1. Lattices

1.1. **Lattices.** A $\mathbb{Z}$-*lattice* (or simply a lattice) L of rank n is a $\mathbb{Z}$-module of rank n with a symmetric bilinear form (also called inner product) $\langle\ ,\ \rangle : L \times L \to \mathbb{Q}$. Unless otherwise stated, we shall assume that the inner product is non-degenerate, that is, $\langle x, y\rangle = 0$ for all y implies $x = 0$.

If $\langle x, y\rangle \in \mathbb{Z}$ for all $x, y \in L$, then we say that L is *integral*. We can always think of L as sitting inside a vector space V of dimension n as a discrete subgroup. Let L be a lattice in a vector space V. The inner product extends linearly to V. Define the *dual lattice*

$$L^\vee = \{x \in V : \langle x, y\rangle \in \mathbb{Z} \quad \text{for all} \quad y \in \Gamma\}.$$

If L is integral then $L \subseteq L^\vee$. Say that an integral lattice L is *self dual* if $L = L^\vee$. Write $|x|^2 = \langle x, x\rangle$ and call it the *norm* of x. Let $L(n)$ denote the set of all lattice vectors in L of norm n. An integral lattice L is *even* if $|x|^2$ is an even integer for all $x \in L$.

Exercise: Show that every integral has an even sublattice of index at most 2.

The inner product matrix $((\langle e_i, e_j\rangle))$ of a basis $(e_1, \cdots, e_n)$ of L is called the *Gram matrix* of L and its determinant is denoted by $d(L)$.

Exercise: Show that if L is integral then $[L^\vee : L] = d(L)$.

In particular, L is self dual if and only if $d(L) = 1$. Say that L has *signature* (m, n) if the Gram matrix of L has m positive and n negative eigenvalues. Say that L is *positive definite* if L has signature $(n, 0)$ and that L is *Lorentzian* if L has signature $(n, 1)$ for some $n \geq 1$.

1.2. **Examples.** In the first two examples below, the inner product is the usual one on $\mathbb{R}^n$. All the lattices below are even. The subscripts usually denotes rank.

(1) Let A_n be the set of all $(x_0, \cdots, x_n) \in \mathbb{Z}^{n+1}$ such that $\sum_i x_i = 0$.
(2) Let D_n be the set of all $(x_1, \cdots, x_n) \in \mathbb{Z}^n$ such that $\sum_i x_i \equiv 0 \bmod 2$.
(3) Let m, n be natural numbers. Let $\mathbb{R}^{m,n}$ be the real vector space of dimension $(m+n)$ with the inner product of signature (m, n):

$$|x|^2 = x_1^2 + \cdots + x_m^2 - x_{m+1}^2 - \cdots - x_{m+n}^2$$

where $x = (x_1, \cdots, x_{m+n}) \in \mathbb{R}^{m,n}$. Let $m - n \equiv 0 \bmod 8$. Define

$$II_{m,n} = \{x \in \mathbb{R}^{m,n} : x_j \in \mathbb{Z} \ \text{ for all } \ j \ \text{ or } \ (x_j + \tfrac{1}{2}) \in \mathbb{Z} \ \text{ for all } \ j, \ \sum_j x_j \equiv 0 \bmod 2\}.$$

Then $II_{m,n}$ is an even self dual integral lattice whenever $(m - n) \equiv 0 \bmod 8$. In fact we have the following theorem: *Even self dual integral lattices of signature (m, n) exist only if $m - n \equiv 0 \bmod 8$. If m and n are both non-zero then there is a unique such lattice, namely $II_{m,n}$.* (see [Se])
(4) Let $E_8 = II_{8,0}$. Let E_7 be the orthogonal complement of any norm 2 vector vector in E_8. Let E_6 be the orthogonal complement of any copy of A_2 lattice sitting in E_8.

(5) Let $\rho = (0, 1, 2, 3, .., 24, 70) \in II_{25,1}$. Then ρ is a primitive null vector so $\rho \in \rho^{\perp}$ and $\Lambda := \rho^{\perp}/\rho\mathbb{Z}$ is a positive definite even self dual lattice of rank 24. This is the famous Leech lattice.

1.3. **Roots.** A *root* of L is a lattice vector s with $|s|^2 > 0$ such that reflection R_s in s is an isometry of L. Here R_s is the isometry of the vector space V that fixes $s^{\perp}$, called the *mirror* of reflection, and takes s to $-s$. A formula for R_s is

$$R_s(v) = v - 2\langle v, s\rangle s/|s|^2.$$

The *reflection group* $\mathrm{Ref}(L)$ of L is the subgroup of $\mathrm{Aut}(L)$ generated by the reflections in the roots of L. A *root lattice* is a lattice generated by its roots. The lattices A_n, D_n, E_6, E_7, E_8 above are root lattices.
Digression: In contrast Leech lattice has no roots. In fact the Leech lattice is the only positive definite self dual lattice of dimension < 32 that has no roots. So its reflection group is trivial. However the automorphism group $O(\Lambda)$ has size 8315553613086720000. It is one of the 26 sporadic finite simple groups in the classification theorem. Its discovery in the sixties lead to the discovery of quite a few more sporadic groups leading up to the largest sporadic finite simple group, the Monster, which has size almost 8×10^{53}.

1.4. **Root syetems of lattices:** A *(reduced) root system* Φ in an inner product space V is a nonempty spanning subset of V such that such that for all $s, s' \in \Phi$,we have $s\mathbb{R}\cap\Phi = \{s, -s\}$, $R_s(\Phi) = \Phi$ and $2\langle s, s'\rangle/\langle s, s\rangle \in \mathbb{Z}$. We say that Φ is *simply laced* if all the roots of L have norm 2.

Exercise: If L is an even integral lattice, then norm 2 vectors are roots of L and $L(2)$ is a simply laced root system.

In particular $A_n(2)$, $D_n(2)$, $E_6(2)$, $E_7(2)$, $E_8(2)$ are finite simply laced root systems. (finite because the lattices a positive definite). In fact: *any finite simply laced root system in $\mathbb{R}^n$ is isomorphic to an "orthogonal direct sum" of these ADE root systems.*

Exercise: Suppose u and v are two non-proportional roots in a positive definite simply laced root system Φ.
(a) Show that the angle between them is $\pi/3, \pi/2$ or $2\pi/3$.
(b) Conclude that if $\langle u, v\rangle > 0$, then in fact $\langle u, v\rangle = 2$ and $u - v = R_v(u)$ is a root.
(c) Show that the only rank 2 simply laced positive definite root systems are $A_1(2) \times A_1(2)$ and $A_2(2)$.

1.5. Here are some questions that might interest a person while thinking about lattices.

(1) What are the interesting lattices? This might mean:
   - what are the 26 dimensional lattices with $d(\Gamma) = 3$
   - what are the "positive definite root lattices"? We shall (almost) answer this one
   - what the the even self dual lattices? When the signature is indefinite, already know the answer. For positive definite lattices the story is a lot more interesting. For $m = 8$, there is a unique one $E_8 = II_{8,0}$. For $m = 16$, there are two: $E_8 \oplus E_8$ and $II_{16,0}$ (the Barne's-Wall lattice). The dimension $m = 24$ is ofcourse the

most interesting case. There are exactly 24 such lattices in dimension 24 called the Niemeier Lattices, the most famous among them the Leech lattice Λ which relates to all sorts of exotic objects like Golay codes, Mathiew groups, Conway groups, Monster.. (see [CS] for a lot more on this). In dimension 32 there are there are more than 80 million, these have not been classified

(2) What the the lattices with nice symmetry groups. Also interesting lattices give rise to interesting symmetry, for example, the weyl groups are symmetries of the root lattices, some sporadic finite simple groups come from lattices like $O(\Lambda)$, many interesting discrete subgroups acting on hyperbolic spaces appear as reflection groups of lattices...
(3) How to count the number of lattice points of a given norm? These investigation leads to theta functions and modular forms.
(4) Questions about lattices relate to many problems in :
   - number theory: theory of quadratic forms, since Gauss, representing integers as sums of squares, since Jacobi.
   - algebraic topology: the intersection forms on the middle cohomology, or the torsion part of cohomology..
   - sphere packing: For example D_4, E_8, Λ give the densest sphere packing in their dimension.
   - kissing number problem, covering problem, error correcting codes... (see [CS]). The leech lattice has kissing number 196560.

1.6**. From the root lattice to Dynkin diagram and back.** Let $\Phi \subseteq V$ be a simply laced positive definite root system with all roots of norm 2. Fix a linear functional $l : V \to \mathbb{R}$ that does not vanish on any root. Let $\Phi_+ = \{r \in \Phi \colon l(r) > 0\}$.; these are called the *positive* roots. A positive root is called *simple* if it cannot be written as a sum of two positive roots. Let Δ be a system of simple roots.

Exercise: Show that if $s, s' \in \Delta$ and $s \neq s'$ then $\langle s, s' \rangle \leq 0$.

Exercise: Each root can be written as a unique integer linear combination of simple roots with all coefficients of the same sign. In particular, Δ is a basis of V.

sketch of proof. The linear independence of simple roots is the following Euclidean geometry exercise: if all the angles between a set of vectors in Euclidean space obtuse then they are linearly independent. For suppose there was a linear dependence relation. Such a relation can be written in the from $\sum_i c_i s_i = \sum_j d_j s_j$ where the s_i's and s_j's are all distinct and c_i's and d_i's are all positive. Let $v = \sum_i c_i s_i$. Argue that $\langle v, v \rangle < 0$...

If possible, among the positive roots that cannot be written as positive integral linear combination of Δ, choose one, let's call it r, such that $l(r)$ is minimal. Then r is not simple so we can write $r = r_1 + r_2$ where both $r_1, r_2 \in \Phi_+$. Then $l(r_1), l(r_2)$ are strictly less than $l(r)$, so they can be written as positive integer linear combo of Δ. But then so can r... $\square$

Recall from the exercise above that the dot product of two distinct simple roots can be either 0 or -1. Make a graph whose vertex set is Δ. Two vertices s, s' are joined if and only if $\langle s, s' \rangle = -1$. This is the Dynkin diagram of Φ. So each simply laced root positive definite root lattice L or the corresponding root system $L(2)$ give us a Dynkin diagram. The root system is indecomposible if and only if the Dynkin diagram is connected. Now one has to classify the connected Dynkin diagrams. This is a pleasant combinatorial exercise once we know the affine diagrams and their magic numberings and hence know that they cannot appear in any Dynkin diagram for a positive definite root system. This produces the list A_n,D_n, E_6, E_7, E_8. So these are the only indecomposible positive definite simply laced root systems. All the other ones are obtained as orthogonal direct sums of these.

1.7**. The ubiquity of ADE's:** This ADE list occurs as the result of many related classification problem in mathematics. Some examples are Root lattices, Coxeter systems, finite type quivers, finite subgroups of $SU(2)$, Du-Val singularities, (the last two are connected via "McKay correspondence"), Simple complex Lie algebras... In case of Lie algebras, you only get the simply laced ones but the rest can be obtained by folding.

1.8**. A geometric description of the simple roots:** Let L be a root lattice with root system $\Phi = L(2)$. The reflection group $\mathrm{Ref}(L)$ is generated by the reflections in the roots Φ. Let $\mathcal{M}$ be the union of the mirrors of Φ. Choose a component W of $V - \mathcal{M}$. One can show that this is a fundamental domain for $\mathrm{Ref}(L)$. This is called a Weyl chamber. Choose the roots that are orthogonal to the walls of the Weyl chamber. These give a set of simple roots and their negatives. It follows from this description that the reflections in the simple roots generate $Ref(L)$. In the ADE examples, the group $Ref(L)$ is known as the Weyl group.

1.9**. Getting back the root lattice from the Dynkin diagram:** Let Δ be a simply laced Dynkin diagram. Let L be the free $\mathbb{Z}$ module with basis indexed by Δ and the inner product defined as follows: for all $s, s' \in \Delta$, we have $\langle s, s \rangle = 2$, and $\langle s, s' \rangle = -1$ if (s, s') is an edge of Δ and $\langle s, s' \rangle = 0$ otherwise. This recovers the root lattice L and then $L(2)$ recovers the root system.

References

[CS] J. H. Conway and N. J. A. Sloane, *Sphere packings, lattices and groups.*

[MH] J. Milnor and D.Husemoller, *Symmetric bilinear forms.*

[Mu] D. Mumford, *Tata lectures on theta (I).*

[Se] J. P. Serre, *A course in arithmetic.*