- Homework 4 is due February 16 at the beginning of class.
- Write the problem statement followed by a proof or solution.
- List problems in the same order they were given.
- If you skip a problem, include the problem statement with no solution.

1. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear operator whose matrix in the standard ordered basis of $\mathbb{R}^2$ is
$$A = \begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix}.$$
   (a) Prove that the only subspaces of $\mathbb{R}^2$ invariant under T are $\mathbb{R}^2$ and the zero subspace.
   (b) Let $U : \mathbb{C}^2 \to \mathbb{C}^2$ be the linear map whose matrix in the standard ordered basis of $\mathbb{C}^2$ is A. Show that U does have 1-dimensional invariant subspaces.

2. Let V be a finite-dimensional vector space and W be an invariant subspace for a linear map $T : V \to V$. Prove that the minimal polynomial for the restricted linear map $T\big|_W : W \to W$ divides the minimal polynomial for T.

3. Let V be the space of $n \times n$-matrices over a field $\mathbb{F}$, and let A be a fixed $n \times n$-matrix over $\mathbb{F}$. Define a linear operator T on V by $T(B) = AB - BA$. Prove that if A is a nilpotent matrix, then T is a nilpotent operator.

4. Let V be a finite-dimensional vector space and let $T, S \in \mathrm{End}(V)$ be commuting linear maps. Suppose there are ordered bases $\mathcal{B}$ and $\mathcal{C}$ for V such that $[T]_{\mathcal{B}}$ and $[S]_{\mathcal{C}}$ are diagonal. Show that there is an ordered basis $\mathcal{D}$ for V such that $[T]_{\mathcal{D}}$ and $[S]_{\mathcal{D}}$ are diagonal.

5. Let A be an $n \times n$-matrix with real entries such that $A^2 + I = 0$. Prove that $n = 2k$ for some $k > 0$, and that furthermore A is similar to a matrix of the block form
$$\begin{bmatrix} 0_k & -I_k \\ I_k & 0_k \end{bmatrix}.$$