

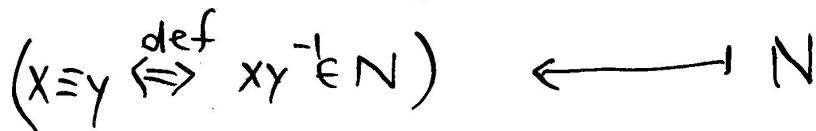
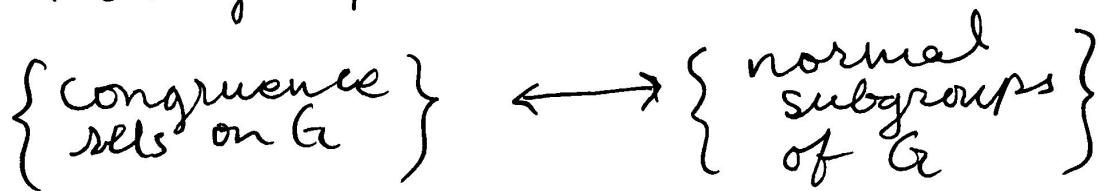
Recap

Quotient Groups

- Monoids = "groups without inverses"  
 Ex.  $(\mathbb{Z}, \cdot, 1)$

-  $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$  can be generalized to  
 $M/\equiv$   $M$  any monoid  
 $\equiv$  a congruence rel. on  $M$ .

- For groups  $G$ :



Note! Given  $N \trianglelefteq G$ , we have  $xy^{-1} \in N \iff x \in Ny$ .

Thus the congruence classes are the (left = right) cosets of  $N$  in  $G$ :

$$[g] = \{h \in G \mid h \in Ng\} = Ng = gN \quad \forall g \in G.$$

Let  $H \leq G$ .

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Notation The set of left cosets of  $H$  in  $G$  is denoted  $G/H$ .

The set of right cosets of  $H$  in  $G$  is denoted  $H \backslash G$ .

$$G/H = \{gH : g \in G\}$$

$$H \backslash G = \{Hg : g \in G\}.$$

Example  $G = Q_8$  quaternion group

$$Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$$

$$i^2 = j^2 = k^2 = -1 ; \quad ijk = -1$$

$$(-1)^2 = 1, \quad (-1)g = g(-1) = -g \quad g \in \{i, j, k\}$$

$H = \{\pm 1\}$  a subgroup of order 2.

$$iH = \{\pm i\} = ~~Hi~~ = Hi$$

$$jH = \{\pm j\} = Hj ; \quad kH = \{\pm k\} = Hk$$

Similarly  $(-i)H = H(-i)$  etc.

and  $(\pm 1)H = H(\pm 1) = H$ . So

$$Q_8/H = \{H, iH, jH, kH\}$$

Theorem Let  $G$  be a group and  $N$  be a normal subgroup of  $G$ .

Then the set of left (=right) cosets of  $N$  in  $G$  can be equipped with a binary operation

$$(gN)(hN) := ghN \quad \forall g, h \in G$$

With respect to this  $G/N$  becomes a group. The identity element is

$$e_{G/N} = eN = N$$

and the inverse of  $gN \in G/N$  is

$$(gN)^{-1} = g^{-1}N.$$

Proof By previous considerations, we already know  $G/N$  is a monoid.

We have  $\forall g \in G$ :

$$(g^{-1}N)(gN) = (g^{-1}g)N = eN = N = e_{G/N}$$

Similarly  $(gN)(g^{-1}N) = N = e_{G/N}$ .

Thus every element of  $G/N$  is a unit. So  $G/N$  is a group.



Example In  $\mathbb{Q}_8/H$  we have

$$(iH)^2 = i^2 H = (-1)H = H$$

$$(iH)(jH) = (ij)H = kH$$

$\mathbb{Q}_8/H$	$H$	$iH$	$jH$	$kH$
$H$	$H$	$iH$	$jH$	$kH$
$iH$	$iH$	$H$	$kH$	$jH$
$jH$	$jH$	$kH$	$H$	$iH$
$kH$	$kH$	$jH$	$iH$	$H$

Compare with  $(\mathbb{Z}_2 \times \mathbb{Z}_2, +)$

$\mathbb{Z}_2 \times \mathbb{Z}_2$	$(0,0)$	$(1,0)$	$(0,1)$	$(1,1)$
$(0,0)$	$(0,0)$			
$(1,0)$		$(0,0)$	$(1,1)$	$(0,1)$
$(0,1)$			$(0,0)$	$(1,0)$
$(1,1)$				$(0,0)$

Defining  $\varphi: \mathbb{Z}_2 \times \mathbb{Z}_2 \rightarrow \mathbb{Q}_8/H$  by

$$\begin{array}{ccc} 0 & 0 & \mapsto 1 \\ 1 & 0 & \mapsto i \\ 0 & 1 & \mapsto j \\ 1 & 1 & \mapsto k \end{array}$$

we see that  $\mathbb{Q}_8/H \cong \mathbb{Z}_2 \times \mathbb{Z}_2$  (isomorphism)

Note 1)  $Q_8/H$  is abelian.

Therefore  $H \times \frac{Q_8}{H}$  is also abelian.

So, in general,  $N \times \frac{G}{N}$  is NOT isomorphic to  $G$ , ( $Q_8$  is non-abelian)

2)  $G/N$  only makes sense <sup>as a set</sup> if  $N \leq G$

For example,  $\frac{\mathbb{Z}_6}{\mathbb{Z}_2}$  makes no sense, since  $\mathbb{Z}_2$  is not a subset of  $\mathbb{Z}_6$ . However,  $\frac{\mathbb{Z}_6}{\langle 3 \rangle}$  does

make sense:  $\langle 3 \rangle = \{0, 3\}$

$$\mathbb{Z}_6 / \langle 3 \rangle = \left\{ \begin{array}{l} \langle 3 \rangle, \\ \text{"} \\ \{0, 3\} \end{array} , \begin{array}{l} 1 + \langle 3 \rangle, \\ \text{"} \\ \{1, 4\} \end{array} , \begin{array}{l} 2 + \langle 3 \rangle \\ \text{"} \\ \{2, 5\} \end{array} \right\}$$

Writing out addition table, we see that  $\mathbb{Z}_6 / \langle 3 \rangle \cong \mathbb{Z}_3$

3) If  $G$  is abelian, any subgroup is normal. Moreover, any subgroup contained in the center of  $G$  is normal. Recall the center of  $G$ :  
 $Z(G) = \{g \in G \mid gx = xg \forall x \in G\}$

4) If  $|G/H| = 2$  then  $H$  is automatically normal:

$$G = H \cup gH \quad g \in G, g \notin H.$$

$$= H \cup Hg$$

$$\text{So } gH = \underset{\substack{\uparrow \\ \text{set complement}}}{G \setminus H} = Hg$$

For ex.  $A_n \trianglelefteq S_n \quad \forall n$