MATH 510, Spring 2024

Homework 1

Due Friday, Jan 26, at 3:20pm. Hand in hard copy at the beginning of class.
You should typeset your solutions in TEX.

- Write the problem statement followed by a proof or solution.
- List problems in the same order they were given.
- If you skip a problem, include the problem statement with no solution.

1. Let W_1 and W_2 be subspaces of a vector space V such that the union $W_1 \cup W_2$ is a subspace of V. Prove that one of the subspaces W_i is contained in the other.

2. Let V be a finite-dimensional space over a field F. Suppose W_1 and W_2 are subspaces of V with $\dim W_1 = \dim W_2$. Prove there is a subspace $U \leq V$ such that $V = W_1 \oplus U = W_2 \oplus U$.

   Hint: In the case $W_1 \neq W_2$, use the previous problem to show there is a vector in V which is not in $W_1 \cup W_2$.

3. Let V be the vector space of all functions from $\mathbb{R}$ to $\mathbb{R}$ (see [HK, §2.1, Example 3]). Let V_{e} be the subset of even functions, $f(-x) = f(x)$; let V_{o} be the subset of odd functions $f(-x) = -f(x)$.

   (a) Prove that V_{e} and V_{o} are subspaces of V.

   (b) Prove that $V_{\mathrm{e}} + V_{\mathrm{o}} = V$.

   (c) Prove that $V_{\mathrm{e}} \cap V_{\mathrm{o}} = \{0\}$.

4. Let $V = \mathbb{R}$ be the set of all real numbers. Regard V as a vector space over the field of *rational* numbers $\mathbb{Q}$, with the usual operations. Prove that this vector space is not finite-dimensional.

5. Let $W_1, W_2, \ldots, W_n$ be subspaces of a vector space V such that $V = W_1 + W_2 + \cdots + W_n$. Suppose that $W_i \cap (W_1 + W_2 + \cdots + \widehat{W_i} + \cdots + W_n) = \{\mathbf{0}_V\}$ (here $\widehat{W_i}$ means the term should be *omitted* from the expression, and $\mathbf{0}_V$ is the zero vector in V) for all i. Show that for each vector $v \in V$ there are *unique* vectors $w_i \in W_i$ such that $v = w_1 + w_2 + \cdots + w_n$.