1) 510 SPRING 2024
EXAM #1

$$T(B) = 2B - Tr(B)I$$

$$T(E_{ii}) = 2E_{ii} - I$$

$$\begin{cases} T(E_{11}) = E_{11} - E_{22} \\ \quad E_{22} \qquad E_{22} - E_{11} \end{cases}$$

$$T(E_{ij}) = 2E_{ij}$$

$$[T] = \begin{bmatrix} 1 & -1 & & \\ -1 & 1 & & \\ & & 2 & \\ & & & 2 \end{bmatrix}$$

$$\tilde{E}_{11} = E_{11} + E_{22} \Rightarrow T(\tilde{E}_{11}) = 0$$
$$\tilde{E}_{22} = E_{11} - E_{22} \qquad (\tilde{E}_{22}) = 2\tilde{E}_{22}$$

$$[T] = \begin{bmatrix} 0 & & & \\ & 2 & & \\ & & 2 & \\ & & & 2 \end{bmatrix} \qquad m_T(x) = x(x-2).$$

2) $\{v_i\}$ basis for V

$\{u_j\}$ basis for U

suppose T inj.

~~Let {~~ Extend $\{T(v_i)\}$ to basis for W.

suppose $T'\left(\sum \lambda_{ij} v_i \otimes u_j\right) = 0$

~~that~~ Then $\sum_{i,j} \lambda_{ij} T(v_i) \otimes u_j = 0$

Since $\{T(v_i) \otimes u_j\}$ is lin indep. all $\lambda_{ij} = 0$.

Converse is easy.

3) Let $\{w_i\}_{i=1}^{r}$ be a basis for $\operatorname{im} T$. For each $v \in V$ we have

$$T(v) = \alpha_1(v) w_1 + \dots + \alpha_r(v) w_r$$

for some scalars $\alpha_i(v) \in \mathbb{F}$.

~~Def~~ Since $\{w_i\}$ are lin indep, $\alpha_i(v)$ is uniquely determined by v.

Also $T(\lambda v + \mu w) = \sum \alpha_i(\lambda v + \mu w) w_i$

$$\lambda T(v) + \mu T(w) =$$

$$= \sum (\lambda \alpha_i(v) + \mu \alpha_i(w)) w_i$$

so α_i are linear.

Define $T_i : V \to V$ by $T_i(v) = \alpha_i(v) W_i$.

Then T_i have rank one and

$T(v) = T_1(v) + \dots + T_r(v)$ so $T = \sum T_i$

4) $A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$

$$xI - A = \begin{bmatrix} x & -1 & \cdots & -1 \\ -1 & x & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & x \end{bmatrix}$$

$$\det(xI-A) = \det \begin{bmatrix} x & -1 & -1 & \cdots & -1 \\ -1-x & x+1 & 0 & \cdots & 0 \\ -1-x & 0 & x+1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1-x & 0 & 0 & \cdots & x+1 \end{bmatrix} =$$

$$= \det \begin{bmatrix} x-(n-1) & -1 & -1 & \cdots & -1 \\ 0 & x+1 & & & \\ \vdots & & x+1 & & \\ \vdots & & & \ddots & \\ 0 & & & & x+1 \end{bmatrix}$$

$= (x-(n-1))(x+1)^{n-1} = C_A(x)$ char. pol.

4) contd.

On the other hand

$$\mathbb{1}^2 = n\,\mathbb{1}$$

$\llcorner$ all 1's matrix

So $A^2 = (\mathbb{1} - I)^2 = \mathbb{1}^2 - 2\mathbb{1} I + I =$

$$= n\mathbb{1} - 2\mathbb{1} + I$$

$$= (n-2)(\mathbb{1} - I) + (n-1) I$$

So $\quad A^2 - (n-2)A - (n-1)I$

$A \notin \mathbb{F} I \quad$ so

$$m_A(x) = x^2 - (n-2)x - (n-1)$$

$$= (x - (n-1))(x+1)$$

Now $f_1(x) f_2(x) \cdots \underbrace{f_n(x)}_{m_A(x)} = c_A(x)$

So $f_1(x) \cdots f_{n-1}(x) \cancel{(x-(n-1))}(x+1) = \cancel{(x-(n-1))}(x+1)^{n-1}$

$f_1(x) \cdots f_{n-1}(x) = (x+1)^{n-2}$

and $f_1(x) \mid f_2(x) \mid \cdots \mid f_{n-1}(x) \mid f_n(x) = (x-(n-1))(x+1)$

$\Rightarrow f_1(x) = 1, \quad f_k(x) = x+1, \; 2 \le k \le n-1$

4) contd.

$$\text{Now} \quad f_p(x) = \prod_{i=1}^{m} (x-\lambda_i)^{k^{(i)}_{n+1-p}}$$

$$= (x-(n-1))^{k^{(1)}_{n+1-p}} (x-(-1))^{k^{(2)}_{n+1-p}}$$

$$\text{So} \quad f_1 = 1 \Rightarrow k^{(1)}_n = k^{(2)}_n = 0$$

$$f_2 = x+1 \Rightarrow k^{(1)}_{n-1} = 0, \quad k^{(2)}_{n-1} = 1$$

$$\vdots \qquad\qquad \vdots$$

$$f_{n-1} = x+1 \qquad k^{(1)}_2 = 0, \quad k^{(2)}_2 = 1$$

$$f_n = (x-(n-1))(x-(-1)) \Rightarrow k^{(1)}_1 = 1, \quad k^{(2)}_1 = 1$$

$$\Rightarrow A \sim \begin{bmatrix} n-1 & & & & \\ & -1 & & & \\ & & -1 & & \\ & & & \ddots & \\ & & & & -1 \end{bmatrix}$$

$$= J_1(n-1) \oplus J_1(-1)^{\oplus(n-1)}$$

5) $A = [a_{ij}]$

$$A J_n(\lambda) = J_n(\lambda) A \iff J_n(0)A = AJ_n(0),$$

$$\uparrow$$

$$J_n(\lambda) = J_n(0) + \underbrace{\lambda I_n}_{\text{commutes with } A}$$

$$J_n(0)A = \begin{bmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix} \begin{bmatrix} \text{---}\,[a_{1j}]\,\text{---} \\ \vdots \\ \text{---}\,[a_{nj}]\,\text{---} \end{bmatrix} =$$

$$= \begin{bmatrix} \text{---}\,[a_{2j}]\,\text{---} \\ \vdots \\ \text{---}\,[a_{nj}]\,\text{---} \\ 0 \cdots\cdots 0 \end{bmatrix}$$ Shifts all rows of A up by 1

Similarly $AJ_n(0) = \begin{bmatrix} 0 & | & & | \\ \vdots & [a_{i1}] & \cdots & [a_{i,n-1}] \\ 0 & | & & | \end{bmatrix}$

shifts all cols 1 step to the right

$$J_n(0)A = AJ_n(0) \iff a_{i+1,j} = a_{i,j-1}$$

$$(a_{ij} := 0 \text{ if } i \notin [1,n] \text{ or } j \notin [1,n]) \quad \forall\, 1 \le i \le n,\ 1 \le j \le n$$

$$\iff a_{i+1,j+1} = a_{ij} \quad \forall\, 1 \le i \le n-1,\ 1 \le j \le n-1$$

$$\text{and } \underset{\forall\, 1 \le i \le n}{a_{i,1} = 0}\ ; \ \underset{\forall\, 1 \le j \le n-1}{a_{n,j} = 0}$$

$$\iff A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{11} & a_{12} & \\ \vdots & 0 & \ddots & a_{12} \\ 0 & \cdots & 0 & a_{11} \end{bmatrix} = a_{11} J_n(0)^0 + a_{12} J_n(0)^1 + \dots + a_{1n} J_n(0)^{n-1}$$

Upper triangular & constant along diagonals.