

MATH 403/503 - L24

Fields of Fractions. (Judson §18.1)

The process of constructing the field \mathbb{Q} of rational numbers from the ring of integers \mathbb{Z} can be generalized.

We start with any integral domain D . Out of D we will construct a field $F_D = \text{Frac } D$.

The elements of F_D will be "fractions" $\frac{a}{b}$, $a, b \in D, b \neq 0$.

Fractions are basically pairs (a, b) , except we have the notion of "equivalent fractions".

Ex. In \mathbb{Q} , $\frac{8}{12} = \frac{2}{3}$. More generally,

$$\frac{a}{b} = \frac{c}{d} \Leftrightarrow ad = cb.$$

To generalize this, consider

$$S = \{ (a, b) \in D \times D \mid b \neq 0 \}$$

Define a relation \sim on S by:

$$(a, b) \sim (c, d) \iff ad = cb.$$

Lemma 1. \sim is an equivalence rel.

Pf i) $ab = ab \implies (a, b) \sim (a, b) \forall (a, b) \in S$

ii) Suppose $(a, b) \sim (c, d)$. Then $ad = cb$.
So $cb = ad$. So $(c, d) \sim (a, b)$.

iii) Suppose $(a, b) \sim (c, d)$, $(c, d) \sim (e, f)$.

Then ① $ad = cb$, ② $cf = ed$.

$$\begin{aligned} \implies acf &\stackrel{\textcircled{2}}{=} aed \\ &= ade \quad (D \text{ comm.}) \\ &\stackrel{\textcircled{1}}{=} cbe \end{aligned}$$

$$D \text{ comm.} \implies c(af - eb) = 0$$

If $c \neq 0$ then $af = eb$ so $(a, b) \sim (e, f)$

If $c = 0$ then $ad = ed = 0 \xrightarrow{d \neq 0} a = e = 0$

so $af = eb (= 0)$ so $(a, b) \sim (e, f)$

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Def \checkmark The fraction
 $\frac{a}{b} := [(a, b)]_{\sim}$ is the equivalence
 class in \mathbb{S} containing (a, b) .
 Let $F_D = S/\sim$ be the set of
 fractions $\frac{a}{b}$ for $a, b \in D, b \neq 0$.

We want to turn F_D into a
 ring. We attempt to define $+$,
 by

$$\frac{a}{b} + \frac{c}{d} := \frac{ad + bc}{bd}$$

$$\frac{a}{b} \cdot \frac{c}{d} := \frac{ac}{bd}$$

Lemma 2 These operations are
 well-defined.

Proof Suppose $\frac{a_1}{b_1} = \frac{a_2}{b_2}, \frac{c_1}{d_1} = \frac{c_2}{d_2}$.

That is, $a_1 b_2 = a_2 b_1, c_1 d_2 = c_2 d_1$
 hold in D . WTS $\frac{a_1 d_1 + b_1 c_1}{b_1 d_1} = \frac{a_2 d_2 + b_2 c_2}{b_2 d_2}$

i.e. $(a_1 d_1 + b_1 c_1) b_2 d_2 = (a_2 d_2 + b_2 c_2) b_1 d_1$
 in D . We have

$$\begin{aligned}
 (a_1 d_1 + b_1 c_1) b_2 d_2 &= \underline{a_1 b_2} d_1 d_2 + b_1 \underline{b_2 c_1} d_2 \quad 4 \\
 &= a_2 \underline{b_1 d_1} d_2 + \underline{b_1 b_2} c_2 \underline{d_1} \\
 &= (a_2 d_2 + b_2 c_2) b_1 d_1
 \end{aligned}$$

This proves $+$ is well-defined.

The case of \cdot is an exercise. \blacksquare

Lemma 3 $(F_D, +, \cdot)$ is a field.

Proof $0_{F_D} = \frac{0}{1}$ and $1_{F_D} = \frac{1}{1}$:

$$\frac{0}{1} + \frac{a}{b} = \frac{0 \cdot b + 1 \cdot a}{1 \cdot b} = \frac{a}{b} \quad \text{Similarly } \frac{a}{b} + \frac{0}{1} = \frac{a}{b}.$$

$$\frac{1}{1} \cdot \frac{a}{b} = \frac{1 \cdot a}{1 \cdot b} = \frac{a}{b} = \frac{a}{b} \cdot \frac{1}{1}$$

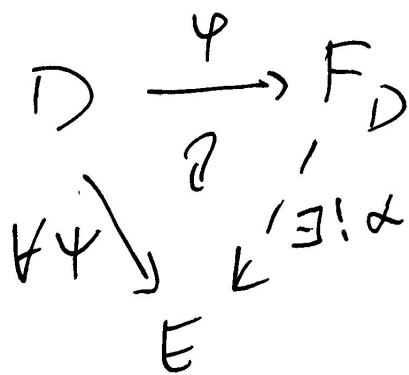
$$-\frac{a}{b} = \frac{-a}{b} : \quad (\text{check})$$

$$\left(\frac{a}{b}\right)^{-1} = \frac{b}{a} : \quad (\text{check})$$

Th. D integral dom

i) Then $\varphi: D \rightarrow F_D$ is an inj
 $d \mapsto \frac{d}{1}$
 ring hom (s.t. $\varphi(d)$ is a unit
 for every $d \neq 0$ in D .)

ii) If E is any field
 with an inj ring hom
 $\psi: D \rightarrow E$, then $\exists \alpha: F_D \rightarrow E$
 s.t. $\psi = \alpha \circ \varphi$ i.e.:



iii*) If (F_D, Φ) is any other pair
 where F_D is a field and $\Phi: D \rightarrow F_D$ is
 an inj. ring hom. satisfying ii)
 then $F_D \cong F_D$.

Proof i) $\varphi(1_D) = \frac{1_D}{1_D} = 1_{F_D}$ and

$$\varphi(d_1 d_2) = \frac{d_1 d_2}{1} = \frac{d_1}{1} \cdot \frac{d_2}{1} = \varphi(d_1) \varphi(d_2)$$

$$\varphi(d_1 + d_2) = \frac{d_1 + d_2}{1} = \frac{d_1}{1} + \frac{d_2}{1} = \varphi(d_1) + \varphi(d_2), \quad \forall d_1, d_2 \in D$$

So φ is a ring homomorphism.

Suppose $\varphi(d) = 0_{F_D} = \frac{0}{1}$. Then $\frac{d}{1} = \frac{0}{1}$.

This implies $d \cdot 1 = 0 \cdot 1 \Rightarrow d = 0$. So $\ker \varphi = \{0\}$

So φ is injective.

ii) (uniqueness) Suppose $\exists \alpha: F_D \rightarrow E$ such that $\psi = \alpha \circ \varphi$. Then $\forall a \in D, b \in D, b \neq 0$

$$\begin{aligned} \alpha\left(\frac{a}{b}\right) &= \alpha\left(\frac{a}{1} \cdot \frac{1}{b}\right) = \alpha\left(\frac{a}{1}\right) \cdot \alpha\left(\frac{1}{b}\right)^{-1} = \\ &= \alpha(\varphi(a)) \alpha(\varphi(b))^{-1} = \psi(a) \psi(b)^{-1}. \end{aligned}$$

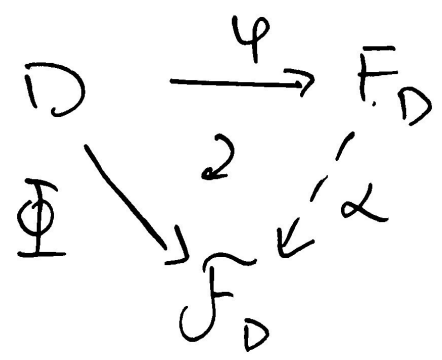
(Existence): Define $\alpha\left(\frac{a}{b}\right) := \psi(a) \psi(b)^{-1}$.

Then α is well-defined (check!)

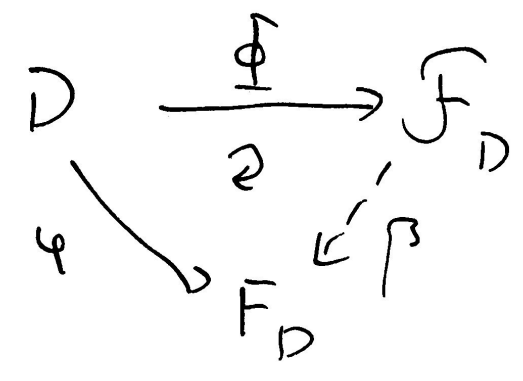
and a ring homomorphism (check!)

satisfying $(\alpha \circ \varphi)(d) = \alpha\left(\frac{d}{1}\right) = \psi(d) \psi(1)^{-1} = \psi(d)$
 $\forall d \in D$.

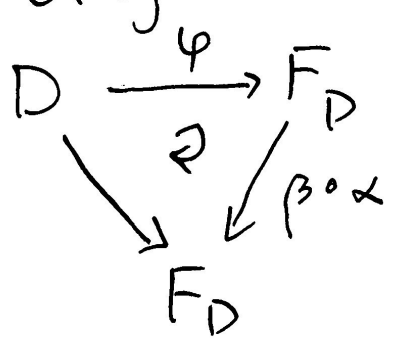
iii*) By ii) for (F_D, ψ) 2
 there is a ring hom $\alpha: F_D \rightarrow \mathcal{F}_D$
 such that



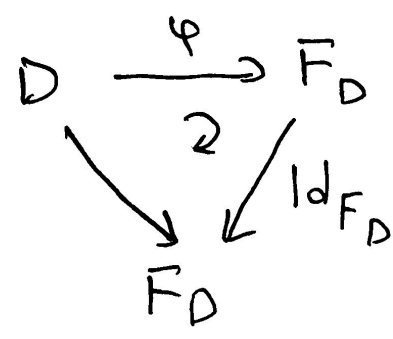
On the other hand, by ii) for (\mathcal{F}_D, ϕ)
 there is a ring hom $\beta: \mathcal{F}_D \rightarrow F_D$
 such that



Now we have two commutative diagrams:



and



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So by uniqueness of " α " for (F_D, φ) in (i), we must have

$$\beta \circ \alpha = \text{Id}_{F_D}.$$

Similarly $\alpha \circ \beta = \text{Id}_{F_D}$ by (ii) for (F_D, φ) . Therefore α is an isomorphism from F_D to F_D .

