Lecture 3

[Etingof §2.3.1]

A representation of an algebra A (also called a left A-module) is a vector space V with an algebra map $\rho : A \to \mathrm{End}(V)$.

A right A-module is a vector space V with an anti-homomorphism $\rho : A \to \mathrm{End}(V)$ i.e. $\rho(ab) = \rho(b)\rho(a)$, $\rho(1) = 1$.

Notation For left A-modules we abbreviate $\rho(a)v$ by $a.v$ or av and for right A-modules we write $v.a$ or va for $\rho(a)v$.
Then

$$a.(b.v) = (ab).v \quad \text{for left } A\text{-mods}$$

$$(v.a).b = v.(ab) \quad \text{for right}$$

If A is commutative, any left A-module becomes a right A-module by $v.a := av$.

Ex. 1) $V=0$

2) $V=A$, $a \cdot b = ab$
product in A

3) $A=k$ then a rep of A is just a vector space

4) $A = k\langle x_1, \dots, x_n\rangle$ then a rep of A is uniquely determined by specifying any n linear operators $\rho(x_1), \dots, \rho(x_n)$.

Def A subrepresentation of a rep. V of A is a subspace $U \subseteq V$ such that $\rho(a)U \subseteq U$ $\forall a \in A$

Ex. $0 = \{0_V\}$ and V

Def A rep $V \neq 0$ of A is irreducible if the only subreps of V are 0 and V.

<u>Def</u> Let V_1, V_2 be reps of an alg A. A homomorphism (or intertwining operator) $\phi: V_1 \to V_2$ is a linear map such that
$$\phi \circ \rho_1(a) = \rho_2(a) \circ \phi \quad \forall a \in A$$
equivalently $\phi(av) = a\phi(v)$ $\forall a \in A \ \forall v \in V_1$

or

$$\begin{tikzcd} V_1 \arrow[r, "\phi"] \arrow[d, "\rho_1(a)"'] & V_2 \arrow[d, "\rho_2(a)"] \\ V_1 \arrow[r, "\phi"] & V_2 \end{tikzcd}$$

The space of all homomorphisms $V_1 \to V_2$ is denoted:
$$\mathrm{Hom}_A(V_1, V_2).$$

If ϕ invertible then ϕ^{-1} is also a homomorphism, and ϕ is an isomorphism.
V_1 V_2 are equivalent or isomorphic if $\exists$ isomorphism $\phi: V_1 \to V_2$.

Def V_1, V_2 reps of A

then $V_1 \oplus V_2$ ← direct sum is a rep of A

with $\rho(a) = \left[\begin{array}{c|c} \rho_1(a) & 0 \\ \hline 0 & \rho_2(a) \end{array}\right]$

equivalently, with

$V_1 \oplus V_2 = \{ (v, w) \mid v \in V_1, w \in V_2 \}$

$a \cdot (v_1, v_2) = (a \cdot v_1, a \cdot v_2)$.

Def A rep V of A is indecomposable if V is not isomorphic to a direct sum of nonzero subreps.

Every irred. rep is indec. but not conversely!

Schur's Lemma Let V_1, V_2 be reps of an alg A.
Let $\phi : V_1 \to V_2$ be a nonzero intertwining operator
Then

1) If V_1 is irreducible, ϕ is injective

2) if V_2 is irreducible, ϕ is surjective

So if both V_i are irr, then ϕ is an isomorphism.

<u>pf</u> (see book)

Lecture 4

[Etingof §2.3, cont'd.]

<u>Cor</u> (Schur's Lemma for $\bar{\mathbb{k}} = \mathbb{k}$)

Let V be a fin. dim'l irrep of A with $\bar{\mathbb{k}} = \mathbb{k}$. Let $\phi : V \to V$ be intertwining op. Then

$$\phi = \xi \, \mathrm{Id}_V$$

for some $\xi \in \mathbb{k}$.

<u>pf</u> ξ eigenvalue of ϕ.

Then $\phi - \xi \, \mathrm{Id} : V \to V$

not iso. So $= 0$.

<u>Cor.</u> A comm alg / $\bar{\mathbb{k}} = \mathbb{k}$

Then every irrep is 1-dim'l

EX 2.3.14

1) $A = k$
The only indec rep is $V = k$

2) $A = k[x]$ $\quad \bar{k} = k$

Irreps: $V_\lambda = k$ with $x.1 = \lambda$
$(\lambda \in k)$

Indecs: Jordan blocks, $\begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}$

<u>Note</u>: Not every indec rep is irr.

3) $A = k[G]$ group alg

$\underline{\mathrm{Rep}}\, A = \underline{\mathrm{Rep}}\, G$

<u>Def</u> A <u>rep</u> of a group G is a v.sp. V with a grp hom $\rho : G \to GL(V)$

$$\underline{\mathrm{Rep}}\, A \xrightarrow{\sim} \underline{\mathrm{Rep}}\, G$$
$$\rho_A \longmapsto \rho_A|_G$$

§2.4 IDEALS

Def A subspace I of an algebra A is a
left ideal if $a \cdot x \in I \; \forall a \in A, x \in I$
right ideal if $x \cdot a \in I \; \forall a \in A, x \in I$
two-sided ideal if it's both a left ideal and a right ideal.

Note Left ideals are the same thing as subrepresentations of the regular rep.

Any right ideal in A is a left ideal in A^{op}

the opposite alg

$$a \underset{op}{\cdot} b := ba$$

flip map

$$m_{A^{op}}(a \otimes b) = (m_A \circ \tau)(a \otimes b)$$

$$m_{A^{op}} = m_A \circ \tau$$

Ex 1) 0, A two-sided ideals

An alg is simple if these are the only two-sided ideals (and $0_A \neq 1_A$)

2) If $\phi : A \to B$ alg map then $\ker \phi$ is a two-sided ideal of A.

3) If $S \subseteq A$ is any subset the two-sided ideal generated by S, denoted $\langle S \rangle$, is

$$\langle S \rangle := \mathrm{Span}_k \left\{ asb \;\middle|\; \begin{array}{l} a, b \in A \\ s \in S \end{array} \right\}$$

Similarly we can define left and right versions:

$$\langle S \rangle_l = \mathrm{Span}\{ as \mid a \in A, s \in S \}$$

$$\langle S \rangle_r = \mathrm{Span}\{ sa \mid s \in S, a \in A \}$$

$I \subset A$ maximal (left/right/2-sided) ideal

if 1) $I \neq A$

2) $I \subseteq J \subsetneq A \Rightarrow I = J$

Fact TFAE

1) every alg has a maximal ideal

2) Axiom of Choice holds

§2.5 Quotients

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§2.6 Gen & Rels.

$$\frac{k\langle x_1 \dots x_n\rangle}{I}$$

$I = \langle S \rangle$

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§2.7

Ex Weyl alg

basis

using $\hookrightarrow \mathrm{End}_k(k[x])$

q-Weyl alg

$U(sl_2)$

ρ faithful if inj

Lecture 5

Examples of Algebras [Etingof §2.7]

1) The Weyl algebra is

$$A_1(k) = k\langle x, y\rangle / \langle yx - xy - 1\rangle$$

2) The q-Weyl algebra ($q \in k^\times = k \setminus \{0\}$) is

$$A_1^q(k) = k\langle x, x^{-1}, y, y^{-1}\rangle / I$$

where I is generated by

$$yx - qxy$$

and the "obvious" elements

$$xx^{-1} - x^{-1}x, \quad xx^{-1} - 1,$$
$$yy^{-1} - y^{-1}y, \quad yy^{-1} - 1$$

3) The enveloping algebra of sl_2 is

$$U(sl_2) = k\langle e, f, h\rangle \Big/ \left\langle \begin{matrix} he - eh - 2e \\ hf - fh + 2f \\ ef - fe - h \end{matrix} \right\rangle$$

4) The n:th Weyl algebra is

$$A_n(k) = \underbrace{A_1(k) \otimes \cdots \otimes A_1(k)}_{n}$$

Thm $\{x^i y^j \mid i,j \in \mathbb{Z}_{\geq 0}\}$ is a basis for $A_1(k)$

Proof 1) $[y, x^k] = k x^{k-1}$, $k \geq 1$

induction on k: ($k=1$ trivial)

We will use the product rule

$$[a, bc] = [a,b]c + b[a,c] \quad \text{(check!)}$$

We have

$$[y, x^k] = [y,x]x^{k-1} + x[y, x^{k-1}]$$
$$= 1 \cdot x^{k-1} + x(k-1)x^{k-2}$$
$$= kx^{k-1}$$

ind. hyp.

2) $A_1(k) = \mathrm{Span}_k \{x^i y^j \mid i,j \in \mathbb{Z}_{\geq 0}\}$

Since x, y generate $A_1(k)$ and belongs to RHS, it suffices to show RHS is a subalgebra of $A_1(k)$. Suffices to show

$$x^i y^j x^k y^l \in RHS$$

For this, it suffices to show $y^j x^k \in RHS$. We have (if $j>0$):

$$y^j x^k = y^{j-1} y x^k = y^{j-1}(x^k y) + [y, x^k] \in RHS$$

by 1) and by induction on j.

3) Linear independence.
We define

$$\varphi : \{x, y\} \longrightarrow \mathrm{End}_k(k[x])$$

by

$$\varphi(x)\, p(x) = x \cdot p(x)$$
$$\varphi(y)\, p(x) = p'(x) = \frac{d}{dx} p(x)$$

By universal property of $k\langle x,y\rangle$, φ extends to an alg map

$$\tilde{\varphi} : k\langle x, y\rangle \longrightarrow \mathrm{End}_k(k[x])$$

We check $\langle yx - xy - 1\rangle \subset \ker\tilde{\varphi}$
Since $\ker\tilde{\varphi}$ is an ideal, it suffices to check $yx - xy - 1 \in \ker\tilde{\varphi}$.
We have

$$\tilde{\varphi}(yx - xy - 1)(p(x)) =$$
$$= \left(\tilde{\varphi}(y)\tilde{\varphi}(x) - \tilde{\varphi}(x)\tilde{\varphi}(y) - \tilde{\varphi}(1)\right)(p(x))$$
$$= (\varphi(y)\varphi(x) - \varphi(x)\varphi(y) - \mathrm{Id})(p(x))$$
$$= \frac{d}{dx}(x p(x)) - x \cdot \frac{d}{dx}(p(x)) - p(x)$$
$$= \underbrace{\cancel{1 \cdot p(x)} + \cancel{x \tfrac{d}{dx}(p(x))}}_{\text{product rule for formal } d/dx} - \cancel{x \tfrac{d}{dx}(p(x))} - \cancel{p(x)} = 0.$$

The point: It suffices to show

$$\{\tilde{\varphi}(x^i y^j) \mid i,j \geq 0\}$$

is lin indep in $\mathrm{End}_k(k[x])$.

$$\tilde{\varphi}(x^i y^j) = x^i \frac{d^j}{dx^j}$$

Suppose

$$\sum_{i,j} \lambda_{ij} x^i \left(\frac{d}{dx}\right)^j = 0$$

We can write it as

$$\sum_j Q_j(x) \left(\frac{d}{dx}\right)^j = 0 \quad (*)$$

where $Q_j(x) \in k[x]$.

Apply to 1:

$$0 = \sum_j Q_j(x) \left(\frac{d}{dx}\right)^j (1) = Q_0(x)$$

Apply instead to x

$$0 = \sum_{j \geq 1} Q_j(x) \left(\frac{d}{dx}\right)^j (x) = Q_1(x)$$

$$\ldots \quad 0 = \sum_{j \geq \ell} Q_j(x) \left(\frac{d}{dx}\right)^j (x^\ell) = \ell! \, Q_\ell(x)$$

Assuming char $k = 0$, we can cancel $\ell!$ and get

$$Q_\ell(x) = 0 \quad \forall \ell.$$

§ 2.9 Lie Algebras.

Def . . .

Ex 2.9.2.

1) abelian

2) assoc alg , $\mathfrak{gl}(V)$

3) $\underset{\text{subspace}}{U} \subset \underset{\text{assoc alg}}{A}$, $[U, U] \subset U$

4) $\mathrm{Der}\, A$

5) $\mathfrak{a} \subset \mathfrak{g}$ $\quad [\mathfrak{a}, \mathfrak{a}] \subset \mathfrak{a}$

$$U(\mathfrak{g}) = T(\mathfrak{g}) \Big/ \Big\langle \big\{ x \otimes y - y \otimes x - [x, y] \,\big|\, x, y \in \mathfrak{g} \big\} \Big\rangle$$

Remark Ado's Thm says any fd Lie alg is a Lie subalg of $\mathfrak{gl}(V)$ for some f.d. V.

Remark Derivations are infinitesimal automorphisms

Ex $\mathbb{R}^3$, $\times$

$sl(n)$, $sl(2)$

$\mathcal{H} = \begin{bmatrix} 0 & * & * \\ & 0 & * \\ & & 0 \end{bmatrix}$

$aff(1) = \begin{bmatrix} * & * \\ 0 & 0 \end{bmatrix}$

$so(n)$ $A^T = -A$

Def A homomorphism $\varphi : \mathfrak{g}_1 \to \mathfrak{g}_2$ of Lie algebras is a linear map st.

$$\varphi([XY]) = [\varphi(x)\,\varphi(y)]$$

A rep of a Lie alg $\mathfrak{g}$ is a v.sp. with

$$\rho : \mathfrak{g} \to \mathfrak{gl}(V) = \mathcal{L}(\mathrm{End}\,V)$$

(a Lie alg homomorphism)

__Ex__ 1) $V = 0$

2) Any V with $\rho(x) = 0 \ \forall x \in \mathfrak{g}$

3) $V = \mathfrak{g}$, $\rho(x)(y) = [x, y]$

adjoint rep. One meaning of the Jacobi Identity is that ρ is a homomorphism.

__Def__ $U(\mathfrak{g})$ basis-dependent version:

$\{x_i\}$ basis with $[x_i, x_j] = \sum_k c_{ij}^k x_k$

Structure constants

Then $U(\mathfrak{g}) := k\langle\{x_i\}\rangle / \text{Rels}$

$$x_i x_j - x_j x_i = \sum_k c_{ij}^k x_k$$

__Ex__ $U(\mathfrak{sl}_2)$ $\mathfrak{sl}_2$ has basis

$e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = e^T$, $h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

These satisfy

$$[e\,f] = h \quad [h\,e] = 2e \quad [h,f] = -2f$$

(check!) so

$$U(sl_2) = \frac{k\langle e, f, h \rangle}{\left(\begin{array}{l} ef - fe = h \\ he - eh = 2e \\ hf - fh = -2f \end{array} \right)}$$

<u>Lie groups</u> are groups which are also smooth manifolds, such that mult $G \times G \to G$ and inverse $G \to G$ are smooth maps. G acts on $C^\infty(G)$ by $(g \cdot \varphi)(h) = \varphi(hg)$, hence on the Lie algebra

$$\mathrm{Vect}(G) := \mathrm{Der}\, C^\infty(G)$$

The Lie algebra of G is

$$\mathfrak{g} = \mathrm{Lie}\, G := \mathrm{Vect}(G)^G$$

There is a bijection between real fd Lie algebras and connected, simply connected Lie groups.

Every real Lie group G has a connected subgroup of the identity element. G° $\text{Lie}\,G = \text{Lie}(G^\circ)$

Every connected Lie group G has a universal cover $\tilde{G}$ which is also a Lie group. $\text{Lie}\,G = \text{Lie}(\tilde{G})$

Ex $\quad O(n)^\circ = SO(n)$

$$\widetilde{SO(n)} = \text{Spin}(n)$$

so-called spin group

$$1 \to \mathbb{Z}/2\mathbb{Z} \to \text{Spin}(n) \to SO(n) \to 1$$

Spin(n) is a double cover of SO(n) (when $n > 2$)

$$\text{Spin}(3) \cong SU(2)$$

§2.8 Quivers.

$$Q = (Q_0, Q_1, s, t) = (I, E, s, t)$$

$$s, t : Q_1 \longrightarrow Q_0$$

source and target maps; arrows; vertices

$$s(h) =: h', \quad t(h) =: h''$$

Ex

$$\begin{tikzcd} \bullet \arrow[r] & \bullet & \bullet \arrow[l] \\ & \bullet \arrow[u] & \end{tikzcd}$$

$$Q_0 = \{a \quad b \quad c \quad d\}$$

$$Q_1 = \{\alpha \quad \beta \quad \gamma\}$$

$s(\alpha) = a \quad t(\alpha) = b$ etc.

A rep of a quiver Q is $\{V_i\}_{i \in Q_0}$ and

$$\{x_h : V_{h'} \to V_{h''}\}_{h \in Q_1}$$

<u>Def</u> The path alg $P_Q = kQ$ of a quiver Q is the vector space with basis all oriented paths in Q (including a trivial path p_i, $i \in I$ at each vertex) and mult def by $ab :=$ path b followed by path a if they connect and 0 otherwise

<u>Remark</u> $\sum_{i \in I} p_i = \mathrm{Id}_{P_Q}$

A subrep of a rep (V_i, x_h) of a quiver is a rep (W_i, y_h) s.t. $W_i \subseteq V_i$ $\forall i \in I$ and $y_h = x_h|_{W_i} : W_{h'} \longrightarrow W_{h''}$ $\forall h \in E$

<u>Def</u> Direct sum, homomorphism of reps.

§3 General Results

§3.1 Subreps in ss reps

Assume $\bar{k} = k$

Def. A semisimple (or completely reducible) rep of an alg A is a direct sum of irreducible reps.

Ex Let V be an n-dim'l irrep of A. Then $Y := \mathrm{End}(V)$ with action of A by left mult. is a semisimple rep of A isomorphic to $nV := V \oplus \cdots \oplus V$. Indeed any basis $\{v_1, \ldots, v_n\}$ for V gives $\mathrm{End}(V) \to nV$ given by

$x \mapsto (xv_1, \ldots, xv_n)$

This is an isomorphism of reps.

Remark If V is semisimple, there is a map

$$\phi : \bigoplus_{X \in \mathrm{Irr} A} \mathrm{Hom}_A(X, V) \otimes X \longrightarrow V$$

where X runs over a set of representatives of the set of iso classes of irreps of A.

The map is given by

$$\phi(g \otimes x \longmapsto g(x))$$

$\forall g \in \mathrm{Hom}_A(X, V)$, $\forall x \in X$, $\forall X \in \mathrm{Irr} A$. Then ϕ is an isomorphism (use Schur's Lemma)

Prp 3.1.4

Let $\mathrm{Irr}\, A = \{V_1, \dots, V_m\}$

Let W be any subrep of $V := \bigoplus n_i V_i$ $(n_i \in \mathbb{Z}_{\geq 0})$

Then $W \cong \bigoplus r_i V_i$ where $r_i \leq n_i$ and the inclusion $W \hookrightarrow V$ is a direct sum of inj. homomorphisms $\phi_i : r_i V_i \to n_i V_i$ given by mult. from the right by some $r_i \times n_i$ matrix X_i with lin. indep. rows:

$$\phi_i(v_1, \dots, v_{r_i}) = (v_1, \dots, v_{r_i}) X_i$$

pf Ind. on $n = \sum_{i=1}^{m} n_i$

$n=1$ clear

Spse $n>1$. Assume $W \neq 0$.

Fix irr subrep $P \subset W$.

(exists by Pr 2.3.15)

By Schur's Lemma, $P \cong V_i$ for some i and $\phi|_P : P \to V$ factors through $n_i V_i$.

Identifying P with V_i, $\phi|_P$ is given by $v \mapsto (vq_1, \dots, vq_{n_i})$ for some $q_\ell \in k \setminus \{0\}$.

Note $GL_i := GL_{n_i}(k)$ acts on $n_i V_i$ by $(v_1, \dots, v_{n_i}) \mapsto (v_1, \dots, v_{n_i}) g$

and by id on $n_j V_j$ $j \neq i$

therefore on the set of subreps of V having property we seek, namely

$$X_i \longmapsto X_i g_i$$

$$X_j \longmapsto X_j \qquad j \neq i$$

Take $g_i \in G_i$ s.t.

$$(q_1, \dots, q_{n_i}) g_i = (1, 0, \dots, 0)$$

Then $W_{g_i} = V_i \oplus W'$ where

$$W' \subset n_1 V_1 \oplus \dots \oplus (n_i - 1) V_i \oplus \dots \oplus n_m V_m$$

is kernel of proj of W_{g_i} to first summand of V_i along the other summands. Done by induction. ∎

Rem Holds for general field & $\dim V = \infty$.

Cor 3.2.1 V fd irrep of A
Let $n > 0$ and let $v_1, \ldots, v_n \in V$ lin indep.
Then $\forall w_1, \ldots, w_n \in V$
$\exists a \in A$ s.t. $av_i = w_i$

Pf Assume the contrary. Then the image of $A \to nV$, $a \mapsto (av_1, \ldots, av_n)$ is a proper subrep so by Prp it corresponds to some $r \times n$-matrix X, $r < n$. Taking $a = 1$, $\exists u_1, \ldots, u_r \in V$ s.t.

$(u_1, \ldots, u_r) X = (v_1, \ldots, v_n)$

Let $(q_1, \ldots, q_n) \in k^n$ be nonzero st $X \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} = 0$

(exists by $r < n$). Then

$$\sum q_i v_i = [u_1 \ldots u_r] X \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} = 0$$

i.e. $\sum q_i v_i = 0 \Rightarrow$ <u>contrad.</u> ∎

<u>Th.</u> (Density Thm)

i) V fd irrep of A then

$\rho : A \to \mathrm{End}(V)$ is surj

ii) $\bigoplus \rho_i : A \to \bigoplus \mathrm{End}(V_i)$ surj if $V = V_1 \oplus \cdots \oplus V_r$, $V_i \not\cong V_j$

Pf i) $B := \rho(A) \subset \mathrm{End}(V)$.

WTS $B = \mathrm{End}(V)$.

Let $c \in \mathrm{End}(V)$

Let v_i basis of V

Let $w_i = c v_i$. By Cor

$\exists\ a \in A$ st

$a v_i = w_i$

Then $a \mapsto c$ so $c \in B$.

$\boxtimes$

Lecture 9, Mon Feb 10

Example

A alg over $\bar{k}=k$.

$V = V_1 \oplus V_2$ direct sum of 2 irreps

$W \subseteq V$ an irreducible subrep.

What can we say about W?

Case 1 $V_1 \not\cong V_2$. Consider the projections

$$\pi_i : V_1 \oplus V_2 \longrightarrow V_i \ , \ i=1,2$$
$$(v_1, v_2) \longmapsto v_i$$

π_i are intertwining operators.

Consider

$$\phi_i := \pi_i\big|_W : W \longrightarrow V_i \qquad i=1,2$$

If $\phi_1 \neq 0$, then it is an isomorphism by Schur's Lemma. So $W \cong V_1 \not\cong V_2$. This forces $\phi_2 = 0$. Symmetrically, if $\phi_2 \neq 0$ then $\phi_1 = 0$.

We conclude that

$$W = V_1 \oplus 0 \quad \text{or} \quad W = 0 \oplus V_2$$

(If both $\phi_1 = \phi_2 = 0$ then $W = 0$ contradicting W irr.)

CASE 2: $V_1 \cong V_2$

As in previous case, at least one of the projections π_i must be nonzero, hence $W \cong V_1 \cong V_2$.

$$\mathrm{End}_A(V) = \mathrm{Hom}_A(V_1 \oplus V_2, V_1 \oplus V_2)$$

$$\cong \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}$$

where

$$H_{ij} = \mathrm{Hom}_A(V_i, V_j)$$

By Schur's Lemma, $\dim_k H_{ij} = 1$. Now $\exists \{e_{ij}\}$ basis for H_{ij} such that

$$\mathrm{End}_A(V) \cong M_2(\quad) \quad \text{(exercise)}$$

$\Rightarrow GL_2(k) = M_2(k)^\times \cong \mathrm{Aut}_A(V)$ is acting on V.

Given any nonzero $w = (v_1, v_2) \in V$ we can find $g \in GL_2(k)$ "rotating" w to $(w_1, 0)$. Composing with π_2 gives non-injective (hence zero, by Schur's Lemma) intertwining operator $W \to 0 \oplus V_2$

This shows $\exists$ automorphism ϕ of V such that

$$\phi(W) = V_1 \oplus 0 \subset V_1 \oplus V_2 = V$$

<u>Cor 3.2.1</u> A alg over $k = \bar{k}$
V fd irrep of A
$v_1, \ldots, v_n \in V$ lin indep. ($\Rightarrow n \leq \dim V$)
Then for any $w_1, \ldots, w_n \in V$ $\exists a \in A$:
$a v_i = w_i \quad \forall i = 1, 2, \ldots, n.$

<u>Thm</u> (Jordan Density Thm) A alg$/k = \bar{k}$
i) V fd irrep of A. Then
$\rho : A \to \mathrm{End}(V)$ is surjective
ii) $V = V_1 \oplus \cdots \oplus V_n$ V_i fd irreps, $V_i \not\cong V_j$ $\forall i \neq j$. Then
$\rho = \oplus \rho_i : A \to \oplus \mathrm{End}(V_i)$ is surjective.

<u>Pf</u> i) $v_1, \ldots, v_n$ basis for V, $c \in \mathrm{End}(V)$. By cor $\exists a$ s.t.
$a v_i = c(v_i)$ $\forall i$. Thus $c = \rho(a)$
ii) $\oplus \mathrm{End}(V_i)$ is semisimple $\cong \oplus d_i V_i$
where $d_i = \dim V_i$. Thus by Prp 3.1.4
$\rho(A) = \oplus \rho_i(A) \overset{i)}{=} \oplus \mathrm{End}(V_i)$ ∎

§3.3

$$A = \bigoplus_{i=1}^{r} \mathrm{Mat}_{d_i}(k) = \begin{bmatrix} \boxed{*} & & \\ & \boxed{*} & \\ & & \ddots \end{bmatrix}$$

Thm The irreps of A are

$V_1 = k^{d_1}, \ldots, V_r = k^{d_r}$

and any fd rep of A is a direct sum of copies of V_i.

Def Dual rep $V^* \in A^{op}\,\underline{\mathrm{Mod}}$.

$(a\xi)(v) = \xi(av)$

Pf: Next time.

Lecture 10, Wed Feb 12, 2025

Proof of Thm (see previous page)

The representations V_i are irreducible: Let $v, w \in k^{d_i}$ be any nonzero vectors. Then there is $a \in \mathrm{Mat}_{d_i}(k)$, $av = 0$. Therefore

$$\underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\in A} \underbrace{\begin{bmatrix} 0 \\ \vdots \\ v \\ \vdots \\ 0 \end{bmatrix}}_{\in V_i} = \underbrace{\begin{bmatrix} 0 \\ \vdots \\ w \\ \vdots \\ 0 \end{bmatrix}}_{\in V_i}$$

Let X be an n-dim'l rep of A. Then X^* is a rep of A^{op}. But

$$A^{op} = \left(\bigoplus \mathrm{Mat}_{d_i}(k)\right)^{op} =$$

$$= \bigoplus \left(\mathrm{Mat}_{d_i}(k)^{op}\right)$$

Using the transpose

$$\cong \bigoplus \mathrm{Mat}_{d_i}(k) = A$$

Fix a basis $\{y_i\}_{i=1}^{n}$ for X^*. Define

$$\phi : \underbrace{A \oplus \cdots \oplus A}_{n} \longrightarrow X^*$$

by $\phi(a_1, \ldots, a_n) = a_1 y_1 + \cdots + a_n y_n$.

Since $k1_A \subseteq A$, ϕ is surjective.

Thus, the dual map

$$\phi^* : X \longrightarrow (A^n)^*$$

is injective.

Now $(A^n)^* \cong A^n$ as representations of A (using left mult, and transpose). Thus X is iso isomorphic to a subrep of A^n.

Since $A \cong \bigoplus_{i=1}^{r} d_i V_i$,

$$A^n \cong \bigoplus_{i=1}^{r} n d_i V_i$$

So $X \cong \bigoplus_{i=1}^{r} m V_i$ by Prop 3.1.4. ∎

§ 3.4 Composition Series.

Every fd rep V of an alg A has a <u>filtration</u>

$0 = V_0 \subset V_1 \subset \cdots \subset V_\ell = V$

such that V_i / V_{i-1} is irreducible $\forall\, i = 1, \ldots, \ell$.

<u>Pf</u> If V irr, done.

otherwise Let V_1 be an irr subrep. Let $W = V/V_1$

$\dim W < \dim V$ so by ind.

$\exists\; 0 = W_0 \subset W_1 \subset \cdots \subset W_\ell = W$

Each W_i has the form

$W_i = V_{i+1}/V_i$ for some subreps

V_{i+1} of V containing V_1.

By 3rd iso thm

$$V_{i+1}/V_i \cong W_i/W_{i-1} \text{ irr}$$

$\forall\, i = 1, \ldots, \ell$.

and $V_1/V_0 \cong V$. irr

$\blacksquare$

§ 3.5 Fin Dim'l Algs

Rad(A)

Def The radical of a finite-dim'l alg A is the set of $a \in A$ acting by zero on all irreps of A.

Prop 3.5.2 $\mathrm{Rad}(A)$ is a two-sided ideal.

Prop 3.5.3 A fin dim'l alg

i) I nilpotent ($I^n = 0$, $n \gg 0$) 2-sided ideal of $A \Rightarrow I \subset \mathrm{Rad}\,A$

ii) $\mathrm{Rad}(A)$ is nilpotent

Pf Let V irrep of A.
Let $v \in V$. Then $Iv \subset V$
is a subrep. If $Iv \neq 0$
then $Iv = V$ so there is
$x \in I$ st $xv = v$. Then
$x^n v = v$ so $x^n \neq 0$ contrad.

ii) $0 = A_0 \subset A_1 \subset \cdots \subset A_n = A$
composition series of
regular rep: A_i / A_{i-1} irr.
Let $x \in \mathrm{Rad}\, A$. Then
$x A_i \subset A_{i-1}$ $\forall i = 1, \ldots, n$
$\Longrightarrow (\mathrm{Rad}\, A)^n = 0$ ∎

Th: A fd alg.

Then $\exists$ only fin many irreps (up to iso).

$V_1, \ldots, V_m$

These are f.d. and

$$A/\mathrm{Rad}A \cong \bigoplus_i \mathrm{End}\, V_i$$

Pf $Av \subset V$ fd subsp.

$\Rightarrow \dim V \le \dim A$.

Spse $V_1, \ldots, V_r$ noniso irreps

then

$$\bigoplus \rho_i : A \to \bigoplus \mathrm{End}\, V_i$$

is surjective by Th 322.

So $r \le \sum_i \dim \mathrm{End}\, V_i \le \dim A$.

Let $\{V_1, \ldots, V_r\}$ all irreps.

By Th 322

$\oplus \rho_i : A \to \oplus \operatorname{End}(V_i)$

is surjective.

Kernel $= \operatorname{Rad}(A)$

by def. $\blacksquare$

Cor $\sum (\dim V_i)^2 \leq \dim A$

Lecture 11

Ex. $A = k[x]/(x^n)$

$\exists !$ irrep $V = k$ with x acting by zero
So $\mathrm{Rad}(A) = (x)$.

Ex $A = \left\{ \begin{bmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{bmatrix} \right\}$ upper-triangular $n\times n$-matrices over k. Then $V = k^n$ has a composition series

$0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$

given by

$$V_i = \mathrm{Span}\{e_1, \ldots, e_i\}$$

$$\begin{array}{cl} \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} & i\text{-th row} \end{array}$$

- Each V_i is a subrep of V
- Each V_i/V_{i-1} is 1-dim'l, hence irreducible.
- $\mathrm{Rad}(A) \supset \{$strictly upper-triangular matrices$\}$
-

Def A fin-dim'l alg A is **semisimple** if $\mathrm{Rad}(A)=0$.

Prop 3.5.8. Let A be a fin-dim'l algebra. TFAE:

1) A is semisimple

2) $\dim A = \sum_{i=1}^{r} (\dim V_i)^2$ where $V_1, V_2, \dots, V_r$ are the irreps of A.

3) $A \cong \bigoplus_{i=1}^{r} \mathrm{Mat}_{d_i}(k)$ for some $d_i \in \mathbb{Z}_{>0}$

4) Any finite-dimensional rep of A is completely reducible.

5) The regular rep. A is completely reducible.

pf (1) $\Leftrightarrow$ (2): By Thm 3.5.4,

$$A/\mathrm{Rad}(A) \cong \bigoplus_i \mathrm{End}(V_i)$$

so $\dim A - \dim \mathrm{Rad}(A) = \sum_i (\dim V_i)^2$

(1) $\Rightarrow$ (3): By Th 3.5.4, if $\mathrm{Rad}(A)=0$ then

$$A \cong \bigoplus \mathrm{Mat}_{d_i}(k)$$

with $d_i = \dim V_i$.

(3) $\Rightarrow$ (1): By Th. 3.3.1 on irreps of $\bigoplus_i \mathrm{Mat}_{d_i}(k)$, no nonzero element acts by zero on all irreps, so $\mathrm{Rad}(A)=0$.

(3) $\Rightarrow$ (4): By Th. 3.3.1.

(4) $\Rightarrow$ (5): Trivial

(5) $\Rightarrow$ (3): Suppose $A = \bigoplus_i n_i V_i$.
By distributive law for $\mathrm{Hom}_A(-,-)$ w.r.t. $\oplus$:

$$\mathrm{End}_A(A) \cong \begin{bmatrix} H_{11} & \cdots & H_{1r} \\ \vdots & & \vdots \\ H_{r1} & \cdots & H_{rr} \end{bmatrix}$$

where $H_{ij} = \mathrm{Hom}_A(n_i V_i, n_j V_j)$.

By Schur's Lemma

$H_{ij} = 0$ for $i \neq j$

and

$H_{ii} \simeq \mathrm{Mat}_{d_i}(k)$ for $i = j$, $d_i = \dim V_i$

so

$$\mathrm{End}_A(A) \cong \bigoplus \mathrm{Mat}_{d_i}(k)$$

Since $\mathrm{End}_A(A) \cong A^{op}$ (Exercise) we get

$$A \cong \mathrm{End}_A(A)^{op} \cong \bigoplus \mathrm{Mat}_{d_i}(k)^{op}$$

$$\cong \bigoplus \mathrm{Mat}_{d_i}(k)$$

using transpose

∎

§ 3.6 Characters

$\rho_V : A \to \mathrm{End}(V)$

Let A be an alg and (V, ρ_V) a fin-dim'l rep of A.

Def The character of V is the map $\chi_V : A \to k$ defined by

$$\chi_V(a) = \mathrm{Tr}(\rho_V(a)) \quad \forall a \in A.$$

Note Let

$$[A,A] = \mathrm{Span}\{[a,b] \mid a,b \in A\}$$

Then $[A,A] \subset \mathrm{Ker}\,\chi_V$ for any fd rep V so we can regard

vector space, not alg

$$\chi_V : \frac{A}{[A,A]} \longrightarrow k$$

Thm i) Characters of distinct fd irreps are linearly indep

ii) If A is fd semisimple, these form a basis for $\left(A/[A,A]\right)^*$.

Pf i) $V_1, \dots, V_r$ noniso irreps

Then $\oplus \rho_i : A \to \oplus \operatorname{End} V_i$

surj. by density Thm.

So $\chi_{V_1}, \dots, \chi_{V_r}$ are lin. indep.

ii) $[\mathrm{Mat}_d, \mathrm{Mat}_d] = \mathfrak{sl}_d$

Then

$$[A, A] = \bigoplus_{i=1}^{r} \mathfrak{sl}_{d_i}(k)$$

so $\dim\left(A/[A,A]\right)^* = r$.

Lecture 12

§ 3.7 Jordan-Hölder Thm

Th: A any alg
V a fd rep of A

Let

$$0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$$

and

$$0 = V_0' \subset V_1' \subset \cdots \subset V_m' = V$$

be two composition series for V

That is,

$$W_i = V_i/V_{i-1} \quad \text{and} \quad W_i' = V_i'/V_{i-1}'$$

are irreducible $\forall i$.

Then $m = n$, and $\exists$ permutation σ such that

$$W_i' \cong W_{\sigma(i)} \quad \forall i = 1, \ldots, n.$$

Proof We assume char $k = 0$ (see book for general case)

$$\chi_V = \sum_{i=1}^{n} \chi_{W_i} = \sum_{i=1}^{m} \chi_{W_i'}$$

But characters of irreps are linearly independent. So the multiplicity of an irrep W among W_i and among W_i' is the same.

Def The "n" is the length of V.

§3.8 Krull–Schmidt Thm

Thm Any fd rep of A can be uniquely (up to isomorphisms and reordering summands) be decomposed into a direct sum of indecomposable reps.

Pf Existence is easy by induction. Suppose

$$V = V_1 \oplus \cdots \oplus V_m = V_1' \oplus \cdots \oplus V_n'.$$

Let $i_s : V_s \to V \quad i_s' : V_s' \to V$

$p_s : V \to V_s \quad p_s' : V \to V_s'$

be the inclusions and projections. Let $\theta_s = p_1 i_s' p_s' i_1 : V_1 \to V_1$. Then

$\sum_{s=1}^{n} \theta_s = \mathrm{Id}_{V_1}$ We will use Lemma:

Lemma 3.8.2

W fd indec rep of A
Then

i) Any homomorphism

$$\theta : W \longrightarrow W$$

is either an isomorphism or nilpotent.

ii) If $\theta_s : W \to W$ $s = 1, \ldots, n$ are nilpotent homomorphisms then so is $\theta := \theta_1 + \cdots + \theta_n$.

<u>Proof</u> i) Generalized eigenspaces of θ are subreps of W and W is their sum. Thus θ can only have one eigenvalue, say λ. If $\lambda = 0$, θ is nilpotent. If $\lambda \neq 0$, θ has $\neq 0$ determinant and thus is an isomorphism.

ii) By induction on n, we may assume $n=2$.

If $\theta = \theta_1 + \theta_2$ is not nilpotent, then by 1) it is an isomorphism. Then

$$Id_W = \theta^{-1}\theta_1 + \theta^{-1}\theta_2$$

$\theta^{-1}\theta_i$ are not isomorphisms (det $=0$) so they are nilpotent.

So $T = Id_W - \theta^{-1}\theta_1$ is an isomorphism (with inverse $1 + T + \cdots + T^{N-1}$, if $(\theta^{-1}\theta_1)^N = 0$)

contradicting that $T = \theta^{-1}\theta_2$ is nilpotent.

∎

(Proof of Krull-Schmidt contd.)

By Lemma, θ_s is an isomorphism for some s.

Then $V_1' = \operatorname{Im}(p_1' i_1) \oplus \operatorname{Ker}(p_1 i_1')$

so, since V_1' is indec,

$f := p_1' i_1 : V_1 \to V_1'$ and

$g := p_1 i_1' : V_1' \to V_1$ are

isomorphisms.

Let $B = \bigoplus_{j>1} V_j$ $\quad B' = \bigoplus_{j>1} V_j'$

Then

$$V = V_1 \oplus B = V_1 \oplus B' \quad (*)$$

Let

$h : B \to B'$ be $B \hookrightarrow V \twoheadrightarrow B'$ (along $(*)$)

Then h is an isomorphism: If $v \in \operatorname{ker} h$ then $v \in V_1'$ so $gv = 0 \Rightarrow v = 0$.

$\dim B = \dim B' \Rightarrow h$ iso.

Now use induction. ∎

Lecture 13 Wed Feb 19, 2025

§4 Representations of finite groups.

G finite group

k algebraically closed field but arbitrary characteristic

$k[G]$ or kG the group algebra of G

We will denote the basis for kG

$\{ g : g \in G \} = G$

Works well when G is <u>written multiplicatively</u>

<u>Ex</u> $G = S_2$ $\quad kS_2 = \mathrm{Span}\{(1), (12)\}$

$G = \mathbb{Z}_n \cong \langle t \rangle$, $t^n = 1$

$k\mathbb{Z}_n = \mathrm{span}\{1, t, \dots, t^{n-1}\}$

$\cong k[t]/(t^n - 1)$

Thm 4.1.1 (Maschke's Thm)

Suppose the characteristic of k does not divide $|G|$
Then
i) $k[G]$ is semisimple

ii) There is an algebra isomorphism

$$\psi : k[G] \to \bigoplus_i \mathrm{End}(V_i)$$

$$g \longmapsto g|_{V_i} := \rho_{V_i}(g)$$

where V_i are the distinct irreps of G. In particular, ψ is an isomorphism of representations, (G acts on both sides by left mult.)

Hence the regular rep. decomposes as

$$k[G] \cong \bigoplus_i (\dim V_i) V_i$$

and $|G| = \sum (\dim V_i)^2$

Proof By Prop 3.5.8, (i) $\Rightarrow$ (ii)

To prove (i), it suffices to show that if V is any fd rep of G and $W \subset V$ any subrep, then there is a subrep $W' \subset V$ such that $V = W \oplus W'$

First, choose any linear complement $\hat{W} \subset V$ of W in V.

Then $V = W \oplus \hat{W}$ as vector spaces but there is no reason $\hat{W}$ is a subrep.

Let $P : V \longrightarrow W$ be the projection onto W along $\hat{W}$. Then $P|_W = \mathrm{Id}_W$, $P|_{\hat{W}} = 0$.

Define the <u>symmetrized projection</u>

$$\bar{P} = \frac{1}{|G|} \sum_{g \in G} \rho(g) P \rho(g^{-1})$$

we can do this since $|G| \in k^\times$.

Now $\bar{P}|_W = \mathrm{Id}_W$ and

$\bar{P}(V) \subseteq W$ so $\bar{P}^2 = \bar{P}$

Let $W' = \ker \bar{P}$

Then $V = W \oplus W'$.

We show $\bar{P}$ is an intertwining operator, hence W' is a subrep.

$\forall h \in G$:

$$\bar{P}\rho(h) = \frac{1}{|G|} \sum_{g \in G} \rho(g) P \rho(g^{-1}h) =$$

{Make the substitution $g = hl$.}

$$= \frac{1}{|G|} \sum_{l \in G} \rho(hl) P \rho(l^{-1} \cancel{h^{-1}} \cancel{h})$$

$$= \rho(h) \bar{P}$$

∎

In fact, the converse holds:

Prop 4.1.2 If $k[G]$ is semisimple, then $\operatorname{char} k \nmid |G|$.

Pf Write $k[G] = \bigoplus_{i=1}^{r} \operatorname{End} V_i$

WLOG $V_1 = k$ is the trivial rep.

Then $k[G] = k \oplus \left(\bigoplus_{i=2}^{r} d_i V_i \right)$, $d_i = \dim V_i$

By Schur's Lemma

$$\operatorname{Hom}_{k[G]}(k, k[G]) = k\Lambda$$

$$\operatorname{Hom}_{k[G]}(k[G], k) = k\varepsilon$$

$\varepsilon : k[G] \to k$, $\Lambda : k \to k[G]$ nonzero maps of reps.

WLOG $\varepsilon(g) = 1$ $\forall g$ and $\Lambda(1) = \sum_{g \in G} g$

Then $\varepsilon \circ \underbrace{\Lambda(1)}_{\neq 0} = \varepsilon\left(\sum g\right) = \sum 1 = |G|$

$\Rightarrow |G| \neq 0$ in k i.e. $\operatorname{char} k \nmid |G|$. ∎

§4.2 Characters

Def A class function
$$f : G \to k$$
is a function that is constant on conjugacy classes.

Th 4.2.1 If $\operatorname{char} k \nmid |G|$, then the set of irreducible characters for G is a basis for the space of class functions.

Corollary The number of irreps of G is equal to the number of conjugacy classes of G.

$\underline{\text{Ex}}$ $G = S_3 \cong D_3$

$$S_3 = \{\underbrace{(1)}_{C_1}, \underbrace{(12), (23), (13)}_{C_2}, \underbrace{(123), (132)}_{C_3}\}$$

There are three conjugacy classes.
So there are 3 irreps.
Always 1 trivial

$$|S_3| = \sum_{i=1}^{r} (\dim V_i)^2$$

$$\Rightarrow \quad 6 = 1^2 + d_2^2 + d_3^2$$

The only solution is

$$6 = 1^2 + 1^2 + 2^2$$

V_1 = trivial

V_2 = sign rep

$V_3 = k^2$, $\rho((12)) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

belong to $k = \bar{k}$
char $k \neq 2, 3$

$$\rho((123)) = \begin{bmatrix} \cos\frac{2\pi}{3} & -\sin\frac{2\pi}{3} \\ \sin\frac{2\pi}{3} & \cos\frac{2\pi}{3} \end{bmatrix} = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$$

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$$F(G,k) = \{ f : G \to k \} \quad \text{all fcns}$$

$$F_c(G,k) = \{ f : G \to k \mid f(ghg^{-1}) = f(h) \ \forall g,h \in G \}$$

The characters $\chi_i := \chi_{V_i}$ of the irreducible reps $V_1, \dots, V_r$ of G are called irreducible characters.

Elements of $F_c(G,k)$ are the class functions of G.

Note: $\dim F_c(G,k) = \#$ conjugacy classes in G

Thm 4.2.1: When $\operatorname{char} k \nmid |G|$, $\{\chi_i\}_{i=1}^{r}$ is a basis for $F_c(G,k)$.

Pf

$$\underbrace{\left(A/[A,A]\right)^*}_{\text{space of all characters of } A} \cong F_c(G,k)$$

(Read in book)

(We previously showed that when A is semisimple, the irreducible characters are a basis for $(A/[A,A])^*$.)

Thus, for $A = k[G]$, when char $k \nmid |G|$, the number of irr. characters of G is equal to the number of conjugacy classes in G.

Character Tables

Example: $G = S_3$

S_3	1	3	2
	(1)	(12)	(123)
$\chi_1 = \chi_{triv}$	1	1	1
$\chi_2 = \chi_{sgn}$	1	-1	1
χ_3	2	0	-1

← size of each conjugacy class

← representatives g_j of conjugacy classes

$\chi_i(g_j)$

$$\rho_3((1)) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \chi_3((1)) = 2$$

$$\rho_3((12)) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow \chi_3((12)) = 0$$

$$\rho_3((123)) = \begin{bmatrix} \cos 2\pi/3 & -\sin \frac{2\pi}{3} \\ \sin 2\pi/3 & \cos 2\pi/3 \end{bmatrix} = \begin{bmatrix} -1/2 & \\ & -1/2 \end{bmatrix}$$

$$\Rightarrow \chi_3((123)) = -1$$

General features:

$$
\begin{array}{c|ccccc}
G & \overset{1}{e} & g & & & \\
\hline
\chi_{triv} & 1 & 1 & 1 & \cdots & 1 \\
\chi_2 & d_2 & * & * & \cdots & * \\
\chi_3 & d_3 & * & \ddots & & \vdots \\
\vdots & & \vdots & & \ddots & \vdots \\
\chi_r & d_r & * & \cdots & & *
\end{array}
$$

where $d_i = \dim V_i$

Example $G = Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$

quaternion group

$ijk = i^2 = j^2 = k^2 = -1 \quad \left(\Rightarrow \begin{array}{l} ij = k = -ji \\ jk = i = -kj \\ ki = j = -ik \end{array} \right)$

The conjugacy classes are

$\{1\} \quad \{-1\} \quad \{\pm i\} \quad \{\pm j\} \quad \{\pm k\}$

So there are $\overline{5}$ irreps one of which is the trivial.

Can they all be 1-dim'l?

No! Then $k[Q_8] \cong \bigoplus_{i=1}^{8} \mathrm{End}(V_i)$

$\cong k^8$ commutative!

$3^2 > 8$ so only way is

$8 = 1^2 + 1^2 + 1^2 + 1^2 + 2^2$

Q_8	1	1	2	2	2
	1	-1	i	j	k
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
$\vdots$	1	1	-1	1	-1
$\vdots$	1	1	-1	-1	1
χ_5	2				

By $Q_8 \twoheadrightarrow Q_8/Z(Q_8) = Q_8/\{\pm 1\} \cong \mathbb{Z}_2 \times \mathbb{Z}_2$

we get 4 1-dim'l irreps from those of $\mathbb{Z}_2 \times \mathbb{Z}_2$!

These all have: $\rho_\ell(-1) = 1$
$\ell = 1, 2, 3, 4$

Since $\rho_\ell(k) = \rho_\ell(i)\rho_\ell(j)$ and they are 1-dim'l,

$$\chi_\ell(k) = \chi_\ell(i)\chi_\ell(j) \quad \ell = 1, \dots, 4$$

so

$$\chi_\ell(i) = \pm 1, \quad \chi_\ell(j) = \pm 1$$

§ 4.4. Duals & $\otimes$

Dual rep:

$$\rho_{V^*}(g) = \rho_V(g^*)^{-1} = \rho_V(g^{-1})^*$$

$$\Rightarrow \chi_{V^*}(g) = \chi_V(g^{-1})$$

Now

$$\chi_V(g) = \sum \lambda_i \quad , \quad \lambda_i \in \mathbb{C}$$
$$|\lambda_i| = 1$$

since $\lambda_i^{|G|} = 1$ by $\rho_V(g)^{|G|} = 1$

$$\chi_{V \otimes W} = \chi_V \chi_W$$

§4.5 Orthogonality of Characters.

Define a positive definite Hermitian inner product on $F_c(G, \mathbb{C})$ by

$$(f_1, f_2) = \frac{1}{|G|} \sum_{g \in G} f_1(g) \overline{f_2(g)}$$

Th 451 $\forall$ reps V, W

$$(\chi_V, \chi_W) = \dim_G(V, W)$$

Thus

$$(\chi_V, \chi_W) = \begin{cases} 1 & V \cong W \\ 0 & V \not\cong W \end{cases}$$

pf. $(\chi_V, \chi_W) = \frac{1}{|G|} \sum \chi_V(g) \chi_{W^*}(g)$

$$= \frac{1}{|G|} \sum \chi_{V \otimes W^*}(g)$$

$$= Tr_{V \otimes W^*} \Big(\underbrace{\frac{1}{|G|} \sum g}_{p \in Z(\mathbb{C}G)} \Big)$$

$$P|_{\text{triv}} = 1$$

$$P|_V = 0$$

V irr $\neq 1$