MATH 403/503 L9

Group Actions

Let G be a group and X be a set. An *action of G on X* is a function

$$G \times X \longrightarrow X$$
$$(g, x) \longmapsto g.x \quad (\text{or } g \cdot x)$$

such that

1) $(gh).x = g.(h.x)$ $\forall g, h \in G$, $\forall x \in X$

2) $e.x = x$ $\forall x \in X$

Remark This is a *left* action. There is also a notion of a *right* action of G on X:

$$X \times G \to X$$
$$(x, g) \longmapsto x.g \quad (\text{or } x \triangleleft g)$$

$$\text{s.t.} \begin{cases} x.(gh) = (x.g).h \\ x.e = x \end{cases}$$

Unless explicitly stated, "action" will mean left action.

Notation That G acts on X is denoted by $G \circlearrowright X$.

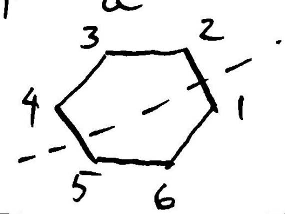
Examples

1) We always have the <u>trivial action</u> of any G on any X, defined by
$$g.x = x \quad \forall g \in G,\ \forall x \in X.$$

2) The symmetric group S_n acts on the set $X = \{1, 2, \ldots, n\}$ by
$$\sigma . a = \sigma(a) \quad \forall \sigma \in S_n,\ a \in X.$$

3) If G acts on X and $H \leq G$, then H also acts on X via restriction.

4) Considering the dihedral group D_n as a subgroup of S_n via
$$r = (1\ 2\ \cdots\ n),$$
$$s = (1\ 2),$$
the S_n-action on $X = \{1, 2, \ldots, n\}$ may be restricted to an action of D_n on the same set. Then X can be thought of as the vertex set of a regular n-gon.



5) Any group G acts on itself by <u>left multiplication</u>:

$$g.h = gh \quad \forall g, h \in G$$

6) We may restrict this to an action of a subgroup $H \leq G$ on G by left mult.:

$$h.a = ha \quad \forall h \in H,\ a \in G,$$

7) Any group G also acts on itself by <u>conjugation</u>:

$$g.h = ghg^{-1} \quad \forall g, h \in G$$

8) If $H \leq G$ and $g \in G$ then $gHg^{-1} \leq G$ (check).
This gives an action of G on the <u>set $Sub(G)$ of all subgroups of G</u>:

$$g.H = gHg^{-1} \quad \forall g \in G \quad \forall H \in Sub(G)$$

Note that $|gHg^{-1}| = |H|$.

Lemma Suppose an action of a group G on a set X is given. Define a relation $\underset{G}{\sim}$ on X by

$$x \underset{G}{\sim} y \iff \exists g \in G : g.x = y$$

Then $\underset{G}{\sim}$ is an equivalence relation on X.

proof Denote $\underset{G}{\sim}$ by $\sim$ for brevity.

1) $e.x = x$ so $x \sim x$ $\forall x \in X$

2) If $x \sim y$ then $g.x = y$, some $g \in G$.

Thus $g^{-1}.(g.x) = g^{-1}.y$

$$\| $$
$$(g^{-1}g).x = e.x = x$$

Hence $y \sim x$.

3) If $x \sim y$, $y \sim z$ then $g.x = y$, $h.y = z$ for some $g, h \in G$. Thus:

$$(hg).x = h.(g.x) = h.y = z$$

So $x \sim z$. $\blacksquare$

Def The equivalence class containing $x \in X$ is denoted
$$\mathcal{O}_x = \{ g.x \mid g \in G \}$$
and is called an <u>orbit</u> (in X under the given action of G). We denote $G\backslash X = \{\mathcal{O}_x \mid x \in X\}$ for the set of orbits w.r.t. a left action, and X/G for the set of orbits wrt a right action.

Examples

1) If G acts trivially on X then every orbit is a singleton $\mathcal{O}_x = \{x\}$
2) When S_n acts on $\{1, 2, \dots, n\}$ by permutations, $(i\,j).\,i = j$ so there is only one orbit

Def If $G \curvearrowright X$ and $\mathcal{O}_x = X$ for some (hence all) $x \in X$, we say the action is <u>transitive</u> or that <u>G acts transitively on X</u>.

So S_n acts transitively on $\{1, \dots, n\}$.
The action of D_n on $\{1, \dots, n\}$ is also transitive.

Examples (contd.)

6) Let $H \leq G$ and let H act on G by left multiplication. Then the orbits are the right cosets of H in G:

$$\mathcal{O}_g = \{\underset{\overset{\shortparallel}{hg}}{h.g} \mid h \in H\} = Hg$$

There is also a <u>right</u> action of H on G by <u>right</u> multiplication. In that case the orbits are the <u>left</u> cosets.

7) When G acts on itself by conjugation, the orbits are the conjugacy classes in G:

$$\mathcal{O}_a = \{gag^{-1} \mid g \in G\} = Cl(a) \quad , a \in G$$

8) When $G \circlearrowright Sub(G)$ we have:

$$\mathcal{O}_H = \{H\} \iff H \trianglelefteq G$$