[HJ §7.5]

Def Let $A, B \in M_{m,n}$. The Hadamard (or Schur) product is

$$A \circ B := [a_{ij} b_{ij}]$$

Ex If A, B are Hermitian then so is $A \circ B$:

$$(A \circ B)^* = [\bar{a}_{ji} \bar{b}_{ji}] \overset{A,B \text{ Herm.}}{=} [a_{ij} b_{ij}] = A \circ B$$

Not true for usual matrix mult:

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 5 \\ 3 & 4 \end{bmatrix}$$

Lemma If $A \in M_n$ is PSD of rank k then $A = v_1 v_1^* + v_2 v_2^* + \cdots + v_k v_k^*$ for some $0 \neq v_i \in \mathbb{C}^n$ that are orthogonal

Pf By Spectral Thm

$$A = U \Lambda U^*$$

$$\Lambda = \operatorname{diag}(\lambda_1, \underbrace{\ldots, \lambda_n}_{\text{last } n-k \text{ zero}}) \qquad \lambda_1, \ldots, \lambda_k \neq 0.$$

Put $v_i = \lambda_i^{1/2} u_i$. Then $\{v_i\}_{i=1}^k$ are orthogonal

$$A = U\Lambda U^* \implies A = \lambda_1 u_1 u_1^* + \dots + \lambda_k u_k u_k^* =$$
$$= v_1 v_1^* + \dots + v_k v_k^* .$$

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<u>Th</u> (Schur Product Theorem)
If $A, B \in M_n$ are PSD, then so is $A \circ B$.
If $A, B \in M_n$ are PD, then so is $A \circ B$.

<u>Proof</u> Write

$$A = v_1 v_1^* + \dots + v_k v_k^*, \quad k = \operatorname{rank} A$$
$$B = w_1 w_1^* + \dots + w_m w_m^*, \quad m = \operatorname{rank} B$$

Note that

$$\underset{[a_{kl} b_{kl}]}{\overset{\|}{A \circ B}} = \sum_{i,j} \underbrace{u_{ij} u_{ij}^*}_{\text{PSD matrices (of rank 1)}} \quad \left(= \left[\sum_{ij} v_{ik} w_{jk} v_{il} w_{jl}\right]_{kl}\right), \quad u_{ij} = v_i \circ w_j \in \mathbb{C}^n$$
$$= [v_{i1} w_{j1} \cdots v_{in} w_{jn}]^T$$

$\implies A \circ B$ is also PSD.

(using $x^*(P+Q)x = \underbrace{x^* P x}_{\geq 0} + \underbrace{x^* Q x}_{\geq 0} \geq 0$)

Suppose A and B are PD.
WTS $A \circ B$ nonsingular. Let $x \in \mathbb{C}^n$
Suppose $(A \circ B)x = 0$. Then

$$0 = x^*(A \circ B)x = \sum_{i,j} \underbrace{x^* u_{ij}}_{\in \mathbb{C}} \underbrace{u_{ij}^* x}_{\in \mathbb{C}} = \sum_{i,j} |x^* u_{ij}|^2$$

$$\implies \forall i,j: \ 0 = |x^* u_{ij}|^2 = |x^*(v_i \circ w_j)|^2 =$$

$= |(x \circ \bar{v}_i)^* w_j|^2$ since 0 for every j and $\{w_j\}$ is an orthogonal basis for $\mathbb{C}^n$

$\Rightarrow \quad x \circ \bar{v}_i = 0 \quad \forall i$

$\Rightarrow \quad v_i^* \cdot x = 0 \quad \forall i \qquad \{v_i\}$ basis

$\Rightarrow \quad x = 0.$

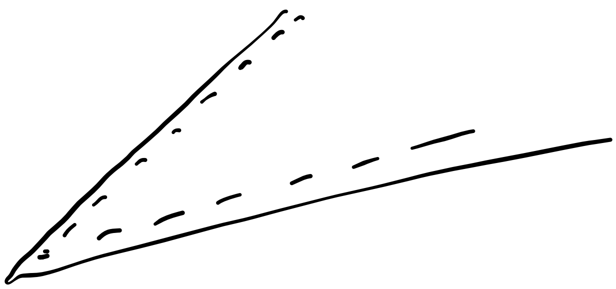
Thus $A \circ B$ nonsing, hence PD. $\blacksquare$

$$\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \circ \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

rank 1   rank 1   rank 0

Wiki: $\operatorname{rank}(A \circ B) \leq (\operatorname{rank} A)(\operatorname{rank} B)$

Exercise: Prove it!



$\lambda > 0 \quad A$ P(S)D

$\Rightarrow \lambda A$ P(S)D

A, B PSD $\Rightarrow A + B$ PSD.

$\Rightarrow$ PSD matrices form a cone in $M_n(\mathbb{C})$

Corollary (Fejer's Thm)

Let $A = [a_{ij}] \in M_n$. Then A is PSD iff $\sum_{i,j} a_{ij} b_{ij} \geq 0$ for all PSD matrices $B = [b_{ij}]$.

Proof Suppose A, B PSD.

Let $\mathbb{1} = [1 \ 1 \ \cdots \ 1]^T \in \mathbb{C}^n$ be the "all ones" vector. By Schur Product Thm $A \circ B$ is PSD. So

$$0 \leq \mathbb{1}^* \underbrace{(A \circ B)}_{[a_{ij} b_{ij}]} \mathbb{1} = \sum_{i,j} a_{ij} b_{ij}$$

Conversely, spse $\sum_{ij} a_{ij} b_{ij} \geq 0 \ \forall$ PSD B.

Let $x \in \mathbb{C}^n$. WTS $x^* A x \geq 0$.

Choose $B = [\overline{x_i} x_j]$. Then B is PSD: $B = \bar{x}\bar{x}^*$. Thus

$$0 \leq \sum_{i,j} a_{ij} \bar{x}_i x_j = x^* A x \Rightarrow A \text{ PSD.}$$

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