Exercise. What happens in (5.3.2) if T is singular?

Exercise. Why must $\|x\| \equiv (|2x_1 - 3x_2|^2 + |x_2|^2)^{1/2}$ be a norm on $\mathbf{C}^2$ (no computations, please!).

New norms can be constructed from old ones by using the notion of duality. This method is discussed at the end of Section (5.4).

Problems

1. If $\|\cdot\|$ is a vector seminorm, show that $\|x\|_T \equiv \|Tx\|$ is also a vector seminorm for any $T \in M_n$. If $\|\cdot\|$ is actually a vector norm, then $\|\cdot\|_T$ is a vector seminorm whose null space is the null space of T.

2. Show that any vector seminorm is of the form $\|\cdot\|_T$ for some vector norm $\|\cdot\|$ and some $T \in M_n$.

5.4 Analytic properties of vector norms

The examples in the preceding two sections make it clear that there are many different functions $\|\cdot\|: V \to \mathbf{R}$ that satisfy the axioms for a norm. It is useful to have many different norms available because one norm may be more convenient or more appropriate than another for a given purpose. For example, the l_2 norm is often convenient to use in optimization problems because it is continuously differentiable (except at the origin). On the other hand, the l_1 norm, while differentiable on a smaller set, is popular in statistics because it leads to estimators that can be more robust than the classical regression estimators. The l_∞ norm is often the most natural one to use, since it directly monitors element-by-element convergence, but, unfortunately, it can be analytically and algebraically awkward to use. In actual applications, the norm on which theory is most naturally based and the norm that is most easily calculated in a given situation may not coincide. It is important, therefore, to know what relationship there may be between two different norms. Fortunately, in the finite-dimensional case all norms are "equivalent" in a certain strong sense.

A basic notion in analysis is that of *convergence of a sequence,* and vector norms can be used to measure convergence of a sequence of vectors.

5.4.1 **Definition.** Let V be a vector space over $\mathbf{R}$ or $\mathbf{C}$ and let $\|\cdot\|$ be a norm on V. We say that the sequence $\{x^{(k)}\}$ of vectors in V *converges* to a vector $x \in V$ with respect to the norm $\|\cdot\|$ if and only if $\|x^{(k)} - x\| \to 0$ as $k \to \infty$.

If $\{x^{(k)}\}$ converges to x with respect to the norm $\|\cdot\|$, we write

$$x^{(k)} \underset{\|\cdot\|}{\to} x \quad \text{or} \quad \lim_{k\to\infty} x^{(k)} = x \quad \text{with respect to} \quad \|\cdot\|$$

It must be made clear which norm is involved in the convergence in question; the issue arises as to whether a given sequence of vectors can converge with respect to one norm but not with respect to another. This ambiguity can happen in an infinite-dimensional vector space.

5.4.2 **Example.** Consider the sequence $\{f_k\}$ of functions in $C[0,1]$ (the vector space of all real-valued or complex-valued continuous functions on $[0,1]$) defined by

$$f_k(x)=0 \qquad 0\le x\le \frac{1}{k}$$

$$f_k(x)=2(k^{3/2}x-k^{1/2}), \qquad \frac{1}{k}\le x\le \frac{3}{2k}$$

$$f_k(x)=2(-k^{3/2}x+2k^{1/2}), \qquad \frac{3}{2k}\le x\le \frac{2}{k}$$

$$f_k(x)=0, \qquad \frac{2}{k}\le x\le 1$$

for $k=2,3,4,\ldots$. One may then calculate that

$$\|f_k\|_1=\tfrac{1}{2}k^{-1/2}\to 0 \quad \text{as} \quad k\to\infty$$

$$\|f_k\|_2=\frac{1}{\sqrt{3}} \quad \text{for all} \quad k$$

$$\|f_k\|_\infty=k^{1/2}\to\infty \quad \text{as} \quad k\to\infty$$

Thus, $\lim_{k\to\infty} f_k=0$ with respect to the L_1 norm but not with respect to the other two norms.

Exercise. Sketch the functions described in the preceding example and verify the assertions made about the L_1, L_2, and L_∞ norms.

Exercise. If $\|\cdot\|$ is a vector norm, if $x^{(k)} \underset{\|\cdot\|}{\to} x$, and if $x^{(k)} \underset{\|\cdot\|}{\to} y$, use the triangle inequality to show that $x=y$. Thus it makes sense to talk about *the* limit of a sequence (if any) with respect to a given norm.

Fortunately, the phenomenon in Example (5.4.2) cannot occur in the case of a finite-dimensional vector space. In order to see this, we need a general lemma about the continuity properties of norms.

5.4.3 **Lemma.** Let $\|\cdot\|$ be a norm on a vector space V over the field **F** (**R** or **C**), and let $x^{(1)}, x^{(2)}, \ldots, x^{(m)} \in V$ be given vectors. The function $g: \mathbf{F}^m \rightarrow \mathbf{R}$ defined by

$$g(z_1, z_2, \ldots, z_m) \equiv \|z_1 x^{(1)} + z_2 x^{(2)} + \cdots + z_m x^{(m)}\|$$

is a uniformly continuous function.

Proof: Let $u = \sum_{i=1}^m u_i x^{(i)}$ and $v = \sum_{i=1}^m v_i x^{(i)}$, and calculate

$$|g(u_1, \ldots, u_m) - g(v_1, \ldots, v_m)| = |\,\|u\| - \|v\|\,| \leq \|u - v\|$$
$$= \left\| \sum_{i=1}^m (u_i - v_i) x^{(i)} \right\| \leq \sum_{i=1}^m |u_i - v_i| \, \|x^{(i)}\| \leq C \max_{1 \leq i \leq m} |u_i - v_i|$$

where $C \equiv m \max_{1 \leq i \leq m} \|x^{(i)}\|$. The first inequality comes from Lemma (5.1.2). Notice that the finite constant C depends only upon the norm $\|\cdot\|$ and the m vectors $x^{(1)}, \ldots, x^{(m)}$. If the vectors $x^{(i)}$ are all the zero vector, there is nothing to show, and, if not, then $C > 0$. In order to have $|g(u_1, \ldots, u_m) - g(v_1, \ldots, v_m)| < \epsilon$, we need only choose $|u_i - v_i| < \epsilon/C$. □

Although the vector space V need not be finite-dimensional for the lemma, it is important that the number of vectors $x^{(i)}$ be finite.

Exercise. Deduce from the lemma that every vector norm on $\mathbf{R}^n$ or $\mathbf{C}^n$ is a uniformly continuous function.

Finite dimensionality of V is, however, essential for the following fundamental fact.

5.4.4 **Theorem.** Let f_1 and f_2 be two real-valued functions on a finite-dimensional vector space V over the field **F** (**R** or **C**), and let $\mathcal{B} = \{x^{(1)}, \ldots, x^{(n)}\}$ be a basis for V. Assume that f_1 and f_2 are

(a) Positive: $f_i(x) \geq 0$ for all $x \in V$, $f_i(x) = 0$ if and only if $x = 0$;
(b) Homogeneous: $f_i(\alpha x) = |\alpha| f_i(x)$ for all $\alpha \in \mathbf{F}$ and all $x \in V$; and
(c) Continuous: $f_i(x(z))$ is continuous on $\mathbf{F}^n$, where

$$z = [z_1, \ldots, z_n]^T \in \mathbf{F}^n \quad \text{and} \quad x(z) \equiv z_1 x^{(1)} + \cdots + z_n x^{(n)}$$

Then there exist finite positive constants C_m and C_M such that

$$C_m f_1(x) \leq f_2(x) \leq C_M f_1(x)$$

for all $x \in V$.

Proof: Define $h(z) \equiv f_2(x(z))/f_1(x(z))$ on the Euclidean unit sphere $S = \{z \in \mathbf{F}^n : \|z\|_2 = 1\}$, a compact set in $\mathbf{F}^n$. Notice that the denominator of

$h(z)$ does not vanish on S by (a), and therefore $h(z)$ is continuous on S by (c). By the Weierstrass theorem (see Appendix E), the continuous function h achieves a finite positive maximum C_M and a positive minimum C_m on the compact set S and hence

$$C_m f_1(x(z)) \le f_2(x(z)) \le C_M f_1(x(z))$$

for all $z \in S$. Because $z/\|z\|_2 \in S$ for every nonzero $z \in \mathbf{F}^n$, (b) ensures that these inequalities hold for all nonzero $z \in \mathbf{F}^n$; they hold trivially for $z=0$ since $f_i(0)=0$. But every $x \in V$ is of the form $x=x(z)$ for some $z \in \mathbf{F}^n$ because $\mathcal{B}$ is a basis, so the asserted inequalities hold for all $x \in V$. □

Definition. Let V be a real or complex vector space. A function $f: V \to \mathbf{R}$ that satisfies the three hypotheses of positivity, homogeneity, and continuity in Theorem (5.4.4) is said to be a *pre-norm*.

The most important example of a class of pre-norms is, of course, the vector norms; Lemma (5.4.3) says that every vector norm satisfies the continuity assumption (c) of Theorem (5.4.4). A pre-norm that satisfies the triangle inequality is a vector norm. Because of the importance of this class, we state the result in this case as the following corollary.

5.4.5 Corollary. Let $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$ be any two vector norms on a finite-dimensional real or complex vector space V. Then there exist finite positive constants C_m and C_M such that $C_m\|x\|_\alpha \le \|x\|_\beta \le C_M\|x\|_\alpha$ for all $x \in V$.

Exercise. How does (5.4.5) break down for vector seminorms?

Exercise. Let $x=[x_1, x_2]^T \in \mathbf{R}^2$ and consider the following norms on $\mathbf{R}^2$: $\|x\|_\alpha \equiv \|[10x_1, x_2]^T\|_\infty$, and $\|x\|_\beta \equiv \|[x_1, 10x_2]^T\|_\infty$. Show that the function $f(x) \equiv (\|x\|_\alpha \|x\|_\beta)^{1/2}$ is a pre-norm on $\mathbf{R}^2$ that is not a norm. See Problem 15 at the end of this section. *Hint:* Consider $f([1,1]^T)$, $f([0,1]^T)$, and $f([1,0]^T)$.

Exercise. If $\|\cdot\|_{\alpha_1}, \ldots, \|\cdot\|_{\alpha_k}$ are vector norms on V, show that $f(x) \equiv (\|x\|_{\alpha_1} \cdots \|x\|_{\alpha_k})^{1/k}$ and $h(x) \equiv \min\{\|x\|_{\alpha_1}, \ldots, \|x\|_{\alpha_k}\}$ are pre-norms on V that are not necessarily vector norms.

One consequence of (5.4.5) is the fact that convergence (in norm) of a sequence of vectors in a finite-dimensional complex vector space is independent of the norm used.

5.4.6 Corollary. If $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$ are vector norms on a finite-dimensional real or complex vector space, and if $\{x^{(k)}\}$ is a given sequence of vectors, then $\lim_{k\to\infty} x^{(k)} = x$ with respect to $\|\cdot\|_\alpha$ if and only if $\lim_{k\to\infty} x^{(k)} = x$ with respect to $\|\cdot\|_\beta$.

Proof: Since $C_m\|x^{(k)}-x\|_\alpha \le \|x^{(k)}-x\|_\beta \le C_M\|x^{(k)}-x\|_\alpha$ for all k, it follows that $\|x^{(k)}-x\|_\alpha \to 0$ if and only if $\|x^{(k)}-x\|_\beta \to 0$ as $k\to\infty$. □

5.4.7 **Definition.** Two norms are said to be *equivalent* if whenever a sequence $\{x^{(k)}\}$ converges to a vector x with respect to the first norm, then it converges to the same vector with respect to the second norm. Thus, (5.4.6) says that *for finite-dimensional real or complex vector spaces, all vector norms are equivalent.* We have seen in Example (5.4.2) that two different norms might not be equivalent on an infinite-dimensional space.

Since all vector norms on $\mathbf{R}^n$ or $\mathbf{C}^n$ are equivalent to $\|\cdot\|_\infty$, we have $\lim_{k\to\infty} x^{(k)} = x$ with respect to any vector norm if and only if

$$\lim_{k\to\infty} x_i^{(k)} = x_i \quad \text{for all} \quad i=1,\ldots,n$$

Componentwise convergence (with respect to any basis) is equivalent to convergence with respect to any vector norm.

Another important consequence of equivalence of all vector norms in the finite-dimensional case is that the unit ball and unit sphere of every vector norm is compact. This fact implies that a continuous complex-valued function on the unit ball of any vector norm is bounded and that it achieves its maximum and minimum if it is real-valued.

5.4.8 **Corollary.** Let V be $\mathbf{R}^n$ or $\mathbf{C}^n$. Let $f(\cdot)$ be a pre-norm on V. The sets $\{x: f(x)\le 1\}$ and $\{x: f(x)=1\}$ are compact. In particular, if $\|\cdot\|$ is a vector norm on V, then the closed unit ball $\{x: \|x\|\le 1\}$ and the unit sphere $\{x: \|x\|=1\}$ are both compact.

Proof: By (5.4.4) there is some $C>0$ such that $\|x\|_2 \le Cf(x)$ for all $x\in V$, so the set $\{x: f(x)\le 1\}$ is a bounded set, which is contained in an ordinary Euclidean ball of radius C centered at the origin. Both of the sets $\{x: f(x)=1\}$ and $\{x: f(x)\le 1\}$ are closed because $f(\cdot)$ is continuous. Since a closed bounded set in R^n or C^n is compact, we are done. □

Often we are not confronted with the problem of determining whether a given sequence $\{x^{(k)}\}$ converges to a given vector x, but rather with determining whether a given sequence $\{x^{(k)}\}$ converges to anything at all. For this reason, one needs to have a convergence criterion that is independent of the limit x to which the sequence converges. If there were such a limit x, then

$$\|x^{(k)}-x^{(j)}\| = \|x^{(k)}-x+x-x^{(j)}\| \le \|x^{(k)}-x\| + \|x-x^{(j)}\| \to 0$$

as $k, j \to \infty$. This is the motivation for the following.

5.4.9 **Definition.** A sequence $\{x^{(k)}\}$ in a vector space V is a *Cauchy sequence* with respect to the vector norm $\|\bullet\|$ if for each $\epsilon > 0$ there is a positive integer $N(\epsilon)$ such that

$$\|x^{(k_1)}-x^{(k_2)}\| \le \epsilon$$

whenever $k_1, k_2 \ge N(\epsilon)$.

5.4.10 **Theorem.** Let $\|\bullet\|$ be a given norm on a finite-dimensional real or complex vector space V, and let $\{x^{(k)}\}$ be a given sequence of vectors in V. The sequence $\{x^{(k)}\}$ converges to a vector in V if and only if it is a Cauchy sequence with respect to the norm $\|\bullet\|$.

Proof: By choosing a basis $\mathcal{B}$ of V and considering the equivalent norm $\|[x]_{\mathcal{B}}\|_\infty$, we see that there is no loss of generality if we assume that $V = \mathbf{R}^n$ or $\mathbf{C}^n$ for some integer n and if we assume that the norm is $\|\bullet\|_\infty$. If $\{x^{(k)}\}$ is a Cauchy sequence, then so is each component sequence $\{x_i^{(k)}\}$ of real or complex numbers for each $i = 1, \ldots, n$. Since a Cauchy sequence of real or complex numbers must have a limit, this means that for each $i = 1, \ldots, n$ there is a scalar x_i such that $\lim_{k\to\infty} x_i^{(k)} = x_i$; it is easy to check that $\lim_{k\to\infty} x^{(k)} = x$, where $x = [x_1, \ldots, x_n]^T$. On the other hand, if there is an x such that $\lim_{k\to\infty} x^{(k)} = x$, then $\|x^{(k_1)}-x^{(k_2)}\| \le \|x^{(k_1)}-x\| + \|x-x^{(k_2)}\|$ and the given sequence is a Cauchy sequence. □

It is a fundamental property of the real and complex fields (used in the proof of the preceding theorem) that a sequence is a Cauchy sequence if and only if it converges to some (real or complex) scalar. This is known as the *completeness property* of the real and complex fields, and we have just shown that the completeness property extends to finite-dimensional real and complex vector spaces with respect to any vector norm. Unfortunately, the completeness property need not hold for vector spaces that are not finite-dimensional.

5.4.11 **Definition.** A vector space V with a norm $\|\bullet\|$ is said to be *complete with respect to the norm* $\|\bullet\|$ if every sequence that is a Cauchy sequence with respect to the norm $\|\bullet\|$ converges to a point of V.

Exercise. Consider the vector space $C[0,1]$ with the L_1 norm $\|f\|_1 = \int_0^1 |f(t)|\, dt$, and consider the sequence of functions $\{f_k\}$ defined by

$f_k(t) = 0,$

$f_k(t) = \frac{k}{2}\left(t - \frac{1}{2}\right.$

$f_k(t) = 1,$

Sketch the functions f_k.
there is no function $f \in$
$\|\bullet\|_1$.

Using the fact that the
or $\mathbf{C}^n$ is compact, we can
new norms from old one

5.4.12 **Definition.** Let

$f^D(y) \equiv \max_{f(x)=1}$ R

is called the *dual norm*

Observe first that the
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Finally, we must ob
tion f^D is well deserved
and it is positive, for if y
that

$f^D(y) = \max_{f(x)=1} |$

$$f_k(t)=0, \qquad 0\le t\le \frac{1}{2}-\frac{1}{k}$$

$$f_k(t)=\frac{k}{2}\left(t-\frac{1}{2}+\frac{1}{k}\right), \qquad \frac{1}{2}-\frac{1}{k}\le t\le \frac{1}{2}+\frac{1}{k}$$

$$f_k(t)=1, \qquad \frac{1}{2}+\frac{1}{k}\le t\le 1$$

Sketch the functions f_k. Show that $\{f_k\}$ is a Cauchy sequence but that there is no function $f\in C[0,1]$ for which $\lim_{k\to\infty} f_k=f$ with respect to $\|\cdot\|_1$.

Using the fact that the unit ball of any vector norm or prenorm on $\mathbf{R}^n$ or $\mathbf{C}^n$ is compact, we can introduce another useful method of generating new norms from old ones.

5.4.12 **Definition.** Let $f(\cdot)$ be a pre-norm on $V=\mathbf{R}^n$ or $\mathbf{C}^n$. The function

$$f^D(y)\equiv \max_{f(x)=1} \operatorname{Re} y^*x$$

is called the *dual norm* of f.

Observe first that the dual norm is a well-defined function on V because $\operatorname{Re} y^*x$ is a continuous function of x for each fixed $y\in V$, and the set $\{x: f(x)=1\}$ is a compact set by (5.4.8). By the Weierstrass theorem, the maximum of $\operatorname{Re} y^*x$ is attained at some point $x_0\in\{x: f(x)=1\}$. If c is a scalar such that $|c|=1$, then by the homogeneity of f we have

$$\begin{aligned}\max_{f(x)=1} |y^*x| &= \max_{f(x)=1}\ \max_{|c|=1} \operatorname{Re} cy^*x\\ &= \max_{f(x)=1}\ \max_{|c|=1} \operatorname{Re} y^*(cx)\\ &=\max_{|c|=1}\ \max_{f(x/c)=1} \operatorname{Re} y^*x = \max_{f(x)=1} \operatorname{Re} y^*x\end{aligned}$$

so an equivalent and sometimes convenient definition for the dual norm is

$$f^D(y)=\max_{f(x)=1} |y^*x| \tag{5.4.12a}$$

Finally, we must observe that the name *dual norm* for the function f^D is well deserved. The function $f^D(\cdot)$ is evidently homogeneous and it is positive, for if $y\neq 0$, we can use the homogeneity of $f(\cdot)$ to show that

$$f^D(y)=\max_{f(x)=1} |y^*x|\ge \left|y^*\frac{y}{f(y)}\right| = \frac{\|y\|_2^2}{f(y)}>0$$

It is perhaps remarkable that even if the function $f(\cdot)$ does not obey the triangle inequality, its dual $f^D(\cdot)$ always does:

$$f^D(y+z) = \max_{f(x)=1} |(y+z)^*x| \le \max_{f(x)=1} [|y^*x| + |z^*x|]$$

$$\le \max_{f(x)=1} |y^*x| + \max_{f(x)=1} |z^*x| = f^D(y) + f^D(z)$$

The dual norm of a pre-norm is therefore always a norm.

Thus, any pre-norm generates a norm by the process of constructing the dual norm. The most common instance of this construction is for a pre-norm that is actually a norm.

A simple inequality for the dual norm is given in the following lemma. We shall see that it is a natural generalization of the Cauchy-Schwarz inequality.

5.4.13 Lemma. Let $f(\cdot)$ be a pre-norm on $V = \mathbf{C}^n$ or $\mathbf{R}^n$. Then

$$|y^*x| \le f(x) f^D(y)$$

$$|y^*x| \le f^D(x) f(y)$$

for all $x, y \in V$.

Proof: If $x \ne 0$, then

$$\left| y^* \frac{x}{f(x)} \right| \le \max_{f(z)=1} |y^*z| = f^D(y)$$

and hence $|y^*x| \le f(x) f^D(y)$. Since this inequality also holds for $x=0$, we are done. The second inequality follows from the first since $|y^*x| = |x^*y|$. □

It is easy to identify the duals of some of the most common vector norms. If $x, y \in \mathbf{C}^n$, then a special case of Hölder's inequality is

$$|y^*x| = \left| \sum_{i=1}^n \bar{y}_i x_i \right| \le \sum_{i=1}^n |\bar{y}_i x_i| \le \max_{1 \le i \le n} |y_i| \sum_{j=1}^n |x_j| = \|y\|_\infty \|x\|_1 \qquad (5.4.14)$$

If y is a given vector, then equality holds in (5.4.14) when x is a unit vector (with respect to $\|\cdot\|_1$) such that $x_i = 1$ for some one value of i for which $|y_i| = \|y\|_\infty$, and $x_i = 0$ otherwise. Similarly, if x is a given nonzero vector, then equality holds in (5.4.14) when y is a unit vector (with respect to $\|\cdot\|_\infty$) such that $y_i = x_i/|x_i|$ for all i such that $x_i \ne 0$ and, $y_i = 0$ otherwise. Thus,

$$(\|y\|_1)^D = \max_{\|x\|_1=1} |y^*x| = \max_{\|x\|_1=1} \|y\|_\infty \|x\|_1 = \|y\|_\infty$$

$$(\|y\|_\infty)^D = \max_{\|x\|_\infty=1} |y^*x| = \max_{\|x\|_\infty=1} \|y\|_1 \|x\|_\infty = \|y\|_1$$

We conclude that $(\|\cdot\|_1)^D = \|\cdot\|_\infty$ and $(\|\cdot\|_\infty)^D = \|\cdot\|_1$.

If we consider the Euclidean norm $\|\cdot\|_2$, a given nonzero vector y, and an arbitrary vector x, then the Cauchy-Schwarz inequality says that

$$|y^*x| = \left|\sum_{i=1}^{n} \bar{y}_i x_i\right| \le \|y\|_2\|x\|_2 \tag{5.4.15}$$

with equality when $x = y/\|y\|_2$. Using the same argument as above for the l_1 and l_∞ norms, we find that $(\|y\|_2)^D = \|y\|_2$, so the Euclidean norm is its own dual.

Exercise. Explain why the inequalities in (5.4.13) are a generalization of the Cauchy-Schwarz inequality (5.1.4).

Notice that for each of the three norms just considered (l_1, l_2, and l_∞), the dual of the dual norm is the original norm. This is no accident; the duality theorem (5.5.14) says that this always happens.

Among these three examples, the only norm that equals its dual is the Euclidean norm. It is not difficult to show that this is also no accident.

5.4.16 **Theorem.** Let $\|\cdot\|$ be a vector norm on $V = \mathbf{R}^n$ or $\mathbf{C}^n$, let $\|\cdot\|^D$ be its dual norm, and let $c > 0$ be given. Then $\|x\| = c\|x\|^D$ for all $x \in V$ if and only if $\|\cdot\| = \sqrt{c}\|\cdot\|_2$. In particular, $\|\cdot\| = \|\cdot\|^D$ if and only if $\|\cdot\|$ is the Euclidean norm $\|\cdot\|_2$.

Proof: If $\|\cdot\| = \sqrt{c}\|\cdot\|_2$ and $x \in V$, then

$$\|x\|^D = \max_{\|y\|=1} |x^*y| = \max_{\|y\|_2 = 1/\sqrt{c}} |x^*y| = \max_{\|y\|_2=1} \left|x^*\frac{y}{\sqrt{c}}\right|$$

$$= \frac{1}{\sqrt{c}} \max_{\|y\|_2=1} |x^*y| = \frac{1}{\sqrt{c}}\|x\|_2^D = \frac{1}{\sqrt{c}}\|x\|_2 = \frac{1}{c}\|x\|$$

for any $x \in V$. Conversely, if $\|\cdot\| = c\|\cdot\|^D$ for some $c > 0$ and if $x \in V$, then (5.4.13) gives the inequality

$$\|x\|_2^2 = |x^*x| \le \|x\|\,\|x\|^D = \frac{1}{c}\|x\|^2$$

so $\|x\| \ge \sqrt{c}\|x\|_2$. We can use this inequality to establish the reverse bound if $x \ne 0$ by considering

$$\frac{1}{c}\|x\| = \|x\|^D = \max_{\|y\|=1} |x^*y| = \max_{y\ne 0} \left|x^*\frac{y}{\|y\|}\right|$$

$$= \max_{y \ne 0} \left|x^*\frac{y}{\|y\|_2}\right| \frac{\|y\|_2}{\|y\|} \le \max_{y\ne 0} \left|x^*\frac{y}{\|y\|_2}\right| \frac{1}{\sqrt{c}}$$

$$= x^*\frac{x}{\|x\|_2}\frac{1}{\sqrt{c}} = \|x\|_2\frac{1}{\sqrt{c}}$$

where we have used the fact that $\|y\|_2/\|y\| \le 1/\sqrt{c}$ for all $y \ne 0$; the Cauchy–Schwarz inequality guarantees that the maximum absolute value of the inner product between a fixed nonzero vector and a Euclidean unit vector occurs when the unit vector is parallel to the given vector. Thus, $\|x\| \le \sqrt{c}\|x\|_2$ for all $x \in V$, which, together with the reverse inequality that we have already proved, shows that $\|x\| = \sqrt{c}\|x\|_2$ for all $x \in V$. The final assertion follows when $c = 1$ and shows that the Euclidean norm is the only norm that equals its dual. □

As final remark, we observe that there is a useful sense in which a *vector*, as well as a vector norm, has a dual.

5.4.17 Definition. Let $x \in \mathbf{C}^n$ be a given vector and let $\|\cdot\|$ be a given vector norm on $\mathbf{C}^n$. The set

$$\{y \in \mathbf{C}^n : \|y\|^D \|x\| = y^* x = 1\}$$

is said to be *the dual of x with respect to* $\|\cdot\|$. An ordered pair of vectors $(x, y) \in \mathbf{C}^n \times \mathbf{C}^n$ is said to be a *dual pair with respect to* $\|\cdot\|$ if y is in the dual of x with respect to the norm $\|\cdot\|$.

It follows from Corollary (5.5.15) that if $\|\cdot\|$ is a vector norm, then the dual of every vector $x \in \mathbf{C}^n$ with respect to $\|\cdot\|$ is nonempty. It could consist of one point or many. If $\|\cdot\| = \|\cdot\|_2$, for example, then the dual of every vector $x \in \mathbf{C}^n$ is the one vector x itself. If $\|\cdot\| = \|\cdot\|_\infty$, on the other hand, the dual of $x = [0, 1]^T$ consists of a single vector, but the dual of $x = [1, 1]^T$ contains infinitely many vectors. See Problem 13.

Problems

1. Note that (5.4.5) may be stated equivalently as

$$C_m(\|\cdot\|_\alpha, \|\cdot\|_\beta) \le \frac{\|x\|_\beta}{\|x\|_\alpha} \le C_M(\|\cdot\|_\alpha, \|\cdot\|_\beta)$$

where $C_m(\cdot, \cdot)$ and $C_M(\cdot, \cdot)$ denote the best possible constants relating the respective norms in (5.4.5). Show that $C_m(\|\cdot\|_\beta, \|\cdot\|_\alpha) = C_M(\|\cdot\|_\alpha, \|\cdot\|_\beta)^{-1}$.

2. Express $C_m(\|\cdot\|_\alpha, \|\cdot\|_\gamma)$ in terms of $C_m(\|\cdot\|_\alpha, \|\cdot\|_\beta)$ and $C_m(\|\cdot\|_\beta, \|\cdot\|_\gamma)$, where the constants involved need not be best possible. Do likewise for C_M.

3. Verify that the accompanying table gives the best bounds $C_M(\|\cdot\|_\alpha, \|\cdot\|_\beta)$ between the l_1, l_2, and l_∞ norms; that is, $\|x\|_\alpha \le C_M \|x\|_\beta$ for all $x \in \mathbf{C}^n$ and for $\alpha, \beta = 1, 2, \infty$. In each case show that the bound is best possible by exhibiting a nonzero vector x such that $\|x\|_\alpha = C_M \|x\|_\beta$.

α \ β	1	2	∞
1	1	$\sqrt{n}$	n
2	1	1	$\sqrt{n}$
∞	1	1	1

What is the table of best lower bounds $\|x\|_\alpha \geq C_m \|x\|_\beta$? *Hint:* See Problem 1.

4. Show that if two norms on a real or complex vector space are equivalent, then they are related by two constants and an inequality as in (5.4.5). *Hint:* Consider $f(x) = 1/\|x\|_\alpha$ on the unit sphere S of $\|\cdot\|_\beta$. If f is unbounded on S, there is a sequence $\{x_N\} \subset S$ with $\|x_N\|_\alpha < 1/N$ and $\|x_N\|_\beta = 1$, which contradicts equivalence of $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$. Notice that this has nothing to do with finite dimensionality or compactness.

5. Show that the functions f_k of (5.4.2) have the property that $f(x) \to 0$ for each x, $\|f_k - f_j\|_1 \to 0$ as $k, j \to \infty$, and for each $k \geq 2$ there is some $J > k$ for which $\|f_k - f_j\|_\infty > k^{1/2}$ for all $j > J$. Thus, a sequence can be convergent in one sense (point-wise), Cauchy in a norm, and not Cauchy in another norm.

6. Let V be a complete real or complex vector space, let $\{x^{(k)}\}$ be a given sequence in V, and let $\|\cdot\|$ be a given vector norm on V. If there is an $M \geq 0$ such that $\sum_{k=1}^{n} \|x^{(k)}\| \leq M$ for all $n = 1, 2, \ldots$, show that the sequence of partial sums $\{y^{(n)}\}$ defined by $y^{(n)} = \sum_{k=1}^{n} x^{(k)}$ converges to a point of V. What theorem about convergence of infinite series of real numbers does this generalize?

7. Show that $\|x\|_\infty = \lim_{p\to\infty} \|x\|_p$ for every $x \in \mathbf{C}^n$.

8. If $\alpha > 0$ and $\|\cdot\|_\alpha \equiv \alpha\|\cdot\|$, show that $(\|\cdot\|_\alpha)^D = (1/\alpha)\|\cdot\|^D$.

9. Show that the dual norm of the l_p norm is the l_q norm for any $p \geq 1$, where q is defined by the relation $1/p + 1/q = 1$. *Hint:* Replace (5.4.14) with the general form of Hölder's inequality.

10. Let $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$ be two given vector norms on $\mathbf{C}^n$, and suppose there is some $C > 0$ such that $\|x\|_\alpha \leq C\|x\|_\beta$ for all $x \in \mathbf{C}^n$. Show that $\|x\|_\beta^D \leq C\|x\|_\alpha^D$ for all $x \in \mathbf{C}^n$. *Hint:*

$$\|x\|_\alpha^D = \max_{\|y\|_\alpha = 1} |y^*x| = \max_{y\neq 0} \left| \frac{y^*x}{\|y\|_\alpha} \right|$$

$$= \max_{y \neq 0} \frac{|y^*x|}{\|y\|_\alpha} \geq \max_{y\neq 0} \frac{|y^*x|}{C\|y\|_\beta} = \frac{1}{C} \max_{\|y\|_\beta = 1} |y^*x|$$