MATH 403/503 L3

Quotient Groups

Recap

- Monoids = "groups without inverses"

  Ex. $(\mathbb{Z}, \cdot, 1)$

- $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ can be generalized to

  $M/\equiv$ M any monoid, $\equiv$ a congruence rel. on M.

- For groups G:

$$\left\{\begin{array}{c}\text{congruence}\\ \text{rels on } G\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{normal}\\ \text{subgroups}\\ \text{of } G\end{array}\right\}$$

$$\equiv \longmapsto N_{\equiv} = [e]$$

$$(x \equiv y \overset{\text{def}}{\Leftrightarrow} xy^{-1} \in N) \longleftarrow N$$

Note: Given $N \trianglelefteq G$, we have $xy^{-1} \in N \Leftrightarrow x \in Ny$.

Thus the congruence classes are the (left = right) cosets of N in G:

$$[g] = \{h \in G \mid h \in Ng\} = Ng = gN \quad \forall g \in G.$$

Let $H \leq G$.

<u>Notation</u> The set of <u>left</u> cosets of H in G is denoted G/H.

The set of <u>right</u> cosets of H in G is denoted $H\backslash G$.

$$G/H = \{gH : g \in G\}$$

$$H\backslash G = \{Hg : g \in G\}.$$

<u>Example</u> $G = Q_8$ quaternion group

$$Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$$

$$i^2 = j^2 = k^2 = -1 \;; \quad ijk = -1$$

$$(-1)^2 = 1, \quad (-1)g = g(-1) = -g \quad g \in \{i, j, k\}$$

$H = \{\pm 1\}$ a subgroup of order 2.

$$iH = \{\pm i\} = Hi$$

$$jH = \{\pm j\} = Hj \;; \quad kH = \{\pm k\} = Hk$$

Similarly $(-i)H = H(-i)$ etc.

and $(\pm 1)H = H(\pm 1) = H$. So

$$Q_8/H = \{H, iH, jH, kH\}$$

Theorem Let G be a group and N be a normal subgroup of G. Then the set of left (=right) cosets $\frac{G}{N}$ of N in G can be equipped with a binary operation

$$(gN)(hN) := ghN \quad \forall g,h \in G$$

With respect to this G/N becomes a group. The identity element is

$$e_{G/N} = eN = N$$

and the inverse of $gN \in G/N$ is

$$(gN)^{-1} = g^{-1}N.$$

Proof By previous considerations, we already know G/N is a monoid. We have $\forall g \in G$:

$$(g^{-1}N)(gN) = (g^{-1}g)N = eN = N = e_{G/N}$$

Similarly $(gN)(g^{-1}N) = N = e_{G/N}$. Thus every element of G/N is a unit. So G/N is a group. $\blacksquare$

<u>Example</u> In Q_8/H we have

$$(iH)^2 = i^2 H = (-1)H = H$$

$$(iH)(jH) = (ij)H = kH$$

Q_8/H	H	iH	jH	kH
H	H	iH	jH	kH
iH	iH	H	kH	jH
jH	jH	kH	H	iH
kH	kH	jk	iH	H

Compare with $(\mathbb{Z}_2 \times \mathbb{Z}_2, +)$

$\mathbb{Z}_2 \times \mathbb{Z}_2$	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
$(0,0)$	$(0,0)$			
$(1,0)$		$(0,0)$	$(1,1)$	$(0,1)$
$(0,1)$			$(0,0)$	$(1,0)$
$(1,1)$				$(0,0)$

Defining $\varphi : \mathbb{Z}_2 \times \mathbb{Z}_2 \longrightarrow Q_8/H$ by $\begin{array}{cc} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{array} \longmapsto \begin{array}{c} 1 \\ i \\ j \\ k \end{array}$

we see that $Q_8/H \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ (isomorphism)

<u>Note</u> 1) Q_8/H is abelian.

Therefore $H \times \frac{Q_8}{H}$ is also abelian.

So, in general, $N \times \frac{G}{N}$ is NOT isomorphic to G. (Q_8 is non-abelian)

2) G/N only makes $\underbrace{\text{sense}}_{\text{as a set}}$ if $N \leq G$

For example, $\frac{\mathbb{Z}_6}{\mathbb{Z}_2}$ makes no sense, since $\mathbb{Z}_2$ is not a subset of $\mathbb{Z}_6$. However, $\frac{\mathbb{Z}_6}{\langle 3 \rangle}$ does make sense: $\langle 3 \rangle = \{0, 3\}$

$$\mathbb{Z}_6/\langle 3 \rangle = \{\underset{\{0,3\}}{\langle 3 \rangle}, \underset{\{1,4\}}{1+\langle 3 \rangle}, \underset{\{2,5\}}{2+\langle 3 \rangle}\}$$

Writing out addition table, we see that $\mathbb{Z}_6/\langle 3 \rangle \cong \mathbb{Z}_3$

3) If G is abelian, any subgroup is normal. Moreover, any subgroup contained in the <u>center</u> of G is normal. Recall the center of G:

$$Z(G) = \{g \in G ; gx = xg \ \forall x \in G\}$$

4) If $|G/H| = 2$ then H is automatically normal:

$$G = H \sqcup gH \qquad g \in G, g \notin H.$$
$$= H \sqcup Hg$$

So $gH = G \setminus H = Hg$

set complement

For ex. $A_n \trianglelefteq S_n \quad \forall n$