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[Reference: Chari-Greenstein Advances, 2007]

The current algebra of $\mathfrak{sl}_2$ and its representations.

Def

Lie algebra: $(\mathfrak{g}, [\cdot,\cdot])$

- $\mathfrak{g}$ vector space
- $[\cdot,\cdot] : \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$ the bracket on $\mathfrak{g}$

(i) $[\cdot,\cdot]$ is bilinear

(ii) $[x,y] = -[y,x]$ $\forall x,y \in \mathfrak{g}$

(iii) $[x[yz]] + [y[zx]] + [z[xy]] = 0$

Example $\mathfrak{g} = \mathfrak{sl}_2 = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \,\middle|\, a+d=0 \right\}$
with bracket
$$[x,y] = x\cdot y - y\cdot x \quad \forall x,y \in \mathfrak{sl}_2.$$

Basis for $\mathfrak{sl}_2$:
$$\left\{ e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$$

So $\dim \mathfrak{sl}_2 = 3$.

Relations:

$$[e,f]=h \ , [h,e]=2e \ , [h,f]=-2f$$

<u>Def</u> The current algebra of a Lie algebra $\mathfrak{g}$ is defined as

$$\mathfrak{g}[t] = \left\{ \sum_{i=0}^{n} a_i t^i \,/\, a_i \in \mathfrak{g} \right\}$$

with bracket determined by

$$[x t^m, y t^n] = [x,y]\, t^{m+n}$$

$\forall x, y \in \mathfrak{g}$, $m, n \in \mathbb{N}$.

<u>Note</u>

1) $\dim \mathfrak{g}[t] = \infty$ for any Lie algebra $\mathfrak{g} \neq 0$.

2) $\mathfrak{g}[t]$ is not simple, i.e. it has nonzero proper <u>ideals</u>:

$$\mathfrak{a}_n = \mathfrak{g}[t] \cdot t^n = \left\{ \sum_{i=n}^{N} a_i t^i \,/\, a_i \in \mathfrak{g} \right\}$$

Then $[\mathfrak{g}[t], \mathfrak{a}_n] \subseteq \mathfrak{a}_n$.

Modules over Lie algebras

Def A $\mathfrak{g}$-module V is a vec.sp. with a bilinear map

$$\mathfrak{g} \times V \to V$$
$$(x, v) \mapsto x.v$$

such that

$$[x,y].v = x.(y.v) - y.(x.v)$$

$\forall x, y \in \mathfrak{g}$ $\forall v \in V$.

Example 1) $V = \left\{ \begin{bmatrix} a \\ b \end{bmatrix} \right\}$ is an sl_2-module via

$$x.v = x \cdot v \quad \text{(matrix mult.)}$$

2) sl_2 is an sl_2-module via
$$x.y = [x,y].$$

3) $V_d = \text{Span}\{ x_1^a x_2^b \mid a+b = d-1 \}$ is a d-dimensional sl_2-module via

$$\left(\sum_{i,j=1}^{2} \lambda_{ij} E_{ij}\right). f = \sum \lambda_{ij} x_i \frac{\partial}{\partial x_j}(f)$$

where $E_{ij} = \begin{array}{cc} \begin{bmatrix} & & \\ & 1 & \\ & & \end{bmatrix} & i \\ j & \end{array}$ matrix units.

Def A $\mathfrak{g}$-module V is simple if the only subspaces $U \subseteq V$ with $\mathfrak{g}.U \subseteq U$ are $\{0\}$ and V.

Ex The modules V_d exhaust all simple f.d. sl_2-modules (up to isomorphisms).

Weyl's Theorem: Any f.d. module over a simple f.d. Lie alg $\mathfrak{g}$ is isomorphic to a direct sum of simple $\mathfrak{g}$-modules.

Ex Any f.d. sl_2-module is isomorphic to $V_{d_1} \oplus V_{d_2} \oplus \cdots \oplus V_{d_k}$, some $d_k \geq 1$.

Question What are the f.d. $sl_2[t]$-modules? (Open problem!)

Def A $\mathfrak{g}[t]$-module V is graded if $V = \bigoplus_{n \in \mathbb{N}} V_n$ and $x t^n . V_m \subseteq V_{m+n}$ for all $x \in \mathfrak{g}$, $n, m \in \mathbb{N}$

V is graded-simple if the only graded submodules are $\{0\}$ and V.

Motivation

Main object:

$U_q(\hat{\mathfrak{g}})$ quantum affine algebras and their modules

$$q \to 1 \rightsquigarrow$$

$\hat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$ affine Kac-Moody alg

$$\text{res} \rightsquigarrow$$

$\mathfrak{g}[t]$ (standard parabolic in $\hat{\mathfrak{g}}$)

Every $\hat{\mathfrak{g}}$-module is a graded $\mathfrak{g}[t]$-module, via restriction.

<u>Theorem</u> The only finite-dimensional graded-simple $sl_2[t]$-modules are the modules $V_d \cdot t^n$ for $n \in \mathbb{N}$. with action

$$xt^m \cdot vt^n = \begin{cases} 0 & m \neq 0 \\ (x.v)t^n, & m = 0 \end{cases}$$

<u>Problem</u>: Weyl's thm fails for $sl_2[t]$.

So direct sums of graded-simple $sl_2[t]$-modules do not give all graded $sl_2[t]$-modules.

Technically,

$\mathrm{Ext}^1(V_a t^r, V_b t^s) = 0$ if $s \neq r+1$ but

$\mathrm{Ext}^1(V_a t^r, V_b t^{r+1}) \cong \mathrm{Hom}_{sl_2}(V_a, sl_2 \otimes V_b)$

$\mathcal{G}$ category of f.d. graded $\mathfrak{g}[t]$-modules.

Chari and Greenstein:

- $\mathcal{G}$ has enough injectives.
- character formula for injective hulls of simples
- Ext^1 between simples
- Simples $\longleftrightarrow P^+ \times \mathbb{Z}_{\geq 0} \subseteq$ affine wt lattice.
- Ext quiver $Q(\Gamma)$ for the algebra $\mathcal{A}(\Gamma) = \mathrm{End}(I_\Gamma)$ where Γ is an interval in $P^+ \times \mathbb{Z}_{\geq 0}$ and I_Γ is the injective cogenerator of the full subcategory of objects with simple subquotients from Γ.

<u>Example</u> 1) $\mathfrak{g} = sl_2$, $r_i \geq 0$, $|r_k - r_{k+1}| = 1$

$\Gamma = \{(\lambda, r_0), (\lambda+\alpha, r_1), \ldots, (\lambda + \ell\alpha, r_\ell)\}$

Then

$$Q(\Gamma): \quad \underset{1}{\bullet} - \underset{2}{\bullet} - \bullet \cdots \bullet - \underset{\ell}{\bullet} \qquad \text{type } A_{\ell+1}$$

and $\mathcal{A}(\Gamma)$ is hereditary.

2) $g = sp_4$, $\theta = \alpha + 2\beta$

$$\begin{tikzcd}
 & (2\theta - \beta, r+1) \arrow[d] & \\
(0, r+1) \arrow[r] & (\theta, r) & (2\theta, r+1) \arrow[l] \\
 & (\theta, r+1) \arrow[u] &
\end{tikzcd}$$