MATH 403/503 L6

<u>Example</u> Let $N = \{(1), \underset{\alpha}{\underset{\shortparallel}{(12)(34)}}, \underset{\beta}{\underset{\shortparallel}{(13)(24)}}, \underset{\gamma}{\underset{\shortparallel}{(14)(23)}}\}$

Claim: $N \leq S_4$.

<u>Pf</u> • $(1) \in N$

• $\alpha^2 = \beta^2 = \gamma^2 = (1)$ so $\alpha^{-1} = \alpha$ etc and N is closed under inverses.

• $\alpha\beta = \gamma = \beta\alpha$ (check) In fact, the product of any two elements of $\{\alpha, \beta, \gamma\}$ equals the third. (1) is the identity element $(1)\alpha = \alpha, \ldots$

So N is closed under the op. in S_4.

By a subgroup criterion, $N \leq S_4$. ∎

In fact, we will show $N \trianglelefteq S_4$.

<u>Def</u> Two elements $x, y \in G$ (a group) are <u>conjugate</u>, written $x \sim y$, if $y = gxg^{-1}$ for some $g \in G$.

<u>Lemma</u> $\sim$ is an equivalence relation

<u>Def</u> The equivalence classes in G w.r.t. $\sim$ are called <u>conjugacy classes</u>.

<u>Notation</u> The conjugacy class containing g is denoted $C\ell(g)$ or $C\ell_G(g)$.

<u>Ex</u> 1) $C\ell(e) = \{e\}$ since $geg^{-1} = e \ \forall g \in G$

2) If G is abelian then $C\ell(x) = \{x\}$ for any $x \in G$.

3) $N \trianglelefteq G$ iff $N \leq G$ and N is a union of conjugacy classes.

<u>Thm</u> In the symmetric group S_n we have

$$\sigma \circ (a_1\ a_2\ \cdots a_k) \circ \sigma^{-1} = (\sigma(a_1)\ \sigma(a_2)\ \cdots \sigma(a_k))$$

for any $\sigma \in S_n$ and any k-cycle $(a_1\, a_2 \cdots a_k)$, $k > 0$.

<u>Proof</u> Let $\tau = LHS$ and $\tau' = RHS$.

$$\tau(\sigma(a_i)) = \sigma \circ (a_1 \cdots a_k) \circ \cancel{\sigma^{-1} \circ \sigma}\,(a_i) = \begin{cases} \sigma(a_{i+1}), & 1 \leq i < k \\ \sigma(a_1), & i = k \end{cases} = \tau'(\sigma(a_i))$$

If $j \in \{1, 2, \ldots, n\} \setminus \{\sigma(a_1), \ldots, \sigma(a_k)\}$ then $\sigma^{-1}(j) \notin \{a_1, \ldots, a_k\}$ so

$$\tau(j) = \sigma \circ (a_1 \cdots a_k)\, \sigma^{-1}(j) = \sigma(\sigma^{-1}(j)) = j = \tau'(j)$$

Thus $\tau = \tau'$. ∎

<u>Def</u> The (cycle) <u>type</u> of $\sigma \in S_n$ is the list of cycle lengths in decreasing (say) order, including 1-cycles, occuring in the disjoint-cycle decomposition of σ.

<u>Ex</u> In S_4

$(123) = (123)(4)$ type $(3,1)$

$(12)(34)$ type $(2,2)$

$(13) = (13)(2)(4)$ $(2,1,1)$

(4312) type (4)

<u>Note</u> The cycle type of any $\sigma \in S_n$ is a partition of n (i.e. the entries sum to n).

<u>Corollary</u> The conjugacy classes in S_n are in bijective correspondence with partitions of n.

<u>Ex</u> $Cl((1)) = \{(1)\}$ $(1) = (1)(2)(3)(4)$ type $(1,1,1,1)$

There are 5 partitions of $n=4$:

$4 = 4$	$(ijkl)$
$= 3+1$	$(ijk)(l) = (ijk)$
$= 2+2$	$(ij)(kl)$
$= 2+1+1$	$(ij)(k)(l) = (ij)$
$= 1+1+1+1$	$(i)(j)(k)(l) = (1)$

Going back to

$$N = \{ (1), \alpha, \beta, \gamma \} \leq S_4$$

we see that

$$N = Cl((1)) \cup Cl((12)(34))$$ union of conj. classes.

So we conclude $N \trianglelefteq S_4$

Ex. Prove that $S_4/N \cong S_3$.

Sol. Method 1: Consider the composition

$$S_3 \xrightarrow{i} S_4 \xrightarrow{\psi} S_4/N$$

$i(\sigma) = \sigma$ (inclusion map, a homomorphism)

$\psi(\sigma) = \sigma N$ (the canonical projection)

Let $\varphi = \psi \circ i$. Compositions of homom's are homomorphisms. So φ is a homom.

Suppose $\sigma \in \ker \varphi$. Then

$N = e_{S_4/N} = \varphi(\sigma) = \sigma N \Rightarrow \sigma \in N.$

But $N \cap S_3 = \{(1)\}$. So $\sigma = (1)$

and $\ker \varphi$ is trivial. So φ is injective. $|S_4/N| = \frac{|S_4|}{|N|} = \frac{4!}{4} = 3! = |S_3|$.

So φ is bijective. Therefore φ is an isomorphism

Method 2: Each $\sigma \in S_4$ sends $\{\alpha, \beta, \gamma\}$ to itself when conjugating by σ, because $\{\alpha,\beta,\gamma\} = Cl(\alpha)$. So we have a map

$$f: S_4 \longrightarrow S_{\{\alpha,\beta,\gamma\}}$$
$$\sigma \longmapsto \begin{pmatrix} \alpha & \beta & \gamma \\ \sigma\alpha\sigma^{-1} & \sigma\beta\sigma^{-1} & \sigma\gamma\sigma^{-1} \end{pmatrix} =: f_\sigma$$

For example

$$\sigma\alpha\sigma^{-1} = \sigma(12)(34)\sigma^{-1} = \sigma(12)\sigma^{-1} \cdot \sigma(34)\sigma^{-1} =$$
$$= (\sigma(1)\,\sigma(2))(\sigma(3)\,\sigma(4))$$

Say $\sigma = (13)$
$$= (3\,2)(1\,4) = (14)(23) = \gamma$$

f is a homomorphism since

$$f_{\sigma\tau}(x) = \sigma\tau x(\sigma\tau)^{-1} = \sigma(\tau x\tau^{-1})\sigma^{-1} =$$
$$= (f_\sigma \circ f_\tau)(x) \quad \text{for all } x \in \{\alpha,\beta,\gamma\}$$

$\operatorname{Ker} f \supseteq N$ since N is abelian.
The remaining elements of S_4 are 3-cycles, none of which fix all α, β, γ.

So $\ker f = N$ and f induces an injective homomorphism

$$\bar{f} : S_4/N \to S_{\{\alpha,\beta,\gamma\}}$$

Again, both sides have order 6, so $\bar{f}$ is an isomorphism. Of course, $S_{\{\alpha,\beta,\gamma\}} \cong S_3$ by renaming letters.

Method 3: Note $|S_4/N| = 6$ and S_4/N is abelian. The only non-abelian group of order 6 is isomorphic to S_3 (Fact/Exercise).
