Definition 2.14.2. The **dual representation** V^* to a representation V of a Lie algebra $\mathfrak{g}$ is the dual space V^* to V with $\rho_{V^*}(x) = -\rho_V(x)^*$.

It is easy to check that these are indeed representations.

Problem 2.14.3. Let V, W, U be finite dimensional representations of a Lie algebra $\mathfrak{g}$. Show that the space $\mathrm{Hom}_{\mathfrak{g}}(V \otimes W, U)$ is isomorphic to $\mathrm{Hom}_{\mathfrak{g}}(V, U \otimes W^*)$. (Here $\mathrm{Hom}_{\mathfrak{g}} := \mathrm{Hom}_{\mathcal{U}(\mathfrak{g})}$.)

2.15. Representations of $\mathfrak{sl}(2)$

This subsection is devoted to the representation theory of $\mathfrak{sl}(2)$, which is of central importance in many areas of mathematics. It is useful to study this topic by solving the following sequence of exercises, which every mathematician should do, in one form or another.

Problem 2.15.1. According to the above, a representation of $\mathfrak{sl}(2)$ is just a vector space V with a triple of operators E, F, H such that $HE - EH = 2E$, $HF - FH = -2F$, $EF - FE = H$ (the corresponding map ρ is given by $\rho(e) = E, \rho(f) = F$, $\rho(h) = H$).

Let V be a finite dimensional representation of $\mathfrak{sl}(2)$ (the ground field in this problem is $\mathbb{C}$).

(a) Take eigenvalues of H and pick one with the biggest real part. Call it λ. Let $\bar{V}(\lambda)$ be the generalized eigenspace corresponding to λ. Show that $E|_{\bar{V}(\lambda)} = 0$.

(b) Let W be any representation of $\mathfrak{sl}(2)$ and let $w \in W$ be a nonzero vector such that $Ew = 0$. For any $k > 0$ find a polynomial $P_k(x)$ of degree k such that $E^k F^k w = P_k(H)w$. (First compute $EF^k w$; then use induction in k.)

(c) Let $v \in \bar{V}(\lambda)$ be a generalized eigenvector of H with eigenvalue λ. Show that there exists $N > 0$ such that $F^N v = 0$.

(d) Show that H is diagonalizable on $\bar{V}(\lambda)$. (Take N to be such that $F^N = 0$ on $\bar{V}(\lambda)$, and compute $E^N F^N v$, $v \in \bar{V}(\lambda)$, by (b). Use the fact that $P_k(x)$ does not have multiple roots.)

(e) Let N_v be the smallest N satisfying (c). Show that $\lambda = N_v - 1$.

(f) Show that for each $N > 0$, there exists a unique up to isomorphism irreducible representation of $\mathfrak{sl}(2)$ of dimension N. Compute the matrices E, F, H in this representation using a convenient basis. (For V finite dimensional irreducible take λ as in (a) and $v \in V(\lambda)$ an eigenvector of H. Show that $v, Fv, \dots, F^\lambda v$ is a basis of V, and compute the matrices of the operators E, F, H in this basis.)

Denote the $(\lambda+1)$-dimensional irreducible representation from (f) by V_λ. Below you will show that any finite dimensional representation is a direct sum of V_λ.

(g) Show that the operator $C = EF + FE + H^2/2$ (the so-called **Casimir operator**) commutes with E, F, H and equals $\frac{\lambda(\lambda+2)}{2}$ Id on V_λ.

Now it is easy to prove the direct sum decomposition. Namely, assume the contrary, and let V be a reducible representation of the smallest dimension, which is not a direct sum of smaller representations.

(h) Show that C has only one eigenvalue on V, namely $\frac{\lambda(\lambda+2)}{2}$ for some nonnegative integer λ (use the fact that the generalized eigenspace decomposition of C must be a decomposition of representations).

(i) Show that V has a subrepresentation $W = V_\lambda$ such that $V/W = nV_\lambda$ for some n (use (h) and the fact that V is the smallest reducible representation which cannot be decomposed).

(j) Deduce from (i) that the eigenspace $V(\lambda)$ of H is $(n+1)$-dimensional. If $v_1, \dots, v_{n+1}$ is its basis, show that $F^j v_i$, $1 \le i \le n+1$, $0 \le j \le \lambda$, are linearly independent and therefore form a basis of V (establish that if $Fx = 0$ and $Hx = \mu x$ for $x \neq 0$, then $Cx = \frac{\mu(\mu-2)}{2} x$ and hence $\mu = -\lambda$).

(k) Define $W_i = \text{span}(v_i, Fv_i, \dots, F^\lambda v_i)$. Show that W_i are subrepresentations of V and derive a contradiction to the fact that V cannot be decomposed.

(l) (Jacobson-Morozov lemma) Let V be a finite dimensional complex vector space and $A : V \to V$ a nilpotent operator. Show that there exists a unique, up to an isomorphism, representation of $\mathfrak{sl}(2)$

on V such that $E = A$. (Use the classification of the representations and the Jordan normal form theorem.)

(m) (Clebsch-Gordan decomposition) Find the decomposition of the representation $V_\lambda \otimes V_\mu$ of $\mathfrak{sl}(2)$ into irreducibles components.

Hint: For a finite dimensional representation V of $\mathfrak{sl}(2)$ it is useful to introduce the character $\chi_V(x) = Tr(e^{xH})$, $x \in \mathbb{C}$. Show that $\chi_{V\oplus W}(x) = \chi_V(x) + \chi_W(x)$ and $\chi_{V\otimes W}(x) = \chi_V(x)\chi_W(x)$. Then compute the character of V_λ and of $V_\lambda \otimes V_\mu$ and derive the decomposition. This decomposition is of fundamental importance in quantum mechanics.

(n) Let $V = \mathbb{C}^M \otimes \mathbb{C}^N$ and $A = J_{0,M} \otimes \mathrm{Id}_N + \mathrm{Id}_M \otimes J_{0,N}$, where $J_{0,n}$ is the Jordan block of size n with eigenvalue zero (i.e., $J_{0,n}e_i = e_{i-1}$, $i = 2, \ldots, n$, and $J_{0,n}e_1 = 0$). Find the Jordan normal form of A using (l) and (m).

2.16. Problems on Lie algebras

Problem 2.16.1 (Lie's theorem)**.** The **commutant** $K(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is the linear span of elements $[x, y]$, $x, y \in \mathfrak{g}$. This is an ideal in $\mathfrak{g}$ (i.e., it is a subrepresentation of the adjoint representation). A finite dimensional Lie algebra $\mathfrak{g}$ over a field k is said to be **solvable** if there exists n such that $K^n(\mathfrak{g}) = 0$. Prove the Lie theorem: if $k = \mathbb{C}$ and V is a finite dimensional irreducible representation of a solvable Lie algebra $\mathfrak{g}$, then V is 1-dimensional.

Hint: Prove the result by induction in dimension. By the induction assumption, $K(\mathfrak{g})$ has a common eigenvector v in V; that is, there is a linear function $\chi : K(\mathfrak{g}) \to \mathbb{C}$ such that $av = \chi(a)v$ for any $a \in K(\mathfrak{g})$. Show that $\mathfrak{g}$ preserves common eigenspaces of $K(\mathfrak{g})$. (For this you will need to show that $\chi([x, a]) = 0$ for $x \in \mathfrak{g}$ and $a \in K(\mathfrak{g})$. To prove this, consider the smallest subspace U containing v and invariant under x. This subspace is invariant under $K(\mathfrak{g})$ and any $a \in K(\mathfrak{g})$ acts with trace $\dim(U)\chi(a)$ in this subspace. In particular $0 = \mathrm{Tr}([x, a]) = \dim(U)\chi([x, a])$.)