Homework 2

(*Due Friday, February 7, 2025*)

1. Let A be an algebra over a field k and let $z \in Z(A)$ be a central element. Let $\rho : A \to \mathrm{End}(V)$ be an indecomposable representation. Show that $\rho(z)$ only has one eigenvalue, equal to the scalar by which z acts on some irreducible subrepresentation of V.

2. Let A be the k-algebra generated by a and b subject to the relations

$$\begin{cases} a^2b - 2aba + ba^2 = 0, \\ b^2a - 2bab + ab^2 = 0. \end{cases}$$

   Prove that $\{[a,b]^i a^j b^k \mid i,j,k \in \mathbb{Z}_{\geq 0}\}$ is a basis for A. (Hint: For linear independence, one way is to show there is a homomorphism from A to algebra $A_1(k)[t] \cong A_1(k) \otimes k[t]$, determined by $a \mapsto y \otimes t,\ b \mapsto x \otimes 1$.)

3. In this problem, let $A = A_1(k)$ be the Weyl algebra over a field k.
   (a) Assume k has characteristic zero. Find all finite-dimensional representations of A. (Hint: Use the fact that, if B and C are square matrices, then $\mathrm{Tr}(BC) = \mathrm{Tr}(CB)$.)
   (b) Assume k has characteristic $p > 0$. Describe the center $Z(A)$ of A. (Hint: First show x^p and y^p are central elements.)
   (c) Assume k has characteristic $p > 0$. Find all finite-dimensional irreducible representations of A. (Hint: If v is an eigenvector of y on V, show that $\{v, xv, \ldots, x^{p-1}v\}$ is a basis for V.)

4. Let $V \neq 0$ be a representation of an algebra A. We say $v \in V$ is *cyclic* if it generates V, that is, $Av = V$. A representation admitting a cyclic vector is said to be *cyclic*. Show that:
   (a) V is irreducible if and only if all nonzero vectors of V are cyclic.
   (b) V is cyclic if and only if it is isomorphic to A/I for some left ideal I of A.