MATH 510 Homework 5 **Name:**

- Homework 5 is due March 8 at the beginning of class.
- *Problem 5 corrected on March 3, 7pm*

1. A permutation matrix is an $n \times n$ matrix in which each column and each row has exactly one non-zero entry, and that entry equals 1. Show that the set of permutation matrices is a subgroup of the group $O(n)$ of real orthogonal matrices. (Subgroup here means a subset which is closed under multiplication, inverses, and contains identity matrix.)

2. Show that if $A \in M_n(\mathbb{C})$ is similar to a unitary matrix, then $A = B^{-1}B^*$ for some nonsingular B.

3. Show that the set of matrices that are similar to unitary matrices is a proper subset of the set of matrices for which A^{-1} is similar to A^*. Hint: Consider the matrix $\mathrm{diag}(2, \frac{1}{2})$.

4. Let A, B be commuting matrices with eigenvalues $\alpha_1, \alpha_2, \ldots, \alpha_n$ and $\beta_1, \beta_2, \ldots, \beta_n$ respectively. If $p(t, s)$ is a polynomial in two variables, show that $p(A, B)$ has eigenvalues $p(\alpha_1, \beta_{i_1}), p(\alpha_2, \beta_{i_2}), \ldots, p(\alpha_n, \beta_{i_n})$ for some permutation $(i_1, i_2, \ldots, i_n)$ of $(1, 2, \ldots, n)$.

5. Let $A \in M_n(\mathbb{C})$ be a matrix such that $\mathrm{Tr}(A^k) = 0$ for all $k \geq 1$. Show that A **is nilpotent**. *Hint:* You may use that the elementary symmetric polynomials
$$e_d = \sum_{1 \leq i_1 < i_2 < \ldots < i_d \leq n} \lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_d}$$
are polynomials in the power sums $p_d = \sum_{i=1}^n \lambda_i^d$.