LIE THEORY SEMINAR Oct 21, 2016

Classification of fin. dim simple Lie algs/$\mathbb{C}$

$$\boxed{\begin{array}{c}\text{Fin-dim simple}\\ \text{Lie algs}/\mathbb{C}\\ \mathfrak{g}\end{array}} \xleftrightarrow{\ \text{I}\ } \boxed{\begin{array}{c}\text{Irreducible}\\ \text{root systems}\\ \Phi\end{array}} \xleftrightarrow{\ \text{II}\ } \boxed{\begin{array}{c}\text{Connected}\\ \text{Dynkin}\\ \text{diagrams}\end{array}}$$

<u>Def</u> A subset Φ of a Euclidean space E is a <u>root system</u> if

(R1) Φ is finite, spans E, $0 \notin E$

(R2) If $\alpha \in \Phi$, $(\mathbb{R}\alpha) \cap \Phi = \{\pm\alpha\}$

(R3) If $\alpha \in \Phi$, the reflection $\sigma_\alpha : E \to E$ leaves Φ invariant: If $\beta \in \Phi$ then

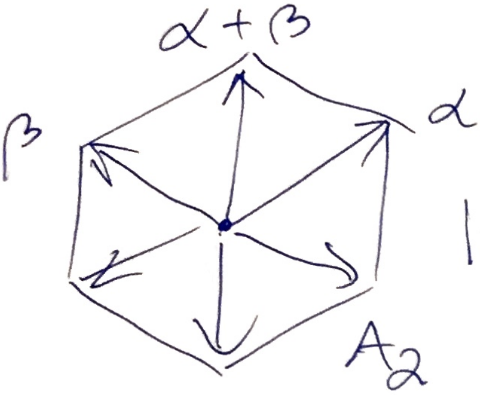
$$\sigma_\alpha(\beta) = \beta - 2\frac{(\beta,\alpha)}{(\alpha,\alpha)}\alpha \in \Phi$$

(R4) If $\alpha, \beta \in \Phi$ then $2\frac{(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$

<u>Def</u> The subgroup W of $GL(E)$ generated by $\{\sigma_\alpha \mid \alpha \in \Phi\}$ is the <u>Weyl group</u> of Φ.

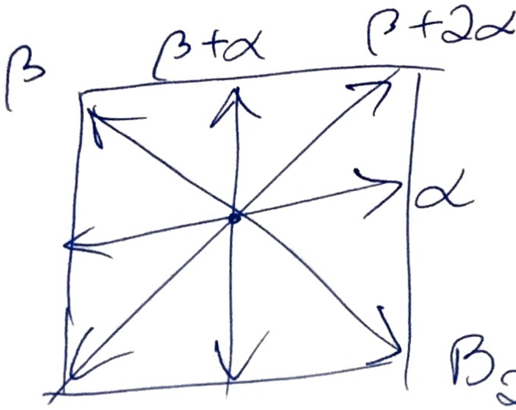
Examples

A_1 $\quad \Phi = \{\alpha, -\alpha\}$



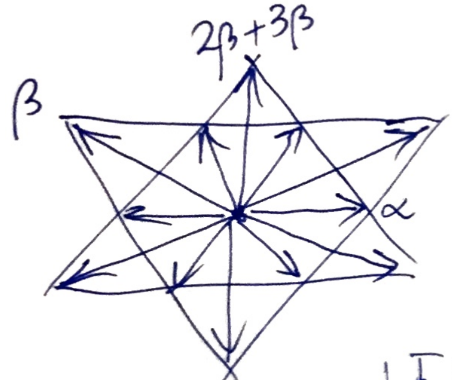
$|\Phi| = 6$

A_2



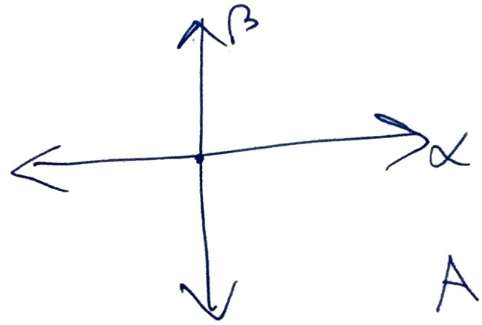
$B_2 \cong C_2$

$|\Phi| = 8$



$|\Phi| = 12$

G_2



$\Phi_1 \perp \Phi_2$

$\Phi = \Phi_1 \cup \Phi_2$

reducible

$A_1 \times A_1$

$\Phi = \{\pm\alpha, \pm\beta\}$

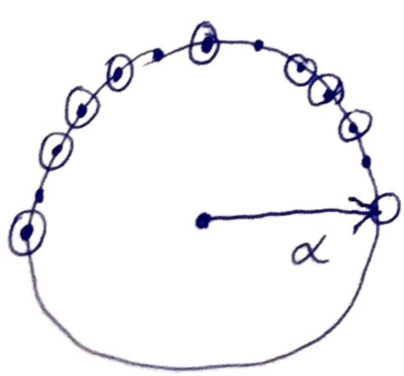
Possible angles between roots

$$\frac{2(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z} \implies 2\frac{\|\beta\|}{\|\alpha\|}\cos\theta \in \mathbb{Z}$$

$$\overset{\alpha\leftrightarrow\beta}{\implies} 4\cos^2\theta \in \mathbb{Z}$$

$$\implies \cos\theta = \pm\tfrac{1}{2}\sqrt{n}, \ n \in \mathbb{Z}^+$$

$$|\cos\theta| \le 1 \Rightarrow n = 0, 1, 2, 3, 4$$



I) $\qquad [x,y]=0 \; \forall x,y\in\mathfrak{h}$ $\quad$ Cartan subalg $\qquad$ 2

Thm $\exists$ abelian subalg $\mathfrak{h}\subseteq\mathfrak{g}$ s.t.

1) $\mathfrak{h}\subseteq h_1\subseteq\mathfrak{g}$, h_1 ab $\Rightarrow \mathfrak{h}=h_1$ (maximal ab.)

2) $\forall h\in\mathfrak{h}$ $\quad \mathrm{ad}\, h:\mathfrak{g}\to\mathfrak{g}$ is diagonalizable
$$x\mapsto[h,x]$$

$\boxed{\dim\mathfrak{h} =: \text{rank } \mathfrak{g}}$

Note If $\mathfrak{h}'$ any other, $\exists\psi\in\mathrm{Aut}(\mathfrak{g}): \psi(\mathfrak{h})=\mathfrak{h}'$.

Note If $h_1, h_2\in\mathfrak{h}$ then

$$(\mathrm{ad}\, h_1)\circ(\mathrm{ad}\, h_2)(x)=[h_1[h_2x]] \overset{\text{Jacobi}}{=}$$

$$=[\underbrace{[h_1 h_2]}_{=0},x]+[h_2,[h_1,x]]=(\mathrm{ad}\, h_2\circ\mathrm{ad}\, h_1)(x)$$

So $\{\mathrm{ad}\, h \mid h\in\mathfrak{h}\}$ is a commuting family of diagonalizable operators on $\mathfrak{g}$

$$\Rightarrow \quad \mathfrak{g}=\mathfrak{h}\oplus\bigoplus_{\alpha\in\Phi}\mathfrak{g}_\alpha$$ Root space decomp.

$$\mathfrak{g}_\alpha=\{x\in\mathfrak{g} \mid [h,x]=\alpha(h)x \;\; \forall h\in\mathfrak{h}\}$$

$$\Phi=\{\alpha\in\mathfrak{h}^*\setminus\{0\} \mid \mathfrak{g}_\alpha\neq 0\}$$

$\boxed{\dim\mathfrak{g} = \dim\mathfrak{h}+|\Phi|}$

rank

Ex $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$

pf Jacobi Identity

Cor If $\alpha, \beta \in \Phi$, $\alpha+\beta \notin \Phi$

then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = 0$

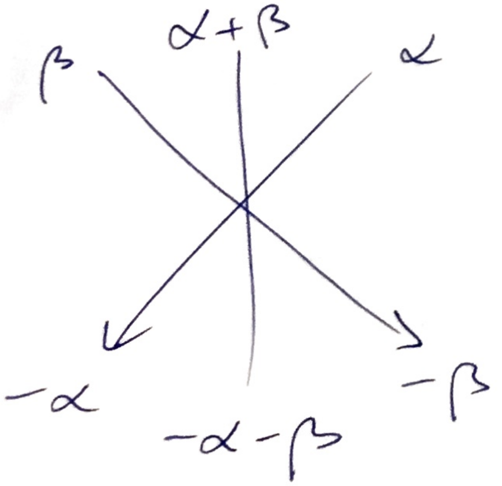
Thm $\exists\, \Delta = \{\alpha_1, \dots, \alpha_\ell\} \subseteq \Phi$ s.t.

1) Δ basis for $\mathfrak{h}^*$

2) $\Phi \subseteq \mathbb{N}\Delta + \mathbb{N}(-\Delta)$

i.e. $\beta = \sum_{i=1}^{\ell} n_i \alpha_i$ where $n_i \in \mathbb{N}\ \forall i$ OR $n_i \in -\mathbb{N}\ \forall i$

Ex



$\Delta = \{\alpha, \beta\}$

If Δ' any other $\exists w \in W$, $w.\Delta' = \Delta$

Let $E = \mathrm{Span}_{\mathbb{R}} \Delta$

Remains: Inner product?

Killing form For $x, y \in \mathfrak{g}$

$$(x, y) := Tr((\mathrm{ad}\, x)(\mathrm{ad}\, y))$$

$\mathrm{ad}\, x : \mathfrak{g} \to \mathfrak{g}$
$(\mathrm{ad}\, x)(z) = [x, z]$

If $\mathfrak{g}$ is a matrix Lie alg can take

$$(x, y) := Tr(\underbrace{x \cdot y}_{\text{matrix product}})$$

Thm $(\cdot,\cdot)$ symmetric non-deg invariant bil. form. Unique up to $\neq 0$ multiple.

Invariant means

$$([z, x], y) + (x, [z, y]) = 0$$

OR: $([x, y], z) = (x, [y, z])$

Ex $(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$ if $\alpha \neq \pm\beta$.

Thm $(\cdot,\cdot)|_{\mathfrak{h}}$ non-deg. $\Rightarrow \mathfrak{h} \cong \mathfrak{h}^*$ via

$h \longmapsto (h, \cdot)$
$h_\alpha \longleftarrow \alpha$

$\Rightarrow \mathfrak{h}^*$ inherits a form $(\alpha, \beta) = (h_\alpha, h_\beta)$

Thm $(E = \mathrm{Span}_{\mathbb{R}} \Delta, (\cdot,\cdot))$ is a root system!

II Dynkin diagram

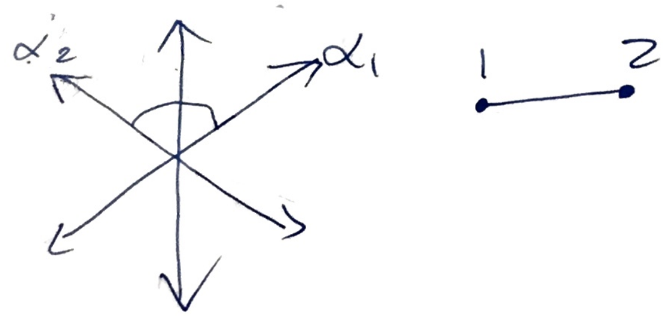
Vertices $\longleftrightarrow$ Δ simple roots

Edges $4\cos^2\theta_{ij}$ edges between α_i, α_j where θ_{ij} angle between them.

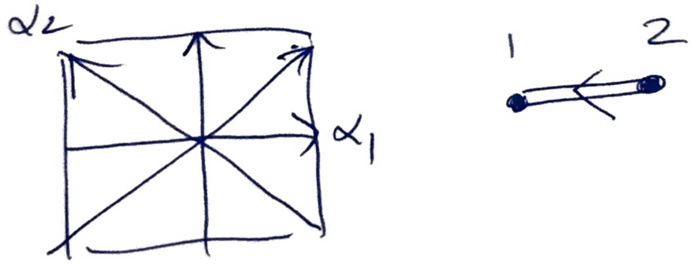
Ex

$A_1 \times A_1$

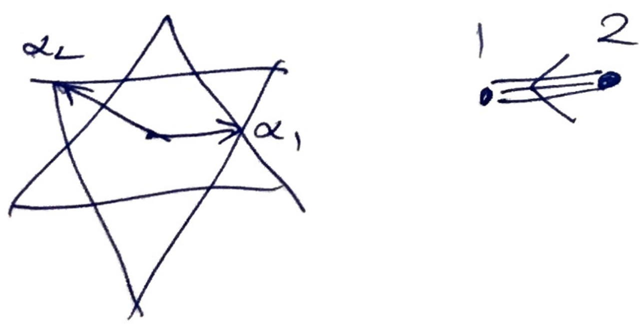
A_2



B_2



G_2



Allows to compute (α_i, α_j) $\forall ij$

$$2\frac{(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)}$$

& recover Φ using reflection σ_α

Direction $<$ indicates shorter root

(Thm Φ irred can only have two root lengths)

Classification Thm The Dynkin diagram of $\mathfrak{g}$ (or Φ) is ~~exactly~~ one of the following:

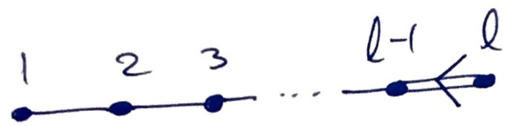
A_ℓ $\quad \ell \geq 1$

$\mathfrak{sl}(\ell+1)$

B_ℓ $\quad \ell \geq 2$

$\mathfrak{so}(2\ell+1)$

C_ℓ $\quad \ell \geq 3$



$\mathfrak{sp}(2\ell)$

D_ℓ $\quad \ell \geq 4$

$\mathfrak{so}(2\ell)$

CLASSICAL

E_6

E_7

E_8

F_4

G_2

EXCEPTIONAL