403/503 - L22

Maximal Ideals

Def An ideal I of a ring R is *maximal* if

1) $I \neq R$ (i.e. I is proper)
2) If J is an ideal of R such that $I \subseteq J \subseteq R$, then $I = J$ or $J = R$.

(In other words, I is a maximal element in the partially ordered set of proper ideals of R, with respect to inclusion.)

Example (0) in not maximal in $\mathbb{Z}$ since $(0) \subsetneq (2) \subsetneq \mathbb{Z}$.

Example If $n \in \mathbb{Z}$, $n \neq 0$, then (n) is maximal iff n is prime:

$(n) \subsetneq (m) \subsetneq \mathbb{Z} \Rightarrow m \neq \pm 1, \pm n$, $m | n \Rightarrow n$ Composite

If $n = ab$, $a \neq \pm 1$, $b \neq \pm 1$, then $(n) \subsetneq (a) \subsetneq \mathbb{Z} \Rightarrow (n)$ not maximal

Example The evaluation homomorphism

$$\varphi_a = ev_{x=a} : R[x] \longrightarrow R$$
$$p(x) \longmapsto p(a)$$

is defined for any commutative ring R and $a \in R$. (More generally, R may be noncommutative, as long as $a \in Z(R)$ center of R).

Note φ_a is surjective (Check!)

We have $\ker \varphi_a = (x-a)$

$(\supseteq)$: $x-a \in \ker\varphi_a$ clear, and $\ker\varphi_a$ is an ideal of $R[x] \Rightarrow (x-a) \subseteq \ker\varphi_a$

$(\subseteq)$: Let $p(x) \in \ker\varphi_a$. Then $p(a)=0$. Using $x^n - a^n = (x-a)(x^{n-1} + ax^{n-2} + \dots + a^{n-1})$ we see that $p(x) - p(a) \in (x-a)$. So $\ker\varphi_a \subseteq (x-a)$.

<u>Example</u> Consider

$$\varphi_0 = ev_{x=0} : \mathbb{Z}[x] \to \mathbb{Z}$$

$$\varphi_0(p(x)) = p(0)$$

As we just saw, $\ker \varphi_0 = (x) \subseteq \mathbb{Z}[x]$

So, by the 1st Isomorphism Theorem for rings,

$$\frac{\mathbb{Z}[x]}{(x)} \cong \mathbb{Z}$$

Since $\mathbb{Z}$ is an integral domain, So is $\frac{\mathbb{Z}[x]}{(x)}$. Therefore (x) is a prime ideal of $\mathbb{Z}[x]$ (by last lecture). However,

$$(x) \subsetneq \underbrace{(2, x)}_{= \{ p(x) \in \mathbb{Z}[x] \mid p(0) \text{ is even} \}} \subsetneq \mathbb{Z}[x]$$

So (x) is not maximal.

Example. Recall $p(x) \in \overset{\text{field}}{\overset{\downarrow}{\mathbb{F}}}[x]$ is irreducible if $p(x) \notin \mathbb{F}$ and

$$p(x) = a(x)\,b(x) \implies a(x) \text{ or } b(x) \text{ is constant,}$$

Just as for $\mathbb{Z}$ (see Example 2), $(p(x))$ is a maximal ideal of $\mathbb{F}[x]$ iff $p(x)$ is irreducible.

Theorem Let R be a commutative ring and M an ideal of R. TFAE:

1) M is maximal
2) $R/_{M}$ is a field.

Before proof we need:

Def The sum of two ideals I, J of a ring R is

$$I + J = \{a+b \mid a \in I,\ b \in J.\}$$

It is the smallest ideal containing both I and J.

Proof

1) $\Rightarrow$ 2). Suppose M is maximal. Let $r+M \in R/M$ be a nonzero element. This means that $r \notin M$. Consider the ideal $M+(r)$. We have $M \subsetneq M+(r) \subseteq R$. Since M is maximal, we must have $M+(r)=R$. In particular $1_R \in M+(r)$. So $\exists a \in M$, $b \in (r)$ with $1_R = a+b$. By def of (r), $b = sr$, some $s \in R$.

So $sr + a = 1_R \Rightarrow sr - 1_R = -a \in M$

$\Rightarrow sr + M = 1_R + M = 1_{R/M}$

$\Rightarrow (s+M)(r+M) = 1_{R/M} = (r+M)(s+M)$

(Since R is comm, R/M is comm.)

$\Rightarrow r+M$ is a unit in R/M.

So R/M is a field.

2) => 1): Suppose R/M is a field.

Suppose $M \subsetneq I \subseteq R$ for some ideal I. WTS $I = R$. It suffices to show $1 \in I$.

Let $a \in I$, $a \notin M$. Then $a+M$ is nonzero in R/M, hence has an inverse, $b+M$, say. Then

$1 + M = (a+M)(b+M) = ab + M$

$\Rightarrow ab - 1 \in M$

Say $ab - 1 = x \in M$

Then $1 = \underbrace{ab}_{\in I \text{ since } a \in I \text{ and } I \text{ is an ideal}} - \underbrace{x}_{\in M \subseteq I} \in I$.