Homework 6

(*Due Friday, March 28, 2025*)

In these exercises, k denotes an algebraically closed field of characteristic zero.

1. (Etingof Problem 5.11.1) Compute the decomposition into irreducibles of all the representations of A_5 induced from the irreducible representations of
   (a) $\mathbb{Z}_2$
   (b) $\mathbb{Z}_3$
   (c) $\mathbb{Z}_5$
   (d) A_4
   (e) $\mathbb{Z}_2 \times \mathbb{Z}_2$

2. (Lorenz Exercise 4.1.1) Let n a positive integer, and $A = kS_n$ the group algebra of the symmetric group S_n. Let $s_i = (i \ \ i+1)$ for $i = 1, 2, \ldots, n-1$ be the adjacent transpositions, and let $X_1, X_2, \ldots, X_{n-1}$ be the Jucys-Murphy elements. Prove that the following relations hold in A:

$$s_i X_i = X_{i+1} s_i - 1, \qquad \forall i = 1, 2, \ldots, n-1;$$
$$s_i X_j = X_j s_i, \qquad \forall j \notin \{i, i+1\}.$$

3. Let $V \in \operatorname{Irr} S_n$ and let $(\cdot, \cdot) : V \times V \to k$ be an S_n-invariant bilinear form (that is, $(\sigma v, \sigma w) = (v, w)$ for all $\sigma \in S_n$ and all $v, w \in V$). Show that the Gelfand-Zetlin basis (v_T) of V is orthogonal: $(v_T, v_{T'}) = 0$ for $T \neq T'$. (Hint: You may use the fact that representations of symmetric groups are self-dual (we have not shown this in class).)