MATH 403/503 - L24

Fields of Fractions. (Judson §18.1)

The process of constructing the field $\mathbb{Q}$ of rational numbers from the ring of integers $\mathbb{Z}$ can be generalized.

We ~~start~~ start with any integral domain D. Out of D we will construct a field $F_D = \mathrm{Frac}\, D$.

The elements of F_D will be "fractions" $\frac{a}{b}$, $a, b \in D$, $b \neq 0$.

Fractions are basically pairs (a, b), except we have the notion of "equivalent fractions".

Ex. In $\mathbb{Q}$, $\frac{8}{12} = \frac{2}{3}$. More generally,

$$\frac{a}{b} = \frac{c}{d} \iff ad = cb.$$

To generalize this, consider

$$S = \{ (a,b) \in D \times D \mid b \neq 0 \}$$

Define a relation $\sim$ on S by:

$$(a,b) \sim (c,d) \Leftrightarrow ad = cb.$$

<u>Lemma 1.</u> $\sim$ is an equivalence rel.

<u>Pf</u> i) $ab = ab \Rightarrow (a,b) \sim (a,b) \ \forall (a,b) \in S$

ii) Suppose $(a,b) \sim (c,d)$. Then $ad = cb$.

So $cb = ad$. so $(c,d) \sim (a,b)$.

iii) Suppose $(a,b) \sim (c,d)$, $(c,d) \sim (e,f)$.

Then ① $ad = cb$, ② $cf = ed$.

$$\begin{aligned} \Rightarrow acf &\overset{②}{=} aed \\ &= ade \quad (D \text{ comm.}) \\ &\overset{①}{=} cbe \end{aligned}$$

$$\xRightarrow{D \text{ comm.}} c(af - eb) = 0$$

If $c \neq 0$ then $af = eb$ so $(a,b) \sim (e,f)$

If $c = 0$ then $ad = ed = 0 \underset{d \neq 0}{\Longrightarrow} a = e = 0$

so $af = eb \ (= 0)$ so $(a,b) \sim (e,f)$

Def The fraction $\frac{a}{b} := [(a,b)]_\sim$ is the equivalence class in S containing (a,b).

Let $F_D = S/\sim$ be the set of fractions $\frac{a}{b}$ for $a, b \in D$, $b \neq 0$.

We want to turn F_D into a ring. We attempt to define $+, \cdot$ by

$$\frac{a}{b} + \frac{c}{d} := \frac{ad+bc}{bd}$$

$$\frac{a}{b} \cdot \frac{c}{d} := \frac{ac}{bd}$$

Lemma 2 These operations are well-defined.

Proof Suppose $\frac{a_1}{b_1} = \frac{a_2}{b_2}$, $\frac{c_1}{d_1} = \frac{c_2}{d_2}$.

That is, $a_1 b_2 = a_2 b_1$, $c_1 d_2 = c_2 d_1$ hold in D. WTS $\frac{a_1 d_1 + b_1 c_1}{b_1 d_1} = \frac{a_2 d_2 + b_2 c_2}{b_2 d_2}$

i.e. $(a_1 d_1 + b_1 c_1) b_2 b_2 = (a_2 d_2 + b_2 c_2) b_1 d_1$ in D. We have!

$$(a_1 d_1 + b_1 c_1) b_2 d_2 = \underbrace{a_1 b_2}_{} d_1 d_2 + b_1 b_2 \underbrace{c_1 d_2}_{}$$

$$= a_2 \underbrace{b_1 d_1}_{} d_2 + \underbrace{b_1}_{} b_2 \, c_2 \underbrace{d_1}_{}$$

$$= (a_2 d_2 + b_2 c_2) b_1 d_1$$

This proves $+$ is well-defined.

The case of $\cdot$ is an exercise. $\blacksquare$

Lemma 3 $(F_D, +, \cdot)$ is a field.

Proof $0_{F_D} = \frac{0}{1}$ and $1_{F_D} = \frac{1}{1}$:

$$\frac{0}{1} + \frac{a}{b} = \frac{0 \cdot b + 1 \cdot a}{1 \cdot b} = \frac{a}{b} \quad \text{similarly } \frac{a}{b} + \frac{0}{1} = \frac{a}{b}.$$

$$\frac{1}{1} \cdot \frac{a}{b} = \frac{1 \cdot a}{1 \cdot b} = \frac{a}{b} = \frac{a}{b} \cdot \frac{1}{1}$$

$$-\frac{a}{b} = \frac{-a}{b} : \quad \text{(check)}$$

$$\left(\frac{a}{b}\right)^{-1} = \frac{b}{a} : \quad \text{(check)}$$

<u>Th.</u> D integral dom

i) Then $\varphi: D \longrightarrow F_D$ is an inj
$d \longmapsto \frac{d}{1}$
ring hom (s.t. $\varphi(d)$ is a unit for every $d \neq 0$ in D.)

ii) If E is any field with an inj ring hom $\psi: D \to E$, then $\exists ! \alpha : F_D \to E$ s.t. $\psi = \alpha \circ \varphi$ i.e.:

$$\begin{tikzcd} D \arrow[r, "\varphi"] \arrow[dr, "\forall \psi"'] & F_D \arrow[d, dashed, "\exists ! \alpha"] \arrow[dl, phantom, "\circlearrowleft"] \\ & E \end{tikzcd}$$

iii*) If $(\mathcal{F}_D, \Phi)$ is any other pair where $\mathcal{F}_D$ is a field and $\Phi: D \to \mathcal{F}_D$ is an inj. ring hom. satisfying ii) then $F_D \cong \mathcal{F}_D$.

Proof i) $\varphi(1_D) = \frac{1_D}{1_D} = 1_{F_D}$ and

$\varphi(d_1 d_2) = \frac{d_1 d_2}{1} = \frac{d_1}{1} \cdot \frac{d_2}{1} = \varphi(d_1)\varphi(d_2)$

$\varphi(d_1 + d_2) = \frac{d_1+d_2}{1} = \frac{d_1}{1} + \frac{d_2}{1}$

$= \varphi(d_1) + \varphi(d_2)$, $\forall d_1, d_2 \in D$

So φ is a ring homomorphism.

Suppose $\varphi(d) = 0_{F_D} = \frac{0}{1}$. Then $\frac{d}{1} = \frac{0}{1}$.

This implies $d \cdot 1 = 0 \cdot 1 \Rightarrow d = 0$. So $\ker\varphi = \{0\}$

So φ is injective.

ii) (uniqueness) Suppose $\exists \alpha : F_D \to E$ such that $\psi = \alpha \circ \varphi$. Then $\forall a \in D, b \in D, b \neq 0$

$\alpha\left(\frac{a}{b}\right) = \alpha\left(\frac{a}{1} \cdot \frac{1}{b}\right) = \alpha\left(\frac{a}{1}\right) \cdot \alpha\left(\frac{b}{1}\right)^{-1} =$

$= \alpha(\varphi(a))\,\alpha(\varphi(b))^{-1} = \psi(a)\psi(b)^{-1}$.

(Existence): Define $\alpha\left(\frac{a}{b}\right) := \psi(a)\psi(b)^{-1}$.

Then α is well-defined (check!)

and a ring homomorphism (check!)

satisfying $(\alpha \circ \varphi)(d) = \alpha\left(\frac{d}{1}\right) = \psi(d)\psi(1)^{-1} = \psi(d)$

$\forall d \in D$.

iii*) By ii) for (F_D, ψ) there is a ring hom $\alpha : F_D \to \mathcal{F}_D$ such that

$$\begin{tikzcd} D \arrow[r, "\varphi"] \arrow[dr, "\Phi"'] & F_D \arrow[d, dashed, "\alpha"] \arrow[dl, phantom, "\circlearrowleft"] \\ & \mathcal{F}_D \end{tikzcd}$$

On the other hand, by ii) for $(\mathcal{F}_D, \Phi)$ there is a ring hom $\beta : \mathcal{F}_D \to F_D$ such that

$$\begin{tikzcd} D \arrow[r, "\Phi"] \arrow[dr, "\varphi"'] & \mathcal{F}_D \arrow[d, dashed, "\beta"] \arrow[dl, phantom, "\circlearrowleft"] \\ & F_D \end{tikzcd}$$

Now we have two commutative diagrams:

$$\begin{tikzcd} D \arrow[r, "\varphi"] \arrow[dr] & F_D \arrow[d, "\beta\circ\alpha"] \arrow[dl, phantom, "\circlearrowleft"] \\ & F_D \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} D \arrow[r, "\varphi"] \arrow[dr] & F_D \arrow[d, "\mathrm{Id}_{F_D}"] \arrow[dl, phantom, "\circlearrowleft"] \\ & F_D \end{tikzcd}$$

So by uniqueness of "α" for (F_D, φ) in i), we must have

$$\beta \circ \alpha = \mathrm{Id}_{F_D}.$$

Similarly $\alpha \circ \beta = \mathrm{Id}_{\mathcal{F}_D}$ by ii) for $(\mathcal{F}_D, \phi)$. Therefore α is an isomorphism from F_D to $\mathcal{F}_D$. ∎