- Homework 3 is due February 9 at the beginning of class.
- Write the problem statement followed by a proof or solution.
- List problems in the same order they were given.
- If you skip a problem, include the problem statement with no solution.

1. Prove the *Second Isomorphism Theorem* for vector spaces, stating that if V is a vector space and $U, W \leq V$, then
$$(U+W)/W \cong U/(U \cap W).$$
(You may assume V is finite-dimensional if you wish.)

2. Let V be a vector space over a field $\mathbb{F}$. The *dual space* of V is defined to be
$$V^* = \mathrm{Hom}(V, \mathbb{F}).$$
   (a) Define a function $\beta : V^* \times W \to \mathrm{Hom}(V, W)$ by $\beta(\xi, w)(v) = \xi(v)w$. Show β is bilinear, hence induces a linear map
$$B : V^* \otimes W \to \mathrm{Hom}(V, W)$$
   satisfying $B(\xi \otimes w)(v) = \xi(v)w$ for all $\xi \in V^*, w \in W, v \in V$.
   (b) Show that the map B is injective. (*Hint:* Choose bases.)
   (c) Show that the image of B consists of all linear maps $T : V \to W$ of finite rank. (*Hint:* For $\supseteq$, choose a basis for $T(V)$.)

3. Let V be a (finite-dimensional, if you wish) vector space. Define $V^{\otimes k}$ for $k > 0$ recursively by $V^{\otimes 1} = V$ and $V^{\otimes k} = V^{\otimes (k-1)} \otimes V$ for $k > 0$. We put $v_1 \otimes v_2 \otimes v_3 = (v_1 \otimes v_2) \otimes v_3$ and similarly with more factors. Let J_k be the subspace of $V^{\otimes k}$ spanned by all vectors $v_1 \otimes v_2 \otimes \cdots \otimes v_k$ where $v_1, v_2, \ldots, v_k \in V$ and $v_i = v_j$ for some $i \neq j$. The *k:th exterior power of* V is defined as
$$\wedge^k V = V^{\otimes k}/J_k.$$
Notation: $v_1 \wedge v_2 \wedge \cdots \wedge v_k = v_1 \otimes v_2 \otimes \cdots \otimes v_k + J_k$.
   (a) If $T : V \to V$ is a linear map, show that $T^{\otimes k} : V^{\otimes k} \to V^{\otimes k}$ (defined recursively by $T^{\otimes 1} = T$, $T^{\otimes s} = T^{\otimes (s-1)} \otimes T$ for $s > 0$) leaves the subspace J_k invariant. Conclude that there is an induced linear map $\wedge^k T : \wedge^k V \to \wedge^k V$.
   (b) If $\dim V = n$, show that $\dim \wedge^k V = \binom{n}{k}$. (*Hint:* By bilinearity, $(u+v) \otimes (u+v) \in J_2$ implies that $u \wedge v + v \wedge u = 0$.)
   (c) Take $V = \mathbb{F}^2$ and $T = T_A$ for an arbitrary $A \in \mathbb{F}^{2\times 2}$. Find the matrix of $\wedge^2 T$ with respect to the basis $\{e_1 \wedge e_2\}$.

4. Let $A \in \mathbb{F}^{m\times n}$ and $B \in \mathbb{F}^{s\times t}$ be matrices. Let $A \otimes B$ be the Kronecker product of the matrices. Prove that $\mathrm{rank}(A \otimes B) = (\mathrm{rank}\, A)(\mathrm{rank}\, B)$.

5. The *trace* of a square matrix $A = [A_{ij}] \in \mathbb{F}^{n\times n}$ is $\mathrm{Tr}\, A = \sum_i A_{ii} \in \mathbb{F}$ (sum of the diagonal elements). Show that if $A \in \mathbb{F}^{n\times n}$ and $B \in \mathbb{F}^{m\times m}$ then $\mathrm{Tr}(A \otimes B) = (\mathrm{Tr}\, A)(\mathrm{Tr}\, B)$.