510 Solutions to Exam #2

1. $$\|A\| = \|AB - BA\|$$
$$\leq \|AB\| + \|BA\| \quad \text{triangle ineq.}$$
$$\leq 2\|A\| \cdot \|B\| \quad \text{submultiplicativity.}$$

Since $A \neq 0$ we have $\|A\| \neq 0$. So

$$\frac{1}{2} \leq \|B\|$$

The above holds for any matrix norm $\|\cdot\|$. Suppose $\rho(B) < \frac{1}{2}$. Let $0 < \varepsilon < \frac{1}{2} - \rho(B)$.
Then $\exists$ matrix norm $\|\cdot\|$ such that

$$\frac{1}{2} \leq \|B\| \leq \rho(B) + \varepsilon < \rho(B) + \frac{1}{2} - \rho(B) = \frac{1}{2}$$

contradiction. Thus $\rho(B) \geq \frac{1}{2}$.

2. A normal $\Rightarrow A = W\Lambda W^*$ for some unitary W, diagonal Λ (by Spectral thm). Then

$$A^* = W\Lambda^* W^*.$$

Now if $\lambda \in \mathbb{C}$ then $|\bar{\lambda}| = \lambda$ so there exist $D = \mathrm{diag}(e^{i\theta_1}, \dots, e^{i\theta_n})$ such that $\Lambda^* = D\Lambda$. Then

$$A^* = WD\Lambda W^* = \underbrace{WDW^*}_{\mathcal{U}} \underbrace{W\Lambda W^*}_{A}.$$

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3. $A^* = \mathcal{U}A \Rightarrow A = A^{**} = (\mathcal{U}A)^* = A^*\mathcal{U}^*$

$\Rightarrow AA^* = (A^*\mathcal{U}^*)(\mathcal{U}A) = A^*A$

So A is normal.

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4. Trivial if $A = 0$. Assume $A \neq 0$.

Then $A = \underbrace{(|||A|||\cdot I_n)}_{\text{hermitian}}\underbrace{\left(\frac{1}{|||A|||}A\right)}_{\text{nonexpanding:}}$ $\quad |||\cdot||| =$ matrix norm induced by $\|\cdot\|$

Let $B = \frac{1}{|||A|||}A$. Then $\forall x \in \mathbb{C}^n \setminus \{0\}$:

$$\|Bx\| = \frac{1}{|||A|||}\|Ax\| \leq \frac{1}{|||A|||}\max_{y\neq 0}\frac{\|Ay\|}{\|y\|}\|x\| = \|x\|$$

and $\|Bx\| \leq \|x\|$ trivially for $x = 0$.

5. By the real spectral thm for any $A \in M_n(\mathbb{R})$ $\exists U \in O(n)$ st

$$U^T A U = \begin{bmatrix} \lambda_1 & & & & & \\ & \ddots & & & & \\ & & \lambda_k & & & \\ & & & \begin{bmatrix} \sigma_1 & -\tau_1 \\ \tau_1 & \sigma_1 \end{bmatrix} & & \\ & & & & \ddots & \\ & & & & & \begin{bmatrix} \sigma_r & -\tau_r \\ \tau_r & \sigma_r \end{bmatrix} \end{bmatrix} \qquad \tau_i \neq 0$$

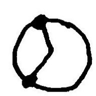
$$A^T = A^2 \iff \begin{cases} \lambda_i = \lambda_i^2 \quad \forall i, \text{ and} \\ \begin{bmatrix} \sigma_i & -\tau_i \\ \tau_i & \sigma_i \end{bmatrix}^T = \begin{bmatrix} \sigma_i & -\tau_i \\ \tau_i & \sigma_i \end{bmatrix}^2 \quad \forall i \end{cases}$$

$\iff \lambda_i \in \{0,1\}$ and $\bar{z}_i = z_i^2$, $z_i = \sigma_i + \tau_i\sqrt{-1}$

$$\Updownarrow$$

$|z_i| = 1$ and $z_i = e^{i\theta_i}$

$e^{-i\theta_i} = e^{i2\theta_i}$

$\iff \cancel{3}\theta_i \in \frac{2\pi}{3}\mathbb{Z}$ 

$\Rightarrow \lambda_i \in \{0,1\}$ $\forall i = 1, .., k$ and $\begin{cases} \sigma_i = \cos\pm\frac{2\pi}{3} = -\frac{1}{2} \\ \tau_i = \sin\pm\frac{2\pi}{3} = \pm\frac{\sqrt{3}}{2} \end{cases}$ $\forall i = 1, .., r$