac–Moody	Realization
$\widehat{\mathfrak{g}} =< e_i, f_i : 0 \leq i \leq n >/$ rel'ns	$\mathfrak{g}[t, t^{-1}]^\sigma \oplus \mathbb{C}c \oplus \mathbb{C}d$
$\tilde{\mathfrak{g}} = [\widehat{\mathfrak{g}}, \widehat{\mathfrak{g}}]/\mathbb{C}c$	Let η be a $\mathbf{a}_j(\delta) \cdot k^{th}$ root of unity
Maximal parabolic: $\widetilde{\mathfrak{p}}_j$	$\tau : \mathfrak{g}^\sigma \simeq \mathfrak{g}^\sigma$ by $x_i^\pm \mapsto \eta^{\pm\delta_{i,j}} x_i^\pm$
$\mathrm{gr}(e_j) = 1$	$\tau : \mathbb{C}[t] \simeq \mathbb{C}[t]$ by $f(t) \mapsto f(\eta^{-1}t)$

$\widetilde{\mathfrak{p}}_j \simeq$ subalgebra of $\mathfrak{g}[t, t^{-1}]^\sigma \simeq \mathfrak{g}[t]^{\sigma\tau}$

Pick j such that $\mathbf{a}_j(\delta) > 1$

Realization

Kac–Moody	Realization
$\widehat{\mathfrak{g}} =< e_i, f_i : 0 \leq i \leq n >/$ rel'ns	$\mathfrak{g}[t, t^{-1}]^\sigma \oplus \mathbb{C}c \oplus \mathbb{C}d$
$\tilde{\mathfrak{g}} = [\widehat{\mathfrak{g}}, \widehat{\mathfrak{g}}]/\mathbb{C}c$	Let η be a $\mathbf{a}_j(\delta) \cdot k^{th}$ root of unity
Maximal parabolic: $\widetilde{\mathfrak{p}}_j$	$\tau : \mathfrak{g}^\sigma \simeq \mathfrak{g}^\sigma$ by $x_i^\pm \mapsto \eta^{\pm\delta_{i,j}} x_i^\pm$
$\mathrm{gr}(e_j) = 1$	$\tau : \mathbb{C}[t] \simeq \mathbb{C}[t]$ by $f(t) \mapsto f(\eta^{-1} t)$

$\widetilde{\mathfrak{p}}_j \simeq$ subalgebra of $\mathfrak{g}[t, t^{-1}]^\sigma \simeq \mathfrak{g}[t]^{\sigma\tau}$

Pick j such that $\mathbf{a}_j(\delta) > 1$

Realization

Kac–Moody	Realization
$\widehat{\mathfrak{g}} =< e_i, f_i : 0 \leq i \leq n >/$ rel'ns	$\mathfrak{g}[t, t^{-1}]^\sigma \oplus \mathbb{C}c \oplus \mathbb{C}d$
$\tilde{\mathfrak{g}} = [\widehat{\mathfrak{g}}, \widehat{\mathfrak{g}}]/\mathbb{C}c$	Let η be a $\mathbf{a}_j(\delta) \cdot k^{th}$ root of unity
Maximal parabolic: $\widetilde{\mathfrak{p}}_j$	$\tau : \mathfrak{g}^\sigma \simeq \mathfrak{g}^\sigma$ by $x_i^\pm \mapsto \eta^{\pm\delta_{i,j}} x_i^\pm$
$\mathrm{gr}(e_j) = 1$	$\tau : \mathbb{C}[t] \simeq \mathbb{C}[t]$ by $f(t) \mapsto f(\eta^{-1} t)$

$\widetilde{\mathfrak{p}}_j \simeq$ subalgebra of $\mathfrak{g}[t, t^{-1}]^\sigma \simeq \mathfrak{g}[t]^{\sigma\tau}$

Pick j such that $\mathbf{a}_j(\delta) > 1$

Realization

Kac–Moody	Realization
$\widehat{\mathfrak{g}} =< e_i, f_i : 0 \leq i \leq n >/$ rel'ns	$\mathfrak{g}[t, t^{-1}]^\sigma \oplus \mathbb{C}c \oplus \mathbb{C}d$
$\tilde{\mathfrak{g}} = [\widehat{\mathfrak{g}}, \widehat{\mathfrak{g}}]/\mathbb{C}c$	Let η be a $\mathbf{a}_j(\delta) \cdot k^{th}$ root of unity
Maximal parabolic: $\widetilde{\mathfrak{p}}_j$	$\tau : \mathfrak{g}^\sigma \simeq \mathfrak{g}^\sigma$ by $x_i^\pm \mapsto \eta^{\pm\delta_{i,j}} x_i^\pm$
$\mathrm{gr}(e_j) = 1$	$\tau : \mathbb{C}[t] \simeq \mathbb{C}[t]$ by $f(t) \mapsto f(\eta^{-1} t)$

$\widetilde{\mathfrak{p}}_j \simeq$ subalgebra of $\mathfrak{g}[t, t^{-1}]^\sigma \simeq \mathfrak{g}[t]^{\sigma\tau}$

Pick j such that $\mathbf{a}_j(\delta) > 1$

Realization

Kac–Moody	Realization
$\widehat{\mathfrak{g}} =< e_i, f_i : 0 \leq i \leq n >/$ rel'ns	$\mathfrak{g}[t, t^{-1}]^\sigma \oplus \mathbb{C}c \oplus \mathbb{C}d$
$\tilde{\mathfrak{g}} = [\widehat{\mathfrak{g}}, \widehat{\mathfrak{g}}]/\mathbb{C}c$	Let η be a $\mathbf{a}_j(\delta) \cdot k^{th}$ root of unity
Maximal parabolic: $\widetilde{\mathfrak{p}}_j$	$\tau : \mathfrak{g}^\sigma \simeq \mathfrak{g}^\sigma$ by $x_i^\pm \mapsto \eta^{\pm\delta_{i,j}} x_i^\pm$
$\mathrm{gr}(e_j) = 1$	$\tau : \mathbb{C}[t] \simeq \mathbb{C}[t]$ by $f(t) \mapsto f(\eta^{-1} t)$

$\widetilde{\mathfrak{p}}_j \simeq$ subalgebra of $\mathfrak{g}[t, t^{-1}]^\sigma \simeq \mathfrak{g}[t]^{\sigma\tau}$

Pick j such that $\mathbf{a}_j(\delta) > 1$

Realization

Kac–Moody	Realization
$\widehat{\mathfrak{g}} =< e_i, f_i : 0 \leq i \leq n >/$ rel'ns	$\mathfrak{g}[t, t^{-1}]^\sigma \oplus \mathbb{C}c \oplus \mathbb{C}d$
$\tilde{\mathfrak{g}} = [\widehat{\mathfrak{g}}, \widehat{\mathfrak{g}}]/\mathbb{C}c$	Let η be a $\mathbf{a}_j(\delta) \cdot k^{th}$ root of unity
Maximal parabolic: $\widetilde{\mathfrak{p}}_j$	$\tau : \mathfrak{g}^\sigma \simeq \mathfrak{g}^\sigma$ by $x_i^\pm \mapsto \eta^{\pm\delta_{i,j}} x_i^\pm$
$\mathrm{gr}(e_j) = 1$	$\tau : \mathbb{C}[t] \simeq \mathbb{C}[t]$ by $f(t) \mapsto f(\eta^{-1}t)$

$\widetilde{\mathfrak{p}}_j \simeq$ subalgebra of $\mathfrak{g}[t, t^{-1}]^\sigma \simeq \mathfrak{g}[t]^{\sigma\tau}$

Pick j such that $\mathbf{a}_j(\delta) > 1$

Global Weyl Module

- For $\lambda \in P_0^+$, $W(\lambda)$ is generated by w_λ with relations:

$$h.w_\lambda = \lambda(h)w_\lambda \quad \mathfrak{n}^+[t]^{\sigma\tau}.w_\lambda = 0, \quad (x_i^- \otimes 1)^{\lambda(h_i)+1}.w_\lambda = 0.$$

- $W(\lambda)$ is a $\mathbb{Z}_+$-graded $\mathfrak{g}[t]^{\sigma\tau}$-module.
- $W(\lambda)$ is irreducible iff $\lambda(h_0) = 0$ or $\lambda(h_i) = 0$ $\forall i \in I \setminus \{j\}$

Global Weyl Module

- For $\lambda \in P_0^+$, $W(\lambda)$ is generated by w_λ with relations:

$$h.w_\lambda = \lambda(h)w_\lambda \quad \mathfrak{n}^+[t]^{\sigma\tau}.w_\lambda = 0, \quad (x_i^- \otimes 1)^{\lambda(h_i)+1}.w_\lambda = 0.$$

- $W(\lambda)$ is a $\mathbb{Z}_+$-graded $\mathfrak{g}[t]^{\sigma\tau}$-module.
- $W(\lambda)$ is irreducible iff $\lambda(h_0) = 0$ or $\lambda(h_i) = 0 \ \forall i \in I \setminus \{j\}$

Global Weyl Module

- For $\lambda \in P_0^+$, $W(\lambda)$ is generated by w_λ with relations:

$$h.w_\lambda = \lambda(h)w_\lambda \quad \mathfrak{n}^+[t]^{\sigma\tau}.w_\lambda = 0, \quad (x_i^- \otimes 1)^{\lambda(h_i)+1}.w_\lambda = 0.$$

- $W(\lambda)$ is a $\mathbb{Z}_+$-graded $\mathfrak{g}[t]^{\sigma\tau}$-module.
- $W(\lambda)$ is irreducible iff $\lambda(h_0) = 0$ or $\lambda(h_i) = 0 \; \forall i \in I \setminus \{j\}$

Global Weyl Module

- For $\lambda \in P_0^+$, $W(\lambda)$ is generated by w_λ with relations:

$$h.w_\lambda = \lambda(h)w_\lambda \quad \mathfrak{n}^+[t]^{\sigma\tau}.w_\lambda = 0, \quad (x_i^- \otimes 1)^{\lambda(h_i)+1}.w_\lambda = 0.$$

- $W(\lambda)$ is a $\mathbb{Z}_+$-graded $\mathfrak{g}[t]^{\sigma\tau}$-module.
- $W(\lambda)$ is irreducible iff $\lambda(h_0) = 0$ or $\lambda(h_i) = 0 \ \forall i \in I \setminus \{j\}$

Global Weyl Module

- For $\lambda \in P_0^+$, $W(\lambda)$ is generated by w_λ with relations:

$$h.w_\lambda = \lambda(h)w_\lambda \quad \mathfrak{n}^+[t]^{\sigma\tau}.w_\lambda = 0, \quad (x_i^- \otimes 1)^{\lambda(h_i)+1}.w_\lambda = 0.$$

- $W(\lambda)$ is a $\mathbb{Z}_+$-graded $\mathfrak{g}[t]^{\sigma\tau}$-module.

- $W(\lambda)$ is irreducible iff $\lambda(h_0) = 0$ or $\lambda(h_i) = 0\ \forall i \in I\setminus\{j\}$

- For $\lambda \in P_0^+$, $W(\lambda)$ is a $(\mathbf{U}(\mathfrak{g}[t]^{\sigma\tau}), \mathbf{A}_\lambda)$-bimodule.
  - $\mathbf{A}_\lambda = \mathbf{U}(\mathfrak{h}[t]^{\sigma\tau})/\mathrm{Ann}_{\mathbf{U}(\mathfrak{h}[t]^{\sigma\tau})} w_\lambda$
- To better description of $W(\lambda)$ we need to describe $\mathbf{A}_\lambda$
- Define $P_{i,r}$ by
  - $(x_i^+ \otimes t)^{(r)}(x_i^- \otimes 1)^{(r)}.w_\lambda = P_{i,r}.w_\lambda$
- The $\{P_{i,r}\}$ also satisfy: $P_{i,0} = 1$,

$$P_{i,r} = -\frac{1}{r}\sum_{p=1}^{r}\left[\left(\sum_{\epsilon=0}^{k-1} h_{i,\epsilon} \otimes t^{pk-\epsilon}\right) P_{i,r-p}\right], \quad r \geq 1.$$

- For $\lambda \in P_0^+$, $W(\lambda)$ is a $(\mathbf{U}(\mathfrak{g}[t]^{\sigma\tau}), \mathbf{A}_\lambda)$-bimodule.
  - $\mathbf{A}_\lambda = \mathbf{U}(\mathfrak{h}[t]^{\sigma\tau})/\mathrm{Ann}_{\mathbf{U}(\mathfrak{h}[t]^{\sigma\tau})} w_\lambda$
- To better description of $W(\lambda)$ we need to describe $\mathbf{A}_\lambda$
- Define $P_{i,r}$ by
  - $(x_i^+ \otimes t)^{(r)}(x_i^- \otimes 1)^{(r)}.w_\lambda = P_{i,r}.w_\lambda$
- The $\{P_{i,r}\}$ also satisfy: $P_{i,0} = 1$,

$$P_{i,r} = -\frac{1}{r}\sum_{p=1}^{r}\left[\left(\sum_{\epsilon=0}^{k-1} h_{i,\epsilon} \otimes t^{pk-\epsilon}\right) P_{i,r-p}\right], \quad r \geq 1.$$

- For $\lambda \in P_0^+$, $W(\lambda)$ is a $(\mathbf{U}(\mathfrak{g}[t]^{\sigma\tau}), \mathbf{A}_\lambda)$-bimodule.
  - $\mathbf{A}_\lambda = \mathbf{U}(\mathfrak{h}[t]^{\sigma\tau})/\mathrm{Ann}_{\mathbf{U}(\mathfrak{h}[t]^{\sigma\tau})} w_\lambda$
- To better description of $W(\lambda)$ we need to describe $\mathbf{A}_\lambda$
- Define $P_{i,r}$ by
  - $(x_i^+ \otimes t)^{(r)}(x_i^- \otimes 1)^{(r)}.w_\lambda = P_{i,r}.w_\lambda$
- The $\{P_{i,r}\}$ also satisfy: $P_{i,0} = 1$,

$$P_{i,r} = -\frac{1}{r}\sum_{p=1}^{r}\left[\left(\sum_{\epsilon=0}^{k-1} h_{i,\epsilon} \otimes t^{pk-\epsilon}\right) P_{i,r-p}\right], \quad r \geq 1.$$

- For $\lambda \in P_0^+$, $W(\lambda)$ is a $(\mathbf{U}(\mathfrak{g}[t]^{\sigma\tau}), \mathbf{A}_\lambda)$-bimodule.
  - $\mathbf{A}_\lambda = \mathbf{U}(\mathfrak{h}[t]^{\sigma\tau}) / \mathrm{Ann}_{\mathbf{U}(\mathfrak{h}[t]^{\sigma\tau})} w_\lambda$
- To better description of $W(\lambda)$ we need to describe $\mathbf{A}_\lambda$
- Define $P_{i,r}$ by
  - $(x_i^+ \otimes t)^{(r)} (x_i^- \otimes 1)^{(r)}.w_\lambda = P_{i,r}.w_\lambda$
- The $\{P_{i,r}\}$ also satisfy: $P_{i,0} = 1$,

$$P_{i,r} = -\frac{1}{r} \sum_{p=1}^{r} \left[\left(\sum_{\epsilon=0}^{k-1} h_{i,\epsilon} \otimes t^{pk-\epsilon} \right) P_{i,r-p} \right], \quad r \geq 1.$$

- For $\lambda \in P_0^+$, $W(\lambda)$ is a $(\mathbf{U}(\mathfrak{g}[t]^{\sigma\tau}), \mathbf{A}_\lambda)$-bimodule.
  - $\mathbf{A}_\lambda = \mathbf{U}(\mathfrak{h}[t]^{\sigma\tau})/\mathrm{Ann}_{\mathbf{U}(\mathfrak{h}[t]^{\sigma\tau})} w_\lambda$
- To better description of $W(\lambda)$ we need to describe $\mathbf{A}_\lambda$
- Define $P_{i,r}$ by
  - $(x_i^+ \otimes t)^{(r)}(x_i^- \otimes 1)^{(r)}.w_\lambda = P_{i,r}.w_\lambda$
- The $\{P_{i,r}\}$ also satisfy: $P_{i,0} = 1$,

$$P_{i,r} = -\frac{1}{r}\sum_{p=1}^{r}\left[\left(\sum_{\epsilon=0}^{k-1} h_{i,\epsilon} \otimes t^{pk-\epsilon}\right) P_{i,r-p}\right], \quad r \geq 1.$$

- For $\lambda \in P_0^+$, $W(\lambda)$ is a $(\mathbf{U}(\mathfrak{g}[t]^{\sigma\tau}), \mathbf{A}_\lambda)$-bimodule.
  - $\mathbf{A}_\lambda = \mathbf{U}(\mathfrak{h}[t]^{\sigma\tau})/\mathrm{Ann}_{\mathbf{U}(\mathfrak{h}[t]^{\sigma\tau})} w_\lambda$
- To better description of $W(\lambda)$ we need to describe $\mathbf{A}_\lambda$
- Define $P_{i,r}$ by
  - $(x_i^+ \otimes t)^{(r)}(x_i^- \otimes 1)^{(r)}.w_\lambda = P_{i,r}.w_\lambda$
- The $\{P_{i,r}\}$ also satisfy: $P_{i,0} = 1$,

$$P_{i,r} = -\frac{1}{r}\sum_{p=1}^{r}\left[\left(\sum_{\epsilon=0}^{k-1} h_{i,\epsilon} \otimes t^{pk-\epsilon}\right) P_{i,r-p}\right], \quad r \geq 1.$$

- For $\lambda \in P_0^+$, $W(\lambda)$ is a $(\mathbf{U}(\mathfrak{g}[t]^{\sigma\tau}), \mathbf{A}_\lambda)$-bimodule.
    - $\mathbf{A}_\lambda = \mathbf{U}(\mathfrak{h}[t]^{\sigma\tau}) / \mathrm{Ann}_{\mathbf{U}(\mathfrak{h}[t]^{\sigma\tau})} w_\lambda$
- To better description of $W(\lambda)$ we need to describe $\mathbf{A}_\lambda$
- Define $P_{i,r}$ by
    - $(x_i^+ \otimes t)^{(r)} (x_i^- \otimes 1)^{(r)}.w_\lambda = P_{i,r}.w_\lambda$
- The $\{P_{i,r}\}$ also satisfy: $P_{i,0} = 1$,

$$P_{i,r} = -\frac{1}{r} \sum_{p=1}^{r} \left[\left(\sum_{\epsilon=0}^{k-1} h_{i,\epsilon} \otimes t^{pk-\epsilon} \right) P_{i,r-p} \right], \quad r \geq 1.$$

$\mathbf{A}_\lambda$

- $\mathbf{A}_\lambda / \operatorname{Jac}(\mathbf{A}_\lambda)$:
  - Generators: $\{P_{i,r_i} : r_i \leq \min\{\lambda(h_i), \lambda(h_0)\}\}$
  - Relations: $P_{1,r_1} \cdots P_{n,r_n}.w_\lambda = 0$ for $\sum_{i=1}^{n} \mathbf{a}_i(\alpha_0) r_i \geq \lambda(h_0) + 1$
- If $\mathfrak{g}$ is of type D_n, $n \neq 4$, then $\operatorname{Jac}(\mathbf{A}_\lambda) = 0$.
- $\mathbf{A}_\lambda / \operatorname{Jac}(\mathbf{A}_\lambda)$ is one-dimensional or infinite-dimensional
- The following are equivalent:
  1. $W(\lambda)$ is finite-dimensional
  2. $\mathbf{A}_\lambda$ is finite-dimensional
  3. $\mathbf{A}_\lambda$ is a local ring

$\mathbf{A}_\lambda$

- $\mathbf{A}_\lambda / \operatorname{Jac}(\mathbf{A}_\lambda)$:
  - Generators: $\{P_{i,r_i} : r_i \leq \min\{\lambda(h_i), \lambda(h_0)\}\}$
  - Relations: $P_{1,r_1} \cdots P_{n,r_n}.w_\lambda = 0$ for $\sum_{i=1}^{n} \mathbf{a}_i(\alpha_0) r_i \geq \lambda(h_0) + 1$
- If $\mathfrak{g}$ is of type D_n, $n \neq 4$, then $\operatorname{Jac}(\mathbf{A}_\lambda) = 0$.
- $\mathbf{A}_\lambda / \operatorname{Jac}(\mathbf{A}_\lambda)$ is one-dimensional or infinite-dimensional
- The following are equivalent:
  1. $W(\lambda)$ is finite-dimensional
  2. $\mathbf{A}_\lambda$ is finite-dimensional
  3. $\mathbf{A}_\lambda$ is a local ring

$\mathbf{A}_\lambda$

- $\mathbf{A}_\lambda / \operatorname{Jac}(\mathbf{A}_\lambda)$:
  - Generators: $\{P_{i,r_i} : r_i \leq \min\{\lambda(h_i), \lambda(h_0)\}\}$
  - Relations: $P_{1,r_1} \cdots P_{n,r_n}.w_\lambda = 0$ for $\sum_{i=1}^{n} \mathbf{a}_i(\alpha_0) r_i \geq \lambda(h_0) + 1$
- If $\mathfrak{g}$ is of type D_n, $n \neq 4$, then $\operatorname{Jac}(\mathbf{A}_\lambda) = 0$.
- $\mathbf{A}_\lambda / \operatorname{Jac}(\mathbf{A}_\lambda)$ is one-dimensional or infinite-dimensional
- The following are equivalent:
  1. $W(\lambda)$ is finite-dimensional
  2. $\mathbf{A}_\lambda$ is finite-dimensional
  3. $\mathbf{A}_\lambda$ is a local ring

$\mathbf{A}_\lambda$

- $\mathbf{A}_\lambda / \operatorname{Jac}(\mathbf{A}_\lambda)$:
  - Generators: $\{P_{i,r_i} : r_i \leq \min\{\lambda(h_i), \lambda(h_0)\}\}$
  - Relations: $P_{1,r_1} \cdots P_{n,r_n}.w_\lambda = 0$ for $\sum_{i=1}^{n} \mathbf{a}_i(\alpha_0) r_i \geq \lambda(h_0) + 1$
- If $\mathfrak{g}$ is of type D_n, $n \neq 4$, then $\operatorname{Jac}(\mathbf{A}_\lambda) = 0$.
- $\mathbf{A}_\lambda / \operatorname{Jac}(\mathbf{A}_\lambda)$ is one-dimensional or infinite-dimensional
- The following are equivalent:
  1. $W(\lambda)$ is finite-dimensional
  2. $\mathbf{A}_\lambda$ is finite-dimensional
  3. $\mathbf{A}_\lambda$ is a local ring

$\mathbf{A}_\lambda$

- $\mathbf{A}_\lambda / \operatorname{Jac}(\mathbf{A}_\lambda)$:
  - Generators: $\{P_{i,r_i} : r_i \leq \min\{\lambda(h_i), \lambda(h_0)\}\}$
  - Relations: $P_{1,r_1} \cdots P_{n,r_n}.w_\lambda = 0$ for $\sum_{i=1}^{n} \mathbf{a}_i(\alpha_0) r_i \geq \lambda(h_0) + 1$
- If $\mathfrak{g}$ is of type D_n, $n \neq 4$, then $\operatorname{Jac}(\mathbf{A}_\lambda) = 0$.
- $\mathbf{A}_\lambda / \operatorname{Jac}(\mathbf{A}_\lambda)$ is one-dimensional or infinite-dimensional
- The following are equivalent:
  1. $W(\lambda)$ is finite-dimensional
  2. $\mathbf{A}_\lambda$ is finite-dimensional
  3. $\mathbf{A}_\lambda$ is a local ring

$\mathbf{A}_\lambda$

- $\mathbf{A}_\lambda / \operatorname{Jac}(\mathbf{A}_\lambda)$:
  - Generators: $\{P_{i,r_i} : r_i \leq \min\{\lambda(h_i), \lambda(h_0)\}\}$
  - Relations: $P_{1,r_1} \cdots P_{n,r_n}.w_\lambda = 0$ for $\sum_{i=1}^{n} \mathbf{a}_i(\alpha_0) r_i \geq \lambda(h_0) + 1$
- If $\mathfrak{g}$ is of type D_n, $n \neq 4$, then $\operatorname{Jac}(\mathbf{A}_\lambda) = 0$.
- $\mathbf{A}_\lambda / \operatorname{Jac}(\mathbf{A}_\lambda)$ is one-dimensional or infinite-dimensional
- The following are equivalent:
  1. $W(\lambda)$ is finite-dimensional
  2. $\mathbf{A}_\lambda$ is finite-dimensional
  3. $\mathbf{A}_\lambda$ is a local ring

$\mathbf{A}_\lambda$

- $\mathbf{A}_\lambda / \operatorname{Jac}(\mathbf{A}_\lambda)$:
  - Generators: $\{P_{i,r_i} : r_i \leq \min\{\lambda(h_i), \lambda(h_0)\}\}$
  - Relations: $P_{1,r_1} \cdots P_{n,r_n}.w_\lambda = 0$ for $\sum_{i=1}^{n} \mathbf{a}_i(\alpha_0) r_i \geq \lambda(h_0) + 1$
- If $\mathfrak{g}$ is of type D_n, $n \neq 4$, then $\operatorname{Jac}(\mathbf{A}_\lambda) = 0$.
- $\mathbf{A}_\lambda / \operatorname{Jac}(\mathbf{A}_\lambda)$ is one-dimensional or infinite-dimensional
- The following are equivalent:
  1. $W(\lambda)$ is finite-dimensional
  2. $\mathbf{A}_\lambda$ is finite-dimensional
  3. $\mathbf{A}_\lambda$ is a local ring

$\mathbf{A}_\lambda$

- $\mathbf{A}_\lambda / \operatorname{Jac}(\mathbf{A}_\lambda)$:
  - Generators: $\{P_{i,r_i} : r_i \leq \min\{\lambda(h_i), \lambda(h_0)\}\}$
  - Relations: $P_{1,r_1} \cdots P_{n,r_n}.w_\lambda = 0$ for $\sum_{i=1}^{n} \mathbf{a}_i(\alpha_0) r_i \geq \lambda(h_0) + 1$
- If $\mathfrak{g}$ is of type D_n, $n \neq 4$, then $\operatorname{Jac}(\mathbf{A}_\lambda) = 0$.
- $\mathbf{A}_\lambda / \operatorname{Jac}(\mathbf{A}_\lambda)$ is one-dimensional or infinite-dimensional
- The following are equivalent:
  1. $W(\lambda)$ is finite-dimensional
  2. $\mathbf{A}_\lambda$ is finite-dimensional
  3. $\mathbf{A}_\lambda$ is a local ring

$\mathbf{A}_\lambda$

- $\mathbf{A}_\lambda / \operatorname{Jac}(\mathbf{A}_\lambda)$:
  - Generators: $\{P_{i,r_i} : r_i \leq \min\{\lambda(h_i), \lambda(h_0)\}\}$
  - Relations: $P_{1,r_1} \cdots P_{n,r_n}.w_\lambda = 0$ for $\sum_{i=1}^{n} \mathbf{a}_i(\alpha_0) r_i \geq \lambda(h_0) + 1$
- If $\mathfrak{g}$ is of type D_n, $n \neq 4$, then $\operatorname{Jac}(\mathbf{A}_\lambda) = 0$.
- $\mathbf{A}_\lambda / \operatorname{Jac}(\mathbf{A}_\lambda)$ is one-dimensional or infinite-dimensional
- The following are equivalent:
  1. $W(\lambda)$ is finite-dimensional
  2. $\mathbf{A}_\lambda$ is finite-dimensional
  3. $\mathbf{A}_\lambda$ is a local ring

Future Work

- Produce examples of finite dimensional Global Weyl Modules
- Find a finite dimensionality condition for Local Weyl Module

Future Work

- Produce examples of finite dimensional Global Weyl Modules

- Find a finite dimensionality condition for Local Weyl Module

Thank you for your time.

References I

J.E. Humpheys
Introduction to Lie Algebras and Representation Theory.
Springer, 1968.

V. Chari and A. Pressley
Weyl Modules for classical and quantum affine algebras
Representation Theory 5 (2001), 191-223 (electronic)

G. Fourier, N. Manning, and P. Senesi
Global Weyl modules for the twisted loop algebra.
Abh. Math. Semin. Univ. Hamb., 83(1):533-82, 2013

References II

V. Chari and G. Fourier, and P. Senesi
Weyl modules for the twisted loop algebras
J. Algebra, 319(12):5016-5038, 2008

G Fourier and D. Kus
Demazure modules and Weyl modules: The twisted current case.
Trans. Amer. Math. Soc., 365(11):6037-6064, 2013