510

Invariant Factors II.

FACT: Every field $\mathbb{F}$ is contained in a (*) alg closed field $\overline{\mathbb{F}}$, called ~~alg~~ closure
(*) unique minimal

Theorem

$$A(x) \sim \begin{bmatrix} f_1 & & \\ & \ddots & \\ & & f_n \end{bmatrix} \quad f_i \mid f_{i+1}, \quad f_i \in \mathbb{F}[x]$$

a) The invariant factors of any $A(x) \in \mathbb{F}[x]^{n\times n}$ are given by

$$f_p(x) = \frac{\delta_p(x)}{\delta_{p-1}(x)} \qquad p = 1, 2, \ldots, n$$

where $\delta_p(x)$ = gcd of all $p\times p$-minors in $A(x)$

b) $$f_p(x) = \prod_{i=1}^{m} (x-\lambda_i)^{n^{(i)}_{n-p+1}}$$

Inv. factors of $xI-A$

where $\overline{\mathbb{F}} \ni \lambda_i$: $n_1^{(i)} \geq \cdots \geq n_{r_i}^{(i)}$ $\quad (n_j^{(i)} = 0,\ j > r_i)$ is the Jordan form data. of $A \in \mathbb{F}[x]$ over $\overline{\mathbb{F}}$.

Corollary a) The invariant factors of $xI - A$ are unique.

b) The Jordan normal form of A over $\overline{\mathbb{F}}$ is unique.

c) $m_A(x) = f_n(x)$, hence $m_A(x) \mid c_A(x)$

$\Rightarrow c_A(A) = 0$ (Cayley-Hamilton)

char. pol. $\det(xI-A) = f_1 f_2 \cdots f_n$

Proof

a) Suppose

$$xI - A \underset{\mathbb{F}[x]^{n\times n}}{\sim} \begin{bmatrix} f_n(x) & & & \\ & f_{n-1}(x) & & \\ & & \ddots & \\ & & & f_1(x) \end{bmatrix}. \quad f_i \mid f_{i+1}$$

Write $f_p(x) = \prod_{i=1}^{m} (x-\lambda_i)^{m_p^{(i)}}$

$m_1^{(i)} \leq \cdots \leq m_{k_r}^{(i)}$.

By Exercise, $\delta_p(x)$ for $\begin{bmatrix} f_1 & & 0 \\ & \ddots & \\ 0 & & f_n \end{bmatrix}$ is the same.

Consider $\delta_p(x)$ for $xI - A$.

The gcd of all $p \times p$ minors is $f_1(x) \cdots f_p(x)$ as $f_1(x) \cdots f_p(x) \mid f_{i_1}(x) \cdots f_{i_p}(x)$ $\forall 1 \leq i_1 \leq \cdots \leq i_p \leq n$.

So $f_1(x) \cdots f_p(x) = \delta_p(x)$,

So $f_p(x) = \dfrac{\delta_p(x)}{\delta_{p-1}(x)}$,

b) $A \in \mathbb{F}^{n\times n}$

Consider $xI - A \in \mathbb{F}[x]^{n\times n}$.

We know there exists an invertible $P \in \overline{\mathbb{F}}^{n\times n}$ such that $A' = P^{-1}AP$ is in Jordan normal form. Then

$P^{-1}(xI - A)P = xI - P^{-1}AP$ which is a block diagonal matrix with blocks

$$\begin{bmatrix} x-\lambda_1 & -1 & & \\ & x-\lambda_1 & -1 & \\ & & \ddots & -1 \\ & & & x-\lambda_1 \end{bmatrix} \quad (*)$$

For each λ_i appearing we have a number of blocks

$\lambda_1 : \quad n_1^{(1)} \geq n_2^{(1)} \geq \cdots \geq n_{r_1}^{(1)} \geq 1$

$\vdots$

$\lambda_m : \quad n_1^{(m)} \geq \cdots \geq n_{r_m}^{(m)} \geq 1$

In (*), add $(x-\lambda_i)$ times 1^{st} row & add to 2^{nd}

$$\begin{bmatrix} x-\lambda_i & -1 & & & \\ (x-\lambda_i)^2 & 0 & -1 & & \\ 0 & 0 & x-\lambda_i & \ddots & \\ & & & \ddots & \end{bmatrix}$$

Next, add $(x-\lambda_1) \times 2^{nd}$ to 3^{rd}. And so on, we get

$$\begin{bmatrix} x-\lambda_i & -1 & & & \\ (x-\lambda_i)^2 & 0 & -1 & & \\ \vdots & & \ddots & \ddots & \\ \vdots & & & \ddots & -1 \\ (x-\lambda_i)^{n_j^{(i)}} & & & & 0 \end{bmatrix}$$

Add $(x-\lambda_i)^k \times (k+1)^{st}$ col to first col. Change signs on cols $2, \dots, n_j^{(i)}$

Now permute columns to get

$$\begin{bmatrix} +1 & & & \cancel{*} 0 \\ & +1 & & \vdots \\ & & \ddots & \vdots \\ & & +1 & \cancel{*} 0 \\ & & & (x-\lambda_i)^{n_j^{(i)}} \end{bmatrix}$$

This shows there are (invertible) elementary matrices $R, S \in \overline{\mathbb{F}}[x]^{n\times n}$ such that

$$B = R(xI - A')S =$$

$$= \begin{bmatrix} (x-\lambda_1)^{n_1^{(1)}} & & & & & & & & \\ & (x-\lambda_1)^{n_2^{(1)}} & & & & & & & \\ & & \ddots & & & & & & \\ & & & (x-\lambda_1)^{n_{r_1}^{(1)}} & & & & & \\ & & & & (x-\lambda_2)^{n_1^{(2)}} & & & & \\ & & & & & \ddots & & & \\ & & & & & & (x-\lambda_2)^{n_{r_2}^{(2)}} & & \\ & & & & & & & \ddots & \\ & & & & & & & (x-\lambda_m)^{n_{r_m}^{(m)}} & \\ & & & & & & & & I_{n-N} \end{bmatrix}$$

where $N = r_1 + r_2 + \cdots + r_m$ = total # of Jordan blocks in A'.

By Exercise, $\delta_p(x)$ for $xI - A$ is the same as $\delta_p(x)$ for B, $0 \le p \le n$.

<u>Now observe</u>: Since B is diagonal, any $p \times p$ minor is zero unless it is a <u>principal</u> minor (subsets $I = J$)

Secondly, if $p \le n - r_1$, then there is a $p \times p$ minor without $x - \lambda_1$

so $\mu_p^{(1)} = 0$ in

$$\delta_p(x) = \prod_{i=1}^{m} (x - \lambda_i)^{\mu_p^{(i)}}$$

However, if $p = n - r_1 + 1$, then every ~~a~~ minimum of $p \times p$-minor contains $(x - \lambda_1)^{n_{r_1}^{(1)}}$, So $\mu_p^{(1)} = n_{r_1}^{(1)}$

Next ~~Each~~ increment of p adds $n_{r_1 - 1}^{(1)}$ etc

$$\mu_p^{(k)} = \begin{cases} 0 & p \le n - r_k \\ n_{r_k}^{(k)} & p = n - r_k + 1 \\ n_{r_k}^{(k)} + n_{r_k - 1}^{(k)} & p = n - r_k + 2 \\ \vdots & \\ n_{r_k}^{(k)} + \cdots + n_1^{(k)} & p = n \end{cases}$$

~~Thus $b_j^{(k)} = \mu^{(k)}$~~

Thus
$$\begin{cases} n_1^{(k)} = \mu_n^{(k)} - \mu_{n-1}^{(k)} \\ \vdots \\ n_{r_k}^{(k)} = \mu_{n-r_k+1}^{(k)} - \mu_{n-r_k}^{(k)} \end{cases}$$

So

$$f_{p+1}(x) = \frac{\delta_{p+1}(x)}{\delta_p(x)} = \prod_{i=1}^{m} (x-\lambda_i)^{\mu_{p+1}^{(i)} - \mu_p^{(i)}} =$$

$$= \prod_{i=1}^{m} (x-\lambda_i)^{n_{n-p}^{(i)}} \quad \cancel{= f_{n+1-(n-p)}(x) = f_{p+1}(x)}$$

Ex. $p = n-1$: $\quad \dfrac{\delta_n(x)}{\delta_{n-1}(x)} = \prod_{i=1}^{m} (x-\lambda_i)^{n_1^{(i)}} = m_A(x).$