MATH 510 Homework 8 **Name:**

- Due April 12 at the beginning of class.

1. If $A \in M_n$ is Hermitian, show that the following three optimization problems all have the same solution:

   (a) $\max\limits_{x^*x=1} x^*Ax$

   (b) $\max\limits_{x^*x\neq 0} \dfrac{x^*Ax}{x^*x}$

   (c) $\max\limits_{x^*Ax=1} \dfrac{1}{x^*x}$ when at least one eigenvalue of A is positive

2. Let $A \in M_n$ have eigenvalues $\{\lambda_i\}$. Show that, even if A is not Hermitian, one has the bounds

$$\min_{x\neq 0} \left|\frac{x^*Ax}{x^*x}\right| \leq |la_i| \leq \max_{x\neq 0} \left|\frac{x^*Ax}{x^*x}\right|$$

3. Use Weyl's Theorem to show that for Hermitian matrices $A, B \in M_n$ one has

$$|\lambda_k(A+B) - \lambda_k(A)| \leq \rho(B), \qquad \forall k = 1, 2, \ldots, n.$$

   (This is an example of a *pertubation theorem* for the eigenvalues of a Hermitian matrix.)

4. A *hyperellipsoid* in $\mathbb{R}^m$ with radii $a_1, \ldots, a_m > 0$ is an image of the set

$$E := \left\{(z_1, \ldots, z_m) \in \mathbb{R}^m : \frac{z_1^2}{a_1^2} + \cdots + \frac{z_m^2}{a_m^2} = 1\right\}$$

   under an orthogonal transformation. A *solid hyperellipsoid* with radii $a_1, \ldots, a_m > 0$ is an image of the set

$$B := \left\{(z_1, \ldots, z_m) \in \mathbb{R}^m : \frac{z_1^2}{a_1^2} + \cdots + \tfrac{z_m^2}{a_m^2} \leq 1\right\}$$

   under an orthogonal transformation. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a *surjective* linear transformation. Prove that the image of the unit sphere

$$S := \{x \in \mathbb{R}^n : \|x\| = 1\} \subseteq \mathbb{R}^n$$

   under T is either a hyperellipsoid (if T is injective) or a solid hyperellipsoid (if T is not injective).

5. Let $A \in M_n$ be Hermitian. Let $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$ be the eigenvalues of A, and $\lambda_{i,1} \le \lambda_{i,2} \le \cdots \le \lambda_{i,n-1}$ the eigenvalues of the $(n-1) \times (n-1)$ principal submatrix Ai' obtained from A by deleting the i:th column and i:th row. Show that the following interlacing inequalities hold:

$$\lambda_1 \le \lambda_{i,1} \le \lambda_2 \le \lambda_{i,2} \le \cdots \le \lambda_{i,n-1} \le \lambda_n$$