- Homework 2 is due February 2 at beginning of class.
- Write the problem statement followed by a proof or solution.
- List problems in the same order they were given.
- If you skip a problem, include the problem statement with no solution.

Corrected on Jan 28, 6:30pm: Problem 1(a) should say $\subseteq$, not $=$. In Problem 5, assume $n > 1$ (or interpret $[0]^0 = [1]$).

1. Let $\mathbb{F}$ be an arbitrary field, and let $A_1, \dots, A_k \in \mathbb{F}^{m\times n}$ be matrices.

   (a) Show that $\operatorname{im}(A_1 + \cdots + A_k) \subseteq \operatorname{im} A_1 + \cdots + \operatorname{im} A_k$, as a sum of subspaces.

   (b) Show that $\operatorname{rank}(A_1 + \cdots + A_k) \leq \operatorname{rank} A_1 + \cdots + \operatorname{rank} A_k$.

2. Let V and W be finite-dimensional vector spaces with ordered bases $\mathcal{B} = (b_1, b_2, \dots, b_n)$ and $\mathcal{C} = (c_1, c_2, \dots, c_m)$, respectively. Let $S : V \to V$ and $T : W \to W$ be linear maps. Define a function $S \oplus T : V \oplus W \to V \oplus W$ (external direct sum) by

   $$(S \oplus T)(v, w) = \big(S(v), T(w)\big)$$

   Let $\mathcal{D} = \big((b_1, 0), \dots, (b_n, 0),\ (0, c_1), \dots, (0, c_m)\big)$ be the corresponding ordered basis for $V \oplus W$.

   (a) Show that $S \oplus T$ is a linear map.

   (b) Show that the matrix $[S \oplus T]_{\mathcal{D}}$ is the direct sum of the matrices $[S]_{\mathcal{B}}$ and $[T]_{\mathcal{C}}$. (Recall that the direct sum of two matrices $A \in F^{n\times n}$ and $B \in F^{m\times m}$ is the $(n+m)\times(n+m)$-block matrix $\begin{bmatrix} A & 0_{n,m} \\ 0_{m,n} & B \end{bmatrix}$ where $0_{a,b} \in F^{a\times b}$ denotes the zero matrix.)

3. Let X be any set and $\mathbb{F}$ be any field. Let $\mathbb{F}^X$ be the set of all functions from X to $\mathbb{F}$ with pointwise operations $(f+g)(x) = f(x)+g(x)$ and $(\lambda f)(x) = \lambda f(x)$. For each $x \in X$, let $e_x \in \mathbb{F}^X$ be the characteristic function on $\{x\}$, defined by

   $$e_x(y) = \begin{cases} 1, & \text{if } y = x, \\ 0, & \text{otherwise.} \end{cases}$$

   The *free $\mathbb{F}$-vector space on X* is defined as $\mathbb{F}X = \operatorname{span}_{\mathbb{F}}\{e_x \mid x \in X\}$.

   (a) Show that $\{e_x \mid x \in X\}$ is a basis for $\mathbb{F}X$.

   (b) Show that for any function $\varphi : X \to W$ from X into some vector space W, there is a unique linear map $\Phi : \mathbb{F}X \to W$ such that $\Phi(e_x) = \varphi(x)$ for all $x \in X$.

   (c) Let $X = U$ where U is some vector space. Explain why, in $\mathbb{F}U$, it is the case that for all $u, v \in U$: $e_{u+v} \neq e_u + e_v$. Similarly, for $1 \neq \lambda \in \mathbb{F}$ and $u \in U$, explain why $e_{\lambda u} \neq \lambda e_u$.

4. Let V, W be finite-dimensional vector spaces with ordered bases $\mathcal{B}, \mathcal{C}$ respectively. Show that a linear transformation $T : V \to W$ is invertible if and only if its matrix $[T]_{\mathcal{B},\mathcal{C}}$ is invertible.

5. Let V be a finite-dimensional vector space of dimension $n > 1$ and suppose $T : V \to V$ is a linear map such that $T^n = 0$ but $T^{n-1} \neq 0$. (Here $T^n = T \circ T \circ \cdots \circ T$ and 0 means the zero in $\mathrm{End}(V)$.) Show that there is a basis $\mathcal{B}$ for V such that

$$[T]_{\mathcal{B}} = \sum_{i=1}^{n-1} E_{i+1,i} = \begin{bmatrix} 0 & \cdots & 0 & 0 \\ 1 & \cdots & 0 & 0 \\ \vdots & \ddots & 0 & 0 \\ 0 & \cdots & 1 & 0 \end{bmatrix}.$$