MATH 510 Homework 6 **Name:**

- Homework 7 is due March 29 at the beginning of class.

1. Let $\|\cdot\|$ be a norm (i.e. a "vector norm" as the book calls it) on $\mathbb{C}^n$. Let $T \in M_n(\mathbb{C})$. Let $\|\cdot\|_T$ be the norm given by $\|x\|_T = \|Tx\|$. Show that if T is an isometry with respect to $\|x\|$, then $\|\cdot\|^{\mathrm{D}} = (\|\cdot\|_T)^{\mathrm{D}}$, where $f(\cdot)^D$ denotes the dual norm of a (pre-)norm $f(\cdot)$.

2. Show that if $x, y \in \mathbb{C}^n$ then $\|y\|_\infty = \max_{\|x\|_1=1} |y^* x|$ and $\|y\|_1 = \max_{\|x\|_\infty=1} |y^* x|$.

3. Prove that a norm $\|\cdot\|$ on a vector space V over $\mathbb{R}$ (same is true for $\mathbb{C}$ but requires an extra step) is derived from an inner product $\langle \cdot, \cdot \rangle$ (in the sense that $\|x\| = \sqrt{\langle x, x \rangle}$ for all $x \in V$) if and only if $\|\cdot\|$ satisfies the *parallelogram identity*

$$\frac{1}{2}\left(\|x+y\|^2 + \|x-y\|^2\right) = \|x\|^2 + \|y\|^2, \qquad \forall x, y \in V.$$

   Hint: When $\|\cdot\|$ is derived from an inner product, $\langle x, y \rangle$ can be expressed in terms of $\|x+y\|$, $\|x\|$, and $\|y\|$. For the $\Leftarrow$ direction, use that formula as the definition of $\langle x, y \rangle$. To show additivity, use the parallelogram identity. For homogeneity $\langle \lambda x, y \rangle = \lambda \langle x, y \rangle$, first show it for $\lambda \in \mathbb{Z}$, then for $\lambda \in \mathbb{Q}$. Use Cauchy-Schwarz and a continuity argument for the general case $\lambda \in \mathbb{R}$.

4. Let $A \in M_n$ be a non-singular matrix such that the upper left $k \times k$-submatrix is singular (i.e. the leading principal $k \times k$-minor is zero) for some $k \in \{1, 2, \ldots, n-1\}$. Show that A cannot be factored as LU where $L \in M_n$ is a lower-triangular matrix and $U \in M_n$ is an upper-triangular matrix. (*Hint:* Start with small k and arbitrary n to see what happens.)

5. Against better judgement, call two matrices $A, B \in M_{m,n}$ *equivalent* if there are non-singular matrices $S \in M_m$ and $T \in M_n$ such that $B = SAT$.

   (a) Show that every matrix $A \in M_{m,n}$ is equivalent to a matrix of the form $\begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix}$ where $k \leq \min\{m, n\}$ and 0 are appropriately sized zero matrices. (*Hint:* Use that row and column operations can be performed using matrix multiplication by elementary matrices and use induction. Alternatively, find appropriate bases for $\mathbb{C}^m, \mathbb{C}^n$.)

   (b) Show that two matrices in $M_{m,n}$ are equivalent if and only if they have the same rank.